The Darwin Magnetic Interaction Energy and its Macroscopic Consequences

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Abstract

In metals and plasmas the Coulomb interaction between mobile charged particles is screened. The main long range interaction between the particles is then the magnetic interaction. When radiation is negligible the simplest way to study this interaction is to use the Darwin approximation. In this way one retains a conservative finite degree of freedom problem. We review the derivation of the Darwin Lagrangian and present careful derivations of the corresponding Hamiltonian in various limits. Our results go beyond those of previous authors in several respects. We point out some consequences of the magnetic interaction energy for the dynamics of charged particles with screened Coulomb interaction. Applications to metallic conduction electrons and to plasmas are considered.

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I. INTRODUCTION

The Coulomb potential energy is known to describe the interaction of charged particles with sufficient accuracy for a wide range of applications, especially in atomic, molecular, and condensed matter physics. In cases where radiation is of importance the electrostatic Coulomb treatment does not suffice and must be replaced by a full treatment of the electromagnetic field obeying Maxwell's equations. It is frequently the case, however, that radiation is not of importance, even though the electrostatic approximation is not good enough. For all these cases one may use the Darwin approximation (Darwin [1], Breitenberger [2]). This approximation, which goes beyond the electrostatic one in giving a correct description of magnetic effects, while retaining a finite degree of freedom conservative problem, seems to be fairly unknown in spite of its wide range of applications. Only a few advanced textbooks [3–5] mention it at all. In atomic physics the corresponding physical effect is described by a perturbation to the Hamiltonian that sometimes is called the Breit [6,7] term. This term, however, is of purely classical origin and is identical to the Darwin magnetic interaction energy, see Breitenberger [2].

Under what circumstances can one expect the Darwin magnetic interaction to be responsible for observable physical effects? In atomic physics the interaction represents a well established perturbation together with several other, purely quantum mechanical perturbations (from spin and statistics). Otherwise neutral systems, such as metals and plasmas, where there are moving charged particles, but in which the Coulomb interaction is screened, should be of special interest [2]. In such systems the magnetic Darwin interaction will be the dominating long range interaction. The reason that very few authors in the past have considered the approach taken in this paper is probably that the concepts of magnetic energy and magnetic force are quite subtle and have caused much conceptual difficulty and speculation [8–11]. It is the purpose of this paper to clear up some of this confusion and to advocate the view that the magnetic interaction energy is responsible both for low temperature superconductivity and for the ubiquity of cosmic magnetic fields.

We first review the derivation of the Darwin Lagrangian. It is usually considered to result from an expansion in the small parameter v/c to second order. While this certainly is one way of viewing it, the conventional way, in fact, its actual validity goes somewhat beyond this. High speeds in themselves need not cause radiation since radiation comes from accelerated dipoles. The Darwin Lagrangian has a *post-Galilean* (Woodcock and Havas [12]) character and it can be regarded as implying Maxwell's equations without time derivatives of the transverse electric field (Kaufman and Rostler [13], Nielson and Lewis [14]).

The wide range of applicability of the Darwin Lagrangian, however, does not extend to its approximate Hamiltonian as derived by Darwin. The Darwin interaction energy need not be small even if the individual terms in it are small. The r^{-1} distance dependence and, the absence of the screening effect that limits the Coulomb interaction, means that it can integrate to considerable amounts, as first pointed out by Trubnikov and Kosachev [15]. Under such circumstances the first order (or simplified) Hamiltonian, that is usually found in the literature, is not qualitatively correct. Apart from v/c there is thus also the important dimensionless parameter NR_0/R, where N is the number of particles, R_0 the classical electron radius, and R the length scale of the system. When this parameter is not small higher order terms must be included in the Hamiltonian. One of our main results is an expression for the second order term in the Hamiltonian, equation (63), that becomes exact in the non-relativistic limit.

We thus first carefully derive various expressions for the Hamiltonian corresponding to the Darwin Lagrangian. New exact, as well as approximate, relativistic as well as non-relativistic expressions, are given. The main result is the non-relativistic second order Hamiltonian

$$\mathcal{H}_{D2} = \sum_i \left[\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}^1_{(i)} + \frac{q_i^2}{2m_i c^2}\boldsymbol{A}^1_{(i)} \cdot \boldsymbol{A}^1_{(i)} \right], \tag{1}$$

where

$$\boldsymbol{A}^1_{(i)} = \sum_{j(\neq i)} \frac{q_j[\boldsymbol{p}_j + (\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]}{2m_j c r_{ij}}. \tag{2}$$

The Hamiltonian that is normally used does not have the last term and thus predicts that the magnetic energy goes to minus infinity as the volume containing a constant current distribution goes to infinity. The Hamiltonian $\mathcal{H}_{D2}$ predicts a positive infinite energy for such a situation and there is thus some hope that it is can be useful in improving our qualitative understanding of the physics of long range magnetic interaction.

Some consequences of this Hamiltonian, corresponding to the Darwin Lagrangian are then indicated. It is pointed out that it predicts a curious r^{-3} repulsive force between moving charged particles. After that results for the conduction electrons in a metal, previously found by us (Essén [16]), are reviewed and elaborated. Finally we discuss applications to the magnetism of plasmas. According to the second order Darwin Hamiltonian (1) magnetic structures are shown to have a typical size $R_m \sim 1/\sqrt{R_0 \varrho_\mathrm{n}}$, where ϱ_n is the effective number density of the effective current producing the magnetic field.

II. THE DARWIN APPROXIMATION AND ITS LAGRANGIAN

Everyone knows that there usually is no need to introduce the electric field explicitly in calculations involving the low energy behavior of charged particles; it is sufficient to use the Coulomb potential energy. The reason is that, at low energies, the electric field is completely determined by the positions of the charged particles so that it does not have any independent degrees of freedom. On the other hand when there are large accelerations the system will radiate and it is necessary to include an independent field. When this happens the energy of the particle system is no longer conserved and no Lagrangian or Hamiltonian involving only the particles can exist.

It turns out that one can regard the Coulomb interaction as the zeroth order term in an expansion in the (small) parameter v/c, where v is a typical speed of the system and c the speed of light. Darwin realized that it is possible to carry this expansion one step further and still have only particle degrees of freedom in the problem. The next non-zero terms that appear are of order $(v/c)^2$ and represent magnetic interactions. In this way the Darwin approximation means that one can include the effects of the magnetic field in the problem without introducing the magnetic field explicitly; all that is needed is a velocity dependent particle-particle interaction.

We now proceed to sketch the derivation of the Darwin Lagrangian. We follow the treatment by Landau and Lifshitz [4,5]. Alternative derivations can be found in [1–3], and from a generalized point of view, in [12]. One can appreciate the subtlety of the derivation by studying Bethe & Fröhlich's [17] slightly erroneous, independent rederivation.

The relativistic Lagrangian of a particle in an external electromagnetic field $(\phi, \boldsymbol{A})$ is

$$L_i(\boldsymbol{r}_i, \boldsymbol{v}_i) = -m_i c^2 \sqrt{1 - \frac{v_i^2}{c^2}} - q_i \phi + \frac{q_i}{c} \boldsymbol{v}_i \cdot \boldsymbol{A}. \tag{3}$$

Now assume that the particle is moving in the field of another particle j. Starting from the retarded potentials, expanding in terms of the small time r_{ij}/c, and finally introducing the Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$) one finds that the field produced at i by j is given by

$$\phi_j(\boldsymbol{r}_i, t) = \frac{q_j}{r_{ij}}, \qquad \boldsymbol{A}_j(\boldsymbol{r}_i, t) = \frac{q_j[\boldsymbol{v}_j + (\boldsymbol{v}_j \cdot \boldsymbol{r}_{ij})\boldsymbol{r}_{ij}/r_{ij}^2]}{2cr_{ij}}, \tag{4}$$

where $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$ and $r_{ij} \equiv |\boldsymbol{r}_{ij}|$. The Coulomb gauge is chosen because it is only in this gauge that the Coulomb interaction is independent of the velocities.

The Lagrangian for particle i in the fields produced by particles j is now

$$L_{(i)} = L_i - \sum_{j(\neq i)} U_{ij} \tag{5}$$

where U_{ij} denotes

$$U_{ij} = q_i \phi_j - \frac{q_i}{c} \boldsymbol{v}_i \cdot \boldsymbol{A}_j \;=\; \frac{q_i q_j}{r_{ij}} - \frac{q_i q_j [\boldsymbol{v}_i \cdot \boldsymbol{v}_j + (\boldsymbol{v}_i \cdot \boldsymbol{e}_{ij})(\boldsymbol{v}_j \cdot \boldsymbol{e}_{ij})]}{2c^2 r_{ij}}. \tag{6}$$

Here we have put $\boldsymbol{e}_{ij} \equiv \boldsymbol{r}_{ij}/r_{ij}$. From this one concludes that the full Lagrangian of the system of particles is $L = (\sum_i L_i - \frac{1}{2}\sum_{j(\neq i)} U_{ij})$. If we define

$$\phi_{(i)} \equiv \sum_{j(\neq i)} \phi_j, \qquad \boldsymbol{A}_{(i)} \equiv \sum_{j(\neq i)} \boldsymbol{A}_j, \qquad U_i \equiv \sum_{j(\neq i)} U_{ij} \;= q_i \phi_{(i)} - \frac{q_i}{c} \boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)}, \tag{7}$$

and

$$U_i^C \equiv q_i \phi_{(i)}, \qquad U_i^D \equiv -\frac{q_i}{c} \boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)}, \tag{8}$$

so that $\phi_{(i)}$ and $\boldsymbol{A}_{(i)}$ represent the internal scalar and vector potential, we can write the Darwin Lagrangian

$$L = \sum_i (L_i - \frac{1}{2} U_i) = \sum_i L_i - \sum_{i<j} U_{ij}. \tag{9}$$

More explicitly we can express it in the form

$$L = \sum_i \left[L_i - \frac{1}{2} \left(U_i^C + U_i^D \right) \right] = \sum_i L_i - \sum_{i<j} \frac{q_i q_j}{r_{ij}} - V_D, \tag{10}$$

where V_D is given by

$$V_D = \frac{1}{2}\sum_i U_i^D = -\frac{1}{2}\sum_i \frac{q_i}{c}\boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)} = -\sum_{i<j} \frac{q_i q_j[\boldsymbol{v}_i \cdot \boldsymbol{v}_j + (\boldsymbol{v}_i \cdot \boldsymbol{e}_{ij})(\boldsymbol{v}_j \cdot \boldsymbol{e}_{ij})]}{2c^2 r_{ij}}, \tag{11}$$

and represents a magnetic interaction energy. The quantities $\boldsymbol{A}_{(i)}$ will be called the *internal vector potential*.

Physically the approximation arises from the full Lagrangian of particles plus electromagnetic fields when the independent degrees of freedom of the fields are neglected. This corresponds to radiation being negligible so that there are no (non-virtual) photons present. The field equations corresponding to this Lagrangian can be shown to differ from Maxwell's full equations in the omission of time-derivatives of the transverse electric field (Kaufman and Rostler [13], Nielson and Lewis [14]). As long as such derivatives are small one can expect the Darwin approximation to be good, independently of the value of v/c.

The velocity dependent part V_D of L is called the Darwin (-Breit) term. That these relativistic terms are of importance even in ordinary macroscopic physics when magnetic phenomena are considered has been shown by Coleman and Van Vleck [8]. They are small when individual particles are considered but easily integrate to macroscopic values [15]. The Darwin Lagrangian (10) can, using very general arguments, be shown to be the best approximately relativistic Lagrangian for classical interacting point particles, that gives the Coulomb interaction in the static limit and that contains a vector interaction [12,18]. This type of relativistic Lagrangian turns out to be singular on a surface in phase space [18,19].

Below we will concentrate on the non-relativistic limit and disregard external fields and electrostatic interactions (these being assumed to lead simply to charge neutrality). The relevant Lagrangian is in this case

$$L_{\rm nr} = \sum_i \left(\frac{1}{2}m_i\boldsymbol{v}_i^2 + \frac{q_i}{2c}\boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)}\right). \tag{12}$$

It is obtained from the full Lagrangian (10) if terms of order $(v/c)^2$ are neglected, except that the internal vector potential is considered to be blown up by the largeness of Avogadros number. This is thus only consistent if there are many particles that contribute to $\boldsymbol{A}_{(i)}$ (or, possibly, if there are very small interparticle distances).

The one-body Hamiltonian corresponding to a one-body Lagrangian L_i is by definition

$$\mathcal{H}_i = \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) \equiv \boldsymbol{\pi}_i \cdot \boldsymbol{v}_i - L_i. \tag{13}$$

Using the Lagrangian of equation (3) the corresponding generalized one-body momentum is

$$\boldsymbol{\pi}_i \equiv \frac{\partial L_i}{\partial \boldsymbol{v}_i} = \frac{m_i\boldsymbol{v}_i}{\sqrt{1 - v_i^2/c^2}} + \frac{q_i}{c}\boldsymbol{A}. \tag{14}$$

The explicit expression for the one-body Hamiltonian is then

$$\mathcal{H}_i = \frac{m_i c^2}{\sqrt{1 - v_i^2/c^2}} + q_i\phi = \sqrt{m_i^2 c^4 + c^2\left(\boldsymbol{\pi}_i - \frac{q_i}{c}\boldsymbol{A}\right)^2} + q_i\phi. \tag{15}$$

The next four sections are devoted to the Hamiltonian corresponding to the many-body Lagrangian L.

III. HAMILTONIAN FOR WEAK VELOCITY DEPENDENT INTERACTIONS

Assume that the one-body Lagrangian of particle i is $L_i = L_i(\boldsymbol{r}_i, \boldsymbol{v}_i)$ and that the total Lagrangian is of the type in equation (9) where

$$U_{ij} = U_{ij}(\boldsymbol{r}_i, \boldsymbol{r}_j, \boldsymbol{v}_i, \boldsymbol{v}_j). \tag{16}$$

is the interaction of particles i and j. The Hamiltonian is by definition,

$$\mathcal{H} = \sum_i \boldsymbol{p}_i \cdot \boldsymbol{v}_i - L \tag{17}$$

where the generalized momentum vector is

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \boldsymbol{v}_i} \equiv (\frac{\partial L}{\partial v_{xi}}, \frac{\partial L}{\partial v_{yi}}, \frac{\partial L}{\partial v_{zi}}). \tag{18}$$

If we now use equation (9) for L and (14) for the one-body generalized momenta, we can write

$$\boldsymbol{p}_i = \boldsymbol{\pi}_i - \sum_{j(\neq i)} \frac{\partial U_{ij}}{\partial \boldsymbol{v}_i} = \boldsymbol{\pi}_i - \frac{\partial U_i}{\partial \boldsymbol{v}_i}. \tag{19}$$

Using equation (13) for the one-body Hamiltonians we then get

$$\mathcal{H} = \sum_i \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) + \sum_{i<j} U_{ij} - \sum_i \frac{\partial U_i}{\partial \boldsymbol{v}_i} \cdot \boldsymbol{v}_i \tag{20}$$

for the many-body Hamiltonian. Note that this Hamiltonian is expressed in terms of the one-body momenta $\boldsymbol{\pi}_i$ instead of the correct many-body momenta (18).

Using formula (19) we can express the one-body Hamiltonian in terms of the generalized momentum

$$\mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) = \mathcal{H}_i\left(\boldsymbol{r}_i, \boldsymbol{p}_i + \frac{\partial U_i}{\partial \boldsymbol{v}_i}\right). \tag{21}$$

We now assume that the velocity dependent part of the interaction is *small* (or that $\mathcal{H}_i$ is linear in $\boldsymbol{\pi}_i$)

$$\mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) \approx \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{p}_i) + \frac{\partial \mathcal{H}_i}{\partial \boldsymbol{\pi}_i} \cdot \frac{\partial U_i}{\partial \boldsymbol{v}_i}. \tag{22}$$

According to one of Hamilton's equations we have

$$\frac{\partial \mathcal{H}_i}{\partial \boldsymbol{\pi}_i} = \boldsymbol{v}_i, \tag{23}$$

(this is also a purely algebraic result) and using this we find that

$$\mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) \approx \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{p}_i) + \frac{\partial U_i}{\partial \boldsymbol{v}_i} \cdot \boldsymbol{v}_i. \tag{24}$$

It should be stressed that smallness here means that $|\partial U_i/\partial \boldsymbol{v}_i| \ll |\boldsymbol{p}_i|$, i.e. weak velocity dependent interaction. It is then further consistent to replace $\boldsymbol{v}_i$ with $\boldsymbol{p}_i/m_i$ to first order.

Inserting (24) into equation (20) we, finally, find that

$$\mathcal{H} \approx \sum_i \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{p}_i) + \sum_{i<j} U_{ij} = \sum_i \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{p}_i) + \sum_{i<j} \frac{q_i q_j}{r_{ij}} + V_D. \tag{25}$$

Note that this expression is the same that one would find in the absence of velocity dependence. This result for $\mathcal{H}$ agrees with a general theorem (Landau and Lifshitz [20]) which states that a small addition to the Lagrangian appears in the Hamiltonian with opposite sign. It is, nevertheless, interesting to see explicitly how this comes about in the present case.

The Darwin term [see equation (8)] has the property

$$\sum_i \frac{\partial U_i^D}{\partial \boldsymbol{v}_i} \cdot \boldsymbol{v}_i = \sum_i (-\frac{q_i}{c} \boldsymbol{A}_{(i)}) \cdot \boldsymbol{v}_i = \sum_i U_i^D = 2V_D \tag{26}$$

so that equation (20), which, assuming the explicit interactions of the previous section, reads

$$\mathcal{H} = \sum_i \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) + \sum_{i<j} \frac{q_i q_j}{r_{ij}} + V_D - \sum_i \frac{\partial U_i^D}{\partial \boldsymbol{v}_i} \cdot \boldsymbol{v}_i, \tag{27}$$

gives us

$$\mathcal{H} = \sum_i \mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i) + \sum_{i<j} \frac{q_i q_j}{r_{ij}} - V_D. \tag{28}$$

This equation can be found in [2,13,15]. It differs from equation (25) in that no approximations have been made. On the other hand it has not yet been expressed in terms of the correct many-body canonical momenta $\boldsymbol{p}_i$, and this explains the sign change in front of V_D as formulae (24) and (26) show. Breit [6,7] had trouble with this sign change of the velocity dependent interaction term, which shows that great care must be taken to ensure correct approximations.

IV. EXACT HAMILTONIANS IN TERMS OF THE INTERNAL VECTOR POTENTIAL

In order to complete the derivation of the Darwin Hamiltonian starting from the exact expression (28), we must now express it entirely in terms of momenta $\boldsymbol{p}_i$ instead of velocities. Using equations (19) and (7) we get

$$\boldsymbol{\pi}_i = \boldsymbol{p}_i - \frac{q_i}{c} \boldsymbol{A}_{(i)} \tag{29}$$

so equation (15) gives us

$$\mathcal{H}_i(\boldsymbol{r}_i, \boldsymbol{\pi}_i(\boldsymbol{p}_i, \boldsymbol{A}_{(i)})) = \sqrt{m_i^2 c^4 + c^2 \left[\boldsymbol{p}_i - \frac{q_i}{c}(\boldsymbol{A} + \boldsymbol{A}_{(i)})\right]^2} + q_i \phi. \tag{30}$$

Formula (28) for the Darwin Hamiltonian can then be written in the more explicit form

$$\mathcal{H} = \sum_i \left\{ \sqrt{m_i^2 c^4 + c^2 \left[\boldsymbol{p}_i - \frac{q_i}{c}(\boldsymbol{A} + \boldsymbol{A}_{(i)})\right]^2} + q_i\phi \right\} + \frac{1}{2}\sum_i q_i \phi_{(i)} + \frac{1}{2}\sum_i \frac{q_i}{c}\boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)}. \quad (31)$$

So far no approximations have been made in the derivation of the Hamiltonian from the Darwin Lagrangian. This expression, however, still contains velocities, explicitly in the last sum, and implicitly in $\boldsymbol{A}_{(i)}$.

In order to concentrate on essentials we assume, from now on, that there are no external fields. We also disregard the internal electric potential and replace it, when necessary, with its main effect: the requirement of charge neutrality. The Hamiltonian that we will consider is thus

$$\mathcal{H} = \sum_i \left[\sqrt{m_i^2 c^4 + c^2 \left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^2} + \frac{1}{2}\frac{q_i}{c}\boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)} \right]. \quad (32)$$

Combining equations (29) and (14) and introducing the notation

$$s(v_i) \equiv \sqrt{1 - \frac{v_i^2}{c^2}} = 1 \Big/ \sqrt{1 + \left[\boldsymbol{p}_i - (q_i/c)\boldsymbol{A}_{(i)}\right]^2 / (m_i c)^2}\,, \quad (33)$$

we find that

$$\boldsymbol{v}_i = \frac{s(v_i)}{m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right). \quad (34)$$

When this is inserted into equation (32) we find that we can express it entirely in terms of $\boldsymbol{p}_i$ and $\boldsymbol{A}_{(i)}$. The result is

$$\mathcal{H} = \sum_i \left\{ \sqrt{m_i^2 c^4 + c^2 \left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^2} + \frac{1}{2}\left[q_i c\, \boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)} - (q_i \boldsymbol{A}_{(i)})^2 \right] \Big/ \sqrt{m_i^2 c^4 + c^2 \left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^2} \right\}. \quad (35)$$

This expression is easily manipulated to the simple expression

$$\mathcal{H} = \sum_i \left[m_i^2 c^4 + c^2 \left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right)\left(\boldsymbol{p}_i - \frac{q_i}{2c}\boldsymbol{A}_{(i)}\right)\right] \Big/ \sqrt{m_i^2 c^4 + c^2 \left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^2}\,, \quad (36)$$

for the Hamiltonian of the Darwin Lagrangian, no approximations made.

Expanding the square root we find in the non-relativistic limit

$$\mathcal{H}_{\rm nr} = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)} \right), \quad (37)$$

compare equation (12) for the corresponding Lagrangian. Here the rest energy has been subtracted. If we go to second order in $[(\boldsymbol{p}_i - q_i\boldsymbol{A}_{(i)}/c)/(m_i c)]^2$, we get the quasi-relativistic Hamiltonian

$$\mathcal{H}_{\rm qr} = \mathcal{H}_{\rm nr} - \sum_i \frac{1}{8m_i^3c^2}\left[\boldsymbol{p}_i^4 - 2\boldsymbol{p}_i^2\left(\frac{q_i}{c}\boldsymbol{p}_i\cdot\boldsymbol{A}_{(i)}\right) + 2\left(\frac{q_i}{c}\boldsymbol{p}_i\cdot\boldsymbol{A}_{(i)}\right)\left(\frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^2 - \left(\frac{q_i}{c}\boldsymbol{A}_{(i)}\right)^4\right], \tag{38}$$

Note that we have not assumed that $q_i\boldsymbol{A}_{(i)}/c$ are small. As mentioned above this should be avoided since $\boldsymbol{A}_{(i)}$ arises from a sum over all particles with terms that have the distance dependence r_{ij}^{-1}. In a macroscopic system there is no a priori reason to assume that the result of such a summation is small.

It is tempting to consider the terms containing $\boldsymbol{A}_{(i)}$ in (37) to represent magnetic energy and, as will be discussed below, it does represent the energy lowering associated with the attraction of parallel currents. On the other hand it is expressed in terms of the internal vector potential (a phase space vector function of $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$) rather than the magnetic field. What is usually called magnetic energy in textbooks is a positive definite quantity (for a clear discussion, see Kovetz [21]); magnetic energy is not normally a well defined concept unless made precise in some, more or less, arbitrary way [2,10]. By contrast the Hamiltonian corresponding to the Darwin Lagrangian *is* perfectly well defined and we will therefore pursue it further below.

V. MOMENTUM FORM OF THE NON-RELATIVISTIC INTERNAL VECTOR POTENTIAL

In order to study the behavior of a non-relativistic system of charges due to its internal magnetic energy we should now express this non-relativistic Hamiltonian (37) as a function of $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$. We must thus express $\boldsymbol{A}_{(i)}$ as a function of these variables. Following Kaufman and Soda [22] we put

$$\mathbf{T}_{ij}\boldsymbol{a} \equiv \frac{1}{2r_{ij}}[\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{e}_{ij})\boldsymbol{e}_{ij}] = \frac{1}{2r_{ij}}(\mathbf{1} + \boldsymbol{e}_{ij}\boldsymbol{e}_{ij})\boldsymbol{a}, \tag{39}$$

and find that $\boldsymbol{A}_{(i)}$ is given by [see equations (4), (7), and (34)]

$$\boldsymbol{A}_{(i)} = \sum_{j(\neq i)}\mathbf{T}_{ij}(q_j\boldsymbol{v}_j/c) = \sum_{j(\neq i)}\mathbf{T}_{ij}[s(v_j)q_j\boldsymbol{p}_j/(m_jc)] - \sum_{j(\neq i)}\mathbf{T}_{ij}[s(v_j)q_j^2\boldsymbol{A}_{(j)}/(m_jc^2)]. \tag{40}$$

This is an implicit expression for the $\boldsymbol{A}_{(i)}$. It should be remembered that it contains $\boldsymbol{A}_{(i)}$ also via the $s(v_j)$ according to formula (33). In the non-relativistic limit this dependence vanishes [$s(v_j)\approx 1$] and the expression can be written in the matrix form

$$\begin{pmatrix} \mathbf{1} & \frac{q_2^2}{m_2c^2}\mathbf{T}_{12} & \cdots & \frac{q_N^2}{m_Nc^2}\mathbf{T}_{1N} \\ \frac{q_1^2}{m_1c^2}\mathbf{T}_{21} & \mathbf{1} & \cdots & \frac{q_N^2}{m_Nc^2}\mathbf{T}_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{q_1^2}{m_1c^2}\mathbf{T}_{N1} & \frac{q_2^2}{m_2c^2}\mathbf{T}_{N2} & \cdots & \mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{(1)} \\ \boldsymbol{A}_{(2)} \\ \vdots \\ \boldsymbol{A}_{(N)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^1_{(1)} \\ \boldsymbol{A}^1_{(2)} \\ \vdots \\ \boldsymbol{A}^1_{(N)} \end{pmatrix} \tag{41}$$

where we have defined

$$\begin{pmatrix} \boldsymbol{A}^1_{(1)} \\ \boldsymbol{A}^1_{(2)} \\ \vdots \\ \boldsymbol{A}^1_{(N)} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{0} & \frac{q_2}{m_2 c}\mathbf{T}_{12} & \cdots & \frac{q_N}{m_N c}\mathbf{T}_{1N} \\ \frac{q_1}{m_1 c}\mathbf{T}_{21} & \mathbf{0} & \cdots & \frac{q_N}{m_N c}\mathbf{T}_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{q_1}{m_1 c}\mathbf{T}_{N1} & \frac{q_2}{m_2 c}\mathbf{T}_{N2} & \cdots & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \vdots \\ \boldsymbol{p}_N \end{pmatrix}. \tag{42}$$

Here N is the number of particles, $\mathbf{1}$ and $\mathbf{0}$ are the 3×3 unit matrix and zero matrix respectively. For convenience we define the following $3N \times 3N$ symmetric matrices

$$\overleftrightarrow{\mathbf{T}} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{T}_{12} & \cdots & \mathbf{T}_{1N} \\ \mathbf{T}_{21} & \mathbf{0} & \cdots & \mathbf{T}_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{T}_{N1} & \mathbf{T}_{N2} & \cdots & \mathbf{0} \end{pmatrix}, \quad \overleftrightarrow{\mathrm{q}} \equiv \begin{pmatrix} q_1\mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & q_2\mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & q_N\mathbf{1} \end{pmatrix}, \quad \overleftrightarrow{\mathrm{m}} \equiv \begin{pmatrix} m_1\mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & m_2\mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & m_N\mathbf{1} \end{pmatrix}, \tag{43}$$

and, in terms of these,

$$\overleftrightarrow{\mathbf{R}} \equiv \overleftrightarrow{\mathrm{q}}^2 \overleftrightarrow{\mathrm{m}}^{-1} c^{-2}, \quad \text{and} \quad \overleftrightarrow{\mathbf{U}} \equiv \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{R}}. \tag{44}$$

If we also define the $3N \times 1$ matrices

$$\vec{\boldsymbol{A}} \equiv \begin{pmatrix} \boldsymbol{A}_{(1)} \\ \boldsymbol{A}_{(2)} \\ \vdots \\ \boldsymbol{A}_{(N)} \end{pmatrix}, \quad \vec{\boldsymbol{A}}^1 \equiv \begin{pmatrix} \boldsymbol{A}^1_{(1)} \\ \boldsymbol{A}^1_{(2)} \\ \vdots \\ \boldsymbol{A}^1_{(N)} \end{pmatrix}, \quad \vec{\boldsymbol{p}} \equiv \begin{pmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \vdots \\ \boldsymbol{p}_N \end{pmatrix}, \tag{45}$$

we can rewrite equation (41) in the matrix form

$$\left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathrm{q}}^2\overleftrightarrow{\mathrm{m}}^{-1}c^{-2}\right)\vec{\boldsymbol{A}} = \left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{R}}\right)\vec{\boldsymbol{A}} = \left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{U}}\right)\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}^1. \tag{46}$$

Here $\overleftrightarrow{\mathbf{1}}$ is the $3N \times 3N$ unit matrix. Equation (42) gives us the following expression for $\vec{\boldsymbol{A}}^1$ in terms of $\overleftrightarrow{\mathbf{U}}$ and $\vec{\boldsymbol{p}}$,

$$\vec{\boldsymbol{A}}^1 = \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathrm{q}}\overleftrightarrow{\mathrm{m}}^{-1}c^{-1}\vec{\boldsymbol{p}} = \overleftrightarrow{\mathbf{U}}\overleftrightarrow{\mathrm{q}}^{-1}c\vec{\boldsymbol{p}}. \tag{47}$$

Using this formula (46) can be solved for $\vec{\boldsymbol{A}}$ in terms of $\vec{\boldsymbol{p}}$ as follows

$$\vec{\boldsymbol{A}} = \left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{U}}\right)^{-1}\overleftrightarrow{\mathbf{U}}\left(\overleftrightarrow{\mathrm{q}}^{-1}c\vec{\boldsymbol{p}}\right). \tag{48}$$

This gives us the desired formula for the $\boldsymbol{A}_{(i)}$ in terms of the $\boldsymbol{p}_i$. One notes that

$$\vec{\boldsymbol{A}} \approx \begin{cases} \overleftrightarrow{\mathbf{U}}\,(\overleftrightarrow{\mathrm{q}}^{-1}c\vec{\boldsymbol{p}}) & \text{for } \|\overleftrightarrow{\mathbf{U}}\| \ll 1 \\ \overleftrightarrow{\mathrm{q}}^{-1}c\vec{\boldsymbol{p}} & \text{for } \|\overleftrightarrow{\mathbf{U}}\| \gg 1 \end{cases} \tag{49}$$

if we denote by $\|\overleftrightarrow{\mathbf{U}}\|$ the norm of the matrix.

If we assume that $\overleftrightarrow{\mathbf{U}}$ is small we can expand equation (48) and, if we define $\vec{\boldsymbol{A}}^{\lambda}$ by

$$\vec{\boldsymbol{A}}^{\lambda} \equiv (-1)^{\lambda-1}(\overleftrightarrow{\mathbf{U}})^{\lambda}\left(\overleftrightarrow{\mathbf{q}}^{-1}c\vec{\boldsymbol{p}}\right) \tag{50}$$

we get

$$\vec{\boldsymbol{A}} = \sum_{\lambda=1}^{\infty} \vec{\boldsymbol{A}}^{\lambda}. \tag{51}$$

This gives us a formal solution of the problem of expressing the internal vector potential in terms of the generalized momenta. Trubnikov and Kosachev [15] approached the problem of finding the Hamiltonian of the Darwin Lagrangian by deriving an expansion of $\boldsymbol{v}_i$ in terms of $\boldsymbol{p}_i$. In the present treatment, based on the Hamiltonian (37), that expansion is not needed.

VI. THE NON-RELATIVISTIC HAMILTONIAN IN TERMS OF GENERALIZED MOMENTA

Let us now return to the non-relativistic Hamiltonian (37). Consider the interaction term in it. By means of formula (51) it can be regarded as a sum of terms of the type

$$I^{\lambda} \equiv -\sum_i \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}^{\lambda}_{(i)} = -\frac{1}{2}(\vec{\boldsymbol{A}}^{\lambda})^T\,(\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}). \tag{52}$$

Here a superscript T indicates matrix transposition. Using formula (50) this gives

$$I^{\lambda} = -\frac{1}{2}[(-1)^{\lambda-1}(\overleftrightarrow{\mathbf{U}})^{\lambda}(\overleftrightarrow{\mathbf{q}}^{-1}c\vec{\boldsymbol{p}})]^T\,(\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}) = -\frac{1}{2}[-\overleftrightarrow{\mathbf{U}}(-1)^{\lambda-2}(\overleftrightarrow{\mathbf{U}})^{\lambda-1}(\overleftrightarrow{\mathbf{q}}^{-1}c\vec{\boldsymbol{p}})]^T\,(\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}) \tag{53}$$

Now using $(\overleftrightarrow{\mathbf{B}}\overleftrightarrow{\mathbf{C}})^T = \overleftrightarrow{\mathbf{C}}^T\,\overleftrightarrow{\mathbf{B}}^T$ we find

$$I^{\lambda} = \frac{1}{2}(\vec{\boldsymbol{A}}^{\lambda-1})^T\,\overleftrightarrow{\mathbf{U}}^T\,(\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}). \tag{54}$$

Since $\overleftrightarrow{\mathbf{R}}$ and $\overleftrightarrow{\mathbf{T}}$ both are symmetric, we find that $\overleftrightarrow{\mathbf{U}}^T = (\overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{R}})^T = \overleftrightarrow{\mathbf{R}}\overleftrightarrow{\mathbf{T}}$ we get

$$I^{\lambda} = \frac{1}{2}(\vec{\boldsymbol{A}}^{\lambda-1})^T\,\overleftrightarrow{\mathbf{R}}\overleftrightarrow{\mathbf{T}}(\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}) = \frac{1}{2}(\vec{\boldsymbol{A}}^{\lambda-1})^T\,\overleftrightarrow{\mathbf{R}}\,(\overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{q}}\overleftrightarrow{\mathbf{m}}^{-1}c^{-1}\vec{\boldsymbol{p}}). \tag{55}$$

According to equation (47) we finally get

$$I^{\lambda} = \frac{1}{2}(\vec{\boldsymbol{A}}^{\lambda-1})^T\,\overleftrightarrow{\mathbf{R}}\,\vec{\boldsymbol{A}}^{1} = \sum_i \frac{q_i^2}{2m_i c^2}\boldsymbol{A}^{1}_{(i)} \cdot \boldsymbol{A}^{\lambda-1}_{(i)}. \tag{56}$$

We have thus proved that, for $\lambda > 1$, we have

$$-\sum_i \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}^{\lambda}_{(i)} = \sum_i \frac{q_i^2}{2m_i c^2}\boldsymbol{A}^{1}_{(i)} \cdot \boldsymbol{A}^{\lambda-1}_{(i)}. \tag{57}$$

Using this, the corresponding term in the Hamiltonian (37) gives us

$$-\sum_i \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \left(\sum_{\lambda=1}^{\infty} \boldsymbol{A}_{(i)}^{\lambda}\right) = -\frac{1}{2}\sum_i \left[\frac{q_i}{m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)}^1 - \frac{q_i^2}{m_i c^2}\boldsymbol{A}_{(i)}^1 \cdot \boldsymbol{A}_{(i)}^1 - \frac{q_i^2}{m_i c^2}\boldsymbol{A}_{(i)}^1 \cdot \left(\sum_{\lambda=2}^{\infty} \boldsymbol{A}_{(i)}^{\lambda}\right)\right]. \tag{58}$$

If we now define

$$\delta \boldsymbol{A}_{(i)} \equiv \boldsymbol{A}_{(i)} - \boldsymbol{A}_{(i)}^1 = \sum_{\lambda=2}^{\infty} \boldsymbol{A}_{(i)}^{\lambda}, \tag{59}$$

the non-relativistic Hamiltonian (37) can be written

$$\mathcal{H}_{\mathrm{nr}} = \sum_i \left[\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)}^1 + \frac{q_i^2}{2m_i c^2}\boldsymbol{A}_{(i)}^1 \cdot \boldsymbol{A}_{(i)}^1 + \frac{q_i^2}{2m_i c^2}\delta \boldsymbol{A}_{(i)} \cdot \boldsymbol{A}_{(i)}^1\right]. \tag{60}$$

In conclusion we will write this

$$\mathcal{H}_{\mathrm{nr}} = \mathcal{H}_D + \mathcal{H}_2 + \delta\mathcal{H} = \mathcal{H}_{D2} + \delta\mathcal{H} \tag{61}$$

where the two first terms in (60) constitute the 'traditional' Darwin Hamiltonian

$$\mathcal{H}_D = T + V_D = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)}^1\right) = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} - \sum_{i<j} \frac{q_i q_j [\boldsymbol{p}_i \cdot \boldsymbol{p}_j + (\boldsymbol{p}_i \cdot \boldsymbol{e}_{ij})(\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})]}{2m_i m_j c^2 r_{ij}}. \tag{62}$$

As the derivation above shows, the third term in (60), which we can split into two and three body interactions as follows

$$\mathcal{H}_2 = \sum_i \frac{q_i^2}{2m_i c^2}\boldsymbol{A}_{(i)}^1 \cdot \boldsymbol{A}_{(i)}^1 = \mathcal{H}_{22} + \mathcal{H}_{23} \tag{63}$$

where,

$$\mathcal{H}_{22} = \sum_{i<j} \frac{q_i^2 q_j^2}{4m_i m_j^2 c^4} \frac{\boldsymbol{p}_j^2 + 3(\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})^2}{r_{ij}^2}, \tag{64}$$

and

$$\mathcal{H}_{23} = \sum_i \frac{q_i^2}{2m_i c^2} \sum_{j<k}{}' \frac{q_j q_k}{2m_j m_k c^2} (\mathbf{T}_{ij}\boldsymbol{p}_j) \cdot (\mathbf{T}_{ik}\boldsymbol{p}_k), \tag{65}$$

is due to second order terms in the expansion (51). The fourth term,

$$\delta\mathcal{H} = \sum_i \frac{q_i^2}{2m_i c^2}\delta \boldsymbol{A}_{(i)} \cdot \boldsymbol{A}_{(i)}^1, \tag{66}$$

is thus due to the remaining, third order and higher, terms in the expansion.

We thus now have a non-relativistic Hamiltonian derived from the Darwin approximation of the retarded potentials that describes the magnetic interaction of charged particles. It has not been assumed that the magnetic effects are small. Its practical feasibility will of course depend on whether one can neglect the unknown, higher order terms, $\delta\mathcal{H}$, and thus use the Hamiltonian $\mathcal{H}_{D2}$ of equation (1). One notes that the qualitative meaning of the interaction term in the traditional Darwin Hamiltonian (62), the attraction of parallel currents, is opposite that of the term $\mathcal{H}_2$. Its second (three-body) part, represents a repulsion of parallel currents.

Alternative derivations of the traditional (simplified) Darwin Hamiltonian (62) can be found in [1,2,4,22]. In atomic physics the Darwin term is often called the Breit [6,7] term; for a derivation from modern quantum electrodynamics, see Greiner [23]. In the past only Trubnikov and Kosachev [15] have seriously considered improvements to (62), but the result (1) appears to be new.

VII. A PECULIAR REPULSIVE R^{-3}-FORCE

The terms V_D and $\mathcal{H}_{23}$ in $\mathcal{H}_{\rm nr}$ both are zero if there is no net current distribution. In this case the main new effect predicted by $\mathcal{H}_{\rm nr}$ comes from the two-body part of $\mathcal{H}_2$ as given in equation (64). It can be rewritten as follows

$$\mathcal{H}_{22} = \sum_j \frac{1}{2m_j} \sum_{i(\neq j)} \frac{q_i^2 q_j^2}{4 m_i m_j c^4 r_{ij}^2} [\boldsymbol{p}_j^2 + 3(\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})^2] \tag{67}$$

If we put

$$\epsilon(r_{ij}) \equiv \frac{q_i^2 q_j^2}{m_i m_j c^4} \frac{1}{4\, r_{ij}^2}, \tag{68}$$

we can write this, interchanging dummy indices and denoting the angle between $\boldsymbol{p}_i$ and $\boldsymbol{e}_{ij}$ by θ_{ij}, as

$$\mathcal{H}_{22} = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} \sum_{j(\neq i)} \epsilon(r_{ij})(1 + 3\cos^2\theta_{ij}). \tag{69}$$

If we now absorb this into the kinetic energy we can rewrite it

$$T' = T + \mathcal{H}_{22} = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} \left[1 + \sum_{j(\neq i)} \epsilon(r_{ij})(1 + 3\cos^2\theta_{ij}) \right] = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} \left[1 + V_i(\boldsymbol{r}_i) \right], \tag{70}$$

where we have defined

$$V_i(\boldsymbol{r}_i) \equiv \sum_{j(\neq i)} \epsilon(r_{ij})(1 + 3\cos^2\theta_{ij}) = \sum_{j(\neq i)} \frac{R_i R_j}{4 r_{ij}^2}(1 + 3\cos^2\theta_{ij}). \tag{71}$$

Here $R_i \equiv q_i^2/(m_i c^2)$ are classical particle radii; for electrons this radius is $R_0 = e^2/(mc^2) \approx 2.82 \times 10^{-15}\,$m.

We thus see that when there are moving charged particles in a system there arises (in this formalism) an r^{-3} repulsive force between the parts that is proportional to the kinetic energy of the particles. A large number of questions then arises. Is this a correct physical result? What observable consequences might this force have? Can they be experimentally verified or falsified? Superficially it seems as if this force should have its largest consequences for stellar interiors, if any. For the moment we have no answers to these questions.

VIII. THE TWO-PARTICLE NON-RELATIVISTIC HAMILTONIAN

In the case of two particles it is possible to derive an exact non-relativistic Hamiltonian. In this case it is possible, and meaningful, to start from formula (48) in the form

$$\begin{pmatrix} \boldsymbol{A}_{(1)} \\ \boldsymbol{A}_{(2)} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \frac{q_2^2}{m_2 c^2}\mathbf{T}_{12} \\ \frac{q_1^2}{m_1 c^2}\mathbf{T}_{21} & \mathbf{1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} & \frac{q_2^2}{m_2 c^2}\mathbf{T}_{12} \\ \frac{q_1^2}{m_1 c^2}\mathbf{T}_{21} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{c}{q_1}\boldsymbol{p}_1 \\ \frac{c}{q_2}\boldsymbol{p}_2 \end{pmatrix}, \tag{72}$$

where $\mathbf{T}_{12} = \mathbf{T}_{21} = (\mathbf{1} + \boldsymbol{e}_{12}\boldsymbol{e}_{12})/(2r_{12})$, and do the explicit matrix inversion and multiplication. After some calculation this gives

$$\begin{pmatrix} \boldsymbol{A}_{(1)} \\ \boldsymbol{A}_{(2)} \end{pmatrix} = [1-\epsilon(r)]^{-1} \begin{pmatrix} -\epsilon(r)\left(\mathbf{1} + \frac{3}{1-4\epsilon(r)}\boldsymbol{e}\boldsymbol{e}\right) & \frac{q_2^2}{m_2 c^2}\frac{1}{2r}\left(\mathbf{1} + \frac{1+2\epsilon(r)}{1-4\epsilon(r)}\boldsymbol{e}\boldsymbol{e}\right) \\ \frac{q_1^2}{m_1 c^2}\frac{1}{2r}\left(\mathbf{1} + \frac{1+2\epsilon(r)}{1-4\epsilon(r)}\boldsymbol{e}\boldsymbol{e}\right) & -\epsilon(r)\left(\mathbf{1} + \frac{3}{1-4\epsilon(r)}\boldsymbol{e}\boldsymbol{e}\right) \end{pmatrix} \begin{pmatrix} \frac{c}{q_1}\boldsymbol{p}_1 \\ \frac{c}{q_2}\boldsymbol{p}_2 \end{pmatrix}. \tag{73}$$

where, $r = r_{12}$, $\boldsymbol{e} = \boldsymbol{e}_{12}$ and, where we defined ϵ in equation (68). Using this, and equation (37) we get the 'exact' two-body, non-relativistic, magnetic Hamiltonian, in the form

$$\mathcal{H} = \frac{1}{1-\epsilon(r)}\left[\sum_{i=1}^{2}\frac{1}{2m_i}\left(\boldsymbol{p}_i^2 + \frac{3\epsilon(r)}{1-4\epsilon(r)}(\boldsymbol{p}_i\cdot\boldsymbol{e})^2\right) - \frac{q_1 q_2}{2m_1 m_2 c^2 r}\left(\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + \frac{1+2\epsilon(r)}{1-4\epsilon(r)}(\boldsymbol{p}_1\cdot\boldsymbol{e})(\boldsymbol{p}_2\cdot\boldsymbol{e})\right)\right], \tag{74}$$

after some further, elementary, calculations. The corresponding relativistic Hamiltonian cannot be calculated in closed form but some exact results on the relativistic two-body problem with magnetic interactions have been obtained by Barut and Craig [24]. Other studies of the relativistic two body problem can be found in Van Alstine and Crater [25,26], Landau and Lifshitz [4], and Achieser and Berestestezki [27] who treat the positronium problem.

Distance scales at which $\epsilon(r)$ is of importance require very high energy. One can thus justly argue that, in the non-relativistic limit that we are considering, we can just as well put $\epsilon(r) = 0$ in (74). Dettwiller [28] used this approximation to study the classical Hydrogen atom. If we do this and also assume that both particles are electrons we get the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{2}\frac{1}{2m}\boldsymbol{p}_i^2 - \frac{e^2}{2m^2c^2 r}\left[\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + (\boldsymbol{p}_1\cdot\boldsymbol{e})(\boldsymbol{p}_2\cdot\boldsymbol{e})\right]. \tag{75}$$

If we make the canonical transformation

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \qquad \boldsymbol{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{76}$$

this Hamiltonian becomes

$$\mathcal{H} = \frac{\boldsymbol{P}^2}{2(2m)} + \frac{\boldsymbol{p}^2}{2(m/2)} - \frac{e^2}{8m^2c^2}\frac{\boldsymbol{P}^2 + (\boldsymbol{P}\cdot\boldsymbol{e})^2}{r} + \frac{e^2}{2m^2c^2}\frac{\boldsymbol{p}^2 + (\boldsymbol{p}\cdot\boldsymbol{e})^2}{r}. \tag{77}$$

This Hamiltonian has the peculiar property that center of mass momentum act as an attractive coupling parameter. If one adds the Coulomb repulsion, e^2/r, one sees that it will always dominate over this attraction, so in vacuum this only leads to the well known stabilization of a relativistic beam of charged particles (see e.g. Wiedemann [29]). In a neutral medium, with a screened Coulomb repulsion, there is nothing remarkable about such a velocity dependent interaction since there is a preferred rest frame.

Consider the free Fermi electron gas, and assume that all states with $|\boldsymbol{k}| < k_{\mathrm{F}}$, the Fermi wave number, are filled, but that there are two electrons on the Fermi surface with $|\boldsymbol{k}| = k_{\mathrm{F}}$. All the electrons inside the Fermi surface have zero net momentum and current density, so only the two on the surface contribute. We now assume that the motion of these is described by the Hamiltonian (77). Clearly the lowest energy is obtained when they have maximum center of mass momentum and this is the case when they have (essentially) the same momentum $\boldsymbol{p} = \hbar\boldsymbol{k} = \hbar k_{\mathrm{F}}\boldsymbol{e}_k$. As an ansatz for the wave function we thus use

$$\Psi(\boldsymbol{R}, \boldsymbol{r}) = \frac{1}{L^3}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_1)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_2)\Phi(\boldsymbol{r}) = \frac{1}{L^3}\exp(\mathrm{i}2k_{\mathrm{F}}\boldsymbol{e}_k\cdot\boldsymbol{R})\Phi(\boldsymbol{r}), \tag{78}$$

where $\Phi(\boldsymbol{r})$ is a symmetric function since the electrons must have opposite spins.

If our ansatz is consistent the relative momentum $\boldsymbol{p}$ must be much smaller than the common plane wave momentum $\boldsymbol{P}$, so we neglect the last term in the Hamiltonian (77) compared to the second to last. If we do this our ansatz leads to the Schrödinger equation

$$\left(2\mathcal{E}_{\mathrm{F}} - \frac{\hbar^2}{2(m/2)}\nabla^2 - \frac{\mathcal{E}_{\mathrm{F}}}{mc^2}\frac{e^2}{r}[1+\cos^2\theta]\right)\Phi(\boldsymbol{r}) = E\Phi(\boldsymbol{r}), \tag{79}$$

where $\mathcal{E}_{\mathrm{F}} \equiv \hbar^2 k_{\mathrm{F}}^2/(2m)$ is the Fermi energy, for the relative motion.

To roughly estimate the properties of the solution we replace $1+\cos^2\vartheta$ by its spherical average: $\overline{1+\cos^2\vartheta} = 4/3$. If we further put $\Delta E \equiv E - 2\mathcal{E}_{\mathrm{F}}$ we get the Hydrogen like equation

$$\left(-\frac{\hbar^2}{2(m/2)}\nabla^2 - \frac{4\mathcal{E}_{\mathrm{F}}}{3mc^2}\frac{e^2}{r}\right)\Phi(\boldsymbol{r}) = \Delta E\,\Phi(\boldsymbol{r}). \tag{80}$$

Because of the r^{-1} character of the potential this equation has bound states independently of the weakness of the interaction. The Bohr-radius and energy of the ground state of this equation are

$$R_{\mathrm{DF}} = \frac{3mc^2}{2\mathcal{E}_{\mathrm{F}}}R_{\mathrm{B}} \approx 8\cdot 10^4\, R_{\mathrm{B}}, \tag{81}$$

and

$$\Delta E = -\frac{8}{9}\frac{[e^2\mathcal{E}_{\mathrm{F}}/(mc^2)]^2}{2\hbar^2} \approx 4\cdot 10^{-10} E_{\mathrm{H}}, \tag{82}$$

respectively, where the numerical values refer to a typical metal in which $\mathcal{E}_{\mathrm{F}} \approx 10\,\mathrm{eV}$, and where R_{B} and E_{H} are the usual Bohr-radius and ground state energy of the Hydrogen atom. (Minor errors in the corresponding results in [16] have been corrected.)

IX. MAGNETIC INTERACTIONS BETWEEN CONDUCTION ELECTRONS

We have seen that the magnetic Hamiltonians have as their first order term, the Darwin term, an interaction that lowers the energy when currents are parallel. This means that in any system where charged particles already have kinetic energy (as in a metal because of the Pauli principle and in a plasma because of the temperature) the energy is lowered if the motions are correlated in such a way that a collective current results.

Heisenberg, long ago, suggested that current flows in the superconducting ground state [30]. The mechanism suggested by Heisenberg was, however, not convincing. The idea that superconductivity might be due to magnetic interactions was first advanced long ago by Frenkel [31]. Frenkel's mechanism was wrong, however, as shown by Bethe and Fröhlich [17]. Later Welker [32] speculated in this direction and in 1939 [33] suggested that the magnetic attraction of parallel currents might be responsible for superconductivity. Welker's specific calculations were, however, also wrong and at that point the scientific community seems to have given up the idea. None of the above authors seem to have been aware of the Darwin Hamiltonian, (even if Bethe and Fröhlich came close to rediscovering it) and without a Hamiltonian it is very hard to do good quantum mechanics. The present author investigated the problem of the metallic ground state using the Darwin Hamiltonian and and the free electron gas model. A rather elementary study (Essén [16]) then shows that the maximum energy lowering (per conduction electron) that can be obtained in fact agrees quite well with the observed energy gap in low temperature superconductors. This investigation is reviewed briefly below.

The ground state of the metallic conduction electrons regarded as a Fermi, free electron gas, is normally considered to be characterized by a single parameter, the Fermi energy $\mathcal{E}_{\rm F}$. If we use periodic (Born-von Karman) boundary conditions, the allowed states are

$$\psi_i(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}} \exp(\mathrm{i}\boldsymbol{k}_i \cdot \boldsymbol{r}), \tag{83}$$

where the wavenumber vectors $\boldsymbol{k}_i$ must obey

$$\boldsymbol{k}_i = \frac{2\pi}{L}(n_{ix}, n_{iy}, n_{iz}) \;\text{ with }\; n_{ix}, n_{iy}, n_{iz} = 0, \pm 1, \pm 2 \ldots . \tag{84}$$

For a given density

$$N/L^3 = \frac{1}{3\pi^2} k_{\rm F}^3, \tag{85}$$

i.e. a given number, N, of electrons, it is, however, very unlikely that the electrons exactly fill the 'shell' (Fermi surface) with $\boldsymbol{k} = k_{\rm F}$. The number of possible states on the Fermi surface is

$$N_s = \frac{2}{\pi} k_{\rm F}^2 L^2 = \frac{6\pi}{k_{\rm F} L} N \tag{86}$$

and an important parameter that characterizes the ground state of the gas is then the fraction

$$\gamma = N_c/N_s \tag{87}$$

of these that are filled. Here N_c is the number of electrons on the Fermi surface (the 'zero temperature conduction' electrons). Thus apart from the Fermi energy (or wave number) the ground state is characterized by the parameter γ. For $\gamma = 0$ or $\gamma = 1$ the ground state is non-degenerate but for other values of γ it is degenerate, $\gamma = \frac{1}{2}$ corresponding to maximal degeneracy.

When the Darwin magnetic interaction energy is included in the Hamiltonian all the various degenerate states, corresponding to different distributions of the N_c $\boldsymbol{k}$-vectors on the Fermi surface, are no longer degenerate. Instead a maximally anisotropic distribution will minimize the energy since such a distribution will correspond to maximal current density. It is easy to make an estimate of the optimum energy that the Darwin term in the Hamiltonian might produce and Essén [16] has shown that, for γ-values near 1/2 the energy lowering per conduction electron is at best

$$\Delta_{\rm D} \equiv -\frac{E_{\rm D}}{N} \approx 1.4\ R_0 k_{\rm F} \mathcal{E}_{\rm F}. \tag{88}$$

Here $R_0 \equiv e^2/(mc^2)$ is the classical electron radius, and $E_{\rm D}$ is the expectation value of the Darwin term V_D in the Hamiltonian for a Hartree wave function consisting of a product of one-electron wave functions (83).

When numerical values are inserted it is found that formula (88) gives values that agree closely with the energy gaps associated with superconductivity for low temperature superconductors. Arguments that the magnetic interaction should be too weak or otherwise unsuitable to explain superconductivity are thus wrong. On the other hand, formula (88) contains no free parameters and would thus be falsified by the recently discovered high temperature superconductivity. It can be shown, however, that the interaction of the conduction electrons with the lattice is qualitatively much like a magnetic interaction [16]. These two effects are therefore likely to both contribute to the phenomenon.

X. STATISTICAL MECHANICS AND MAGNETISM

Early studies of the interaction of magnetism with matter, as reviewed by Van Vleck [34], came to the conclusion that, according to classical statistical mechanics, matter does not interact with the magnetic field (it has zero susceptibility), and that therefore all magnetic effects must be explained by quantum mechanics. This finding is a bit worrying since it is found empirically that cosmic plasmas nearly always are connected with intense magnetic activity [35], while theories of plasma physics usually do not take account of quantum effects. Plasma phenomena, on the other hand, are rarely equilibrium phenomena so the discrepancy is not glaring. More relevantly, however, the zero classical susceptibility proofs did not take account of the Darwin magnetic interaction. Since this interaction lowers the energy for parallel currents it seems as a promising candidate for an explanation of cosmic magnetic fields via classical statistical mechanics. We'll look a bit more closely into this below.

Using classical statistical mechanics one can also show that the current density must be zero, see London [36]. This again neglects magnetic interactions and is contrary to

Heisenberg's suggestion of ground state currents [30] and the findings in [16] reviewed in the previous section. London also claims that Bethe and Fröhlich [17] showed that this still holds if magnetic interactions are included, but this is clearly not correct. Bethe and Fröhlich only studied the effect of the magnetic (Darwin) interaction on the effective mass of the electron, and this effect is, of course, completely negligible. In conclusion thus, when magnetic interactions are included, currents are actually not forbidden, but, on the contrary, in good agreement with classical statistical mechanics, as long as there is kinetic energy present in the system.

Krizan and Havas [37] developed statistical mechanics including the (first order) Darwin term in the Hamiltonian. They applied it to plasmas but had to exclude long range interactions for technical reasons. They try to argue that these should be small but that is not convincing. On the contrary, the (first order) Darwin term will diverge if there is a bulk current density over an extended volume. This divergence was called 'magnetische Katastrophe' by Welker [32]. In metals the divergence is prevented by the current density being essentially two-dimensional [16], but in plasmas there is no such restriction. Trubnikov and Kosachev [15] managed to derive results for plasmas that do not rely on the (simplified) Darwin Hamiltonian (61) but that include the full Hamiltonian without approximation. There is, however, reason to be suspicious about all thermodynamics dealing with magnetic effects caused by the Darwin Hamiltonian since the interaction is long range. It is one of the fundamental assumptions of statistical physics that subsystems are approximately closed, or 'quasi-closed', as discussed by Landau and Lifshitz [38].

The electrostatic interaction is also long range, but in this case the Debye screening makes it effectively short range. For magnetism there is no analog to this screening. This may be one reason why plasmas rarely appear to be near thermal equilibrium. I have not been able to find any analysis of these problems in the literature.

Kaufman and Soda [22] also made a study of statistical mechanics that included the Darwin term. They, however, did not apply it to plasmas. Many authors have applied the Darwin approximation to plasmas via particle code models [13,14,39], i.e. by directly integrating the equations of motion. It is, however, very difficult to draw general conclusions from specific numerical simulations.

An alternative velocity dependent interaction between charged particles, suggested by Weber, has been ruled out [40] as leading to unphysical results when applied to plasma physics. The Darwin interaction on the other hand agrees well with known plasma phenomena as well as with other aspects of charged particle dynamics [39,41].

XI. MAGNETIC SELF ENERGY OF ROTATING SPHERICAL CURRENT DISTRIBUTION

When the Darwin energy is a perturbation, as it is in metals, it is enough to consider the Darwin Hamiltonian (62). If, however, we envisage a situation where the magnetic energy according to the Darwin term seems to diverge, we must also include, at least, the next term in the expansion of $\boldsymbol{A}_{(i)}$ and use the 'magnetic' Hamiltonian $\mathcal{H}_{D2}$, as defined by equations (1) and (2). In this section we calculate the contribution to the three terms of $\mathcal{H}_{D2}$ from the current arising from a rotating spherical distribution of charge. We assume charge neutrality, i.e. that there is a compensating immobile distribution of the opposite charge.

Assume that the number density

$$\varrho_{\rm n} = \begin{cases} \frac{N}{4\pi R^3/3} & \text{if } r \leq R \\ 0 & \text{otherwise} \end{cases} \tag{89}$$

of charged particles with charge e and mass m, rotates with angular velocity

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_z. \tag{90}$$

This means that we assume the momenta to be given by

$$\boldsymbol{p}_i = m\boldsymbol{\omega} \times \boldsymbol{r}_i \tag{91}$$

and that there is a current distribution

$$\boldsymbol{j}(\boldsymbol{r}) = e\varrho_{\rm n}\boldsymbol{\omega} \times \boldsymbol{r} \tag{92}$$

that is proportional to the momentum distribution. From this we can calculate the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}{\rm d}V' = \frac{Ne}{2R}\left(1 - \frac{3}{5}\frac{r^2}{R^2}\right)\frac{1}{c}(\boldsymbol{\omega}\times\boldsymbol{r}) = \frac{2\pi}{5c}\left(\frac{5}{3}R^2 - r^2\right)\boldsymbol{j}(\boldsymbol{r}). \tag{93}$$

This vector potential is chosen to match, at $r = R$, to one that goes to zero as $r \to \infty$. The calculation is elementary but some relevant formulae can be found in Essén [42]. We see that $\nabla \cdot \boldsymbol{A} = 0$ so that we are automatically in the Coulomb gauge.

The total kinetic energy can be calculated to be

$$T = \frac{1}{2m}\sum_{i=1}^{N}\boldsymbol{p}_i^2 = \frac{m}{2}\int(\boldsymbol{\omega}\times\boldsymbol{r})^2\varrho_{\rm n}{\rm d}V = \frac{1}{5}mN(R\omega)^2 \tag{94}$$

and the Darwin magnetic self energy of this current distribution is approximately

$$V_D = -\frac{1}{2mc}\sum_i e\boldsymbol{p}_i\cdot\boldsymbol{A}_{(i)} = -\frac{1}{2c}\int \boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}){\rm d}V = -\frac{2}{35}\frac{(Ne)^2}{R}\left(\frac{\omega R}{c}\right)^2. \tag{95}$$

Here we assume that $\boldsymbol{A} \approx \boldsymbol{A}^1$. Finally we get

$$\mathcal{H}_2 = \frac{e^2}{2mc^2}\sum_i \boldsymbol{A}_{(i)}\cdot\boldsymbol{A}_{(i)} = \frac{e^2}{2mc^2}\int |\boldsymbol{A}(\boldsymbol{r})|^2\varrho_{\rm n}{\rm d}V = \frac{3}{175}\frac{e^4N^3\omega^2}{mc^4}. \tag{96}$$

for the diamagnetic term.

For the total energy, kinetic plus magnetic, one thus finds that $E = \overline{\mathcal{H}_{D2}} = T + V_D + \mathcal{H}_2$ is

$$E = \frac{1}{5}mN(R\omega)^2\left[1 - \frac{2}{7}\left(N\frac{R_0}{R}\right) + \frac{3}{35}\left(N\frac{R_0}{R}\right)^2\right]. \tag{97}$$

Here we have introduced the notation

$$R_0 = \frac{e^2}{mc^2}. \tag{98}$$

for the classical electron radius. If we optimize this energy with respect to the dimensionless parameter $x = NR_0/R$ we find that there is a minimum at $x_{\rm min} = 5/3$ and the value of the energy at this minimum is

$$E_{\rm min} = \frac{16}{21}\left(\frac{1}{5}mN(R\omega)^2\right) = 0.762\,T. \tag{99}$$

Evidently the kinetic energy of the moving particles is reduced by roughly 24%, by the magnetic self energy, if the motion causes flow of an electric current.

The quantity $x = NR_0/R$ corresponds to a given number of particles per unit length. If one assumes instead that there is a constant number density $\varrho_{\rm n}$ of particles that contributes to the effective current density it is more interesting to express x in terms of $\varrho_{\rm n}$. This gives $x = \varrho_{\rm n} 4\pi R^2 R_0/3$. The magnetic energy minimization at $x = 5/3$ is the seen to correspond to roughly to the length scale

$$R_m \sim \frac{1}{\sqrt{R_0\varrho_{\rm n}}}. \tag{100}$$

This result that there is a characteristic length scale associated with the magnetic activity seems to be a new prediction of the Hamiltonian (1). One can, of course, not be completely sure that this is not an artifact of the second order approximation that vanishes in a more exact treatment.

XII. CONCLUSIONS

In mechanics the Hamiltonian formalism often seems like a purely formal, and trivial, reformulation of the Lagrangian one. In quantum mechanics and statistical mechanics, on the other hand, the Hamiltonian is crucial for obtaining energy eigenstates and statistical equilibrium distributions. The (relativistic or non-relativistic) Darwin Lagrangian is one of the few examples for which the reformulation is non-trivial and for which no closed form Hamiltonian is known. It seems likely that no such closed form Hamiltonian can be found, at least not in the relativistic case. This paper has improved the situation for the non-relativistic case. These formal difficulties probably reflect corresponding subtleties in the physical problem.

Several new facts regarding the Hamiltonian corresponding to the Darwin Lagrangian have been presented. The general result for weak velocity dependent interactions as given in equations (16) to (25) appears to be new. The same goes for the relativistic result (36) and the quasi relativistic Hamiltonian (38).

New is also the matrix treatment of the problem of finding the internal vector potential as a function of the generalized momenta, the explicit formula (48), and the expansion (50). The most useful outcome of the matrix formalism, namely the result that the second order term beyond the traditional Darwin Hamiltonian, can be obtained in closed form, is one of the most elegant new results of this paper.

One notes that the qualitative meaning of the Darwin Hamiltonian, the energy lowering due to the attraction of parallel currents, is opposite to that of the new second order term. The unphysical divergence of this energy lowering, as predicted by the traditional Darwin Hamiltonian for constant current densities, is thus prevented by the new term.

The parallel current attraction energy lowering is an effect that is not manifest in the energy when it is expressed in terms of velocities, see equation (28). This means that this is a rather subtle effect related to the behavior of phase space volume elements. This may be one of the reasons that it is not well understood or discussed, in spite of the fact that the attraction of parallel currents is one of the more fundamental elementary facts of electromagnetism and represents one way of measuring current accurately. Extensive arguments that this attraction manifests itself physically as the attractive force behind low temperature superconductivity have been published before by the present author [16], and are thus only briefly touched upon in this paper. That this attraction also might be responsible for the abundance and persistence of cosmic magnetic fields, seems to be a new point of view. It is hoped that it will contribute to a deeper understanding of these, in general, quite difficult problems.

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