Phase space energy of charged particles; spontaneous formation of magnetic structure and new effective forces

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Abstract

The phase space energy of a system of charged particles in the negligible radiation (Darwin) non-relativistic limit is derived. The usefulness of the second order approximation to this Hamiltonian, previously found by the present author, is discussed and given a stronger theoretical foundation. As a result a typical length scale for magnetic structures is found.

The virial theorem is then applied to the second order Hamiltonian. Together with the assumption that magnetism is a first order perturbation it proves that magnetic interaction lowers the energy of a plasma. This indicates that net currents must flow in a plasma and a simple estimate indicates that the spontaneously forming magnetic structures are resistant to thermal disruption.

The effective one-particle Hamiltonian implied by the second order Hamiltonian is calculated. It predicts a new, effective many-body, force that accelerates charged particles near a large plasma. It is conjectured that this effective force could give a simpler explanation for stellar wind and other large scale plasma phenomena.

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I. ON DARWIN'S APPROACH TO ELECTROMAGNETISM

We live in a world build from charged particles. We know the exact theory needed to predict their behavior: Maxwell's equations and the Lorentz force law. Yet, when there are macroscopic numbers of free charged particles, as in a plasma or metal, the resulting coupled equations for particles and fields become so complicated that approximations are necessary. The standard approximation for plasmas is magnetohydrodynamics. In spite of the approximations, it has been very difficult to achieve a useful understanding of plasma behavior using this theory. Here we will take a radically different approach. We will study the finite degree of freedom many-body system that best approximates the real system, using analytical mechanics.

As long as radiation can be neglected a system of charged particles has a conserved energy. Usually the corresponding Hamiltonian is taken to contain the Coulomb, or electrostatic, interaction energy only. In 1920 C. G. Darwin [1] realized that magnetic effects can be included in such a description up to terms that are proportional to $(v/c)^2$. The Darwin approach leads firstly to a Lagrangian [2–6]. Darwin found a first order approximation to the corresponding Hamiltonian [1,7–9] but, for a long time, no exact explicit expression for it was known. The first order Hamiltonian, which gives excellent results for few-body systems (see [10] for a careful treatment of two-body systems) is however not correct for macroscopic systems. The crucial difference between these is that in few-body systems magnetic effects are always relativistic whereas in macroscopic systems this is not normally the case [3].

In 1968 Trubnikov and Kosachev [11] found a formally exact expression for the Darwin Hamiltonian in terms of a series expansion for the canonical momentum. This expansion, however, would not converge for the case of a macroscopic plasma. The present author reconsidered the problem and found a different expansion for the dependence of the vector potential on the canonical momenta. This led to a closed expression for a second order term and a physically reasonable second order Hamiltonian [12]. The question of convergence was not completely resolved so we return to it here. The first part of this paper gives strong arguments that the second order Darwin Hamiltonian, formula (19) or (21), is qualitatively correct and physically useful for macroscopic systems.

The Darwin approximation to the equations of motion for charged particles has been used before in plasma physics with considerable success [13–15]. The Hamiltonian on the other hand has not been studied much in spite of the great importance of the Hamiltonian formalism in statistical physics and the great simplicity and generality of energy considerations. The reason for this has been the uselessness of the first order Hamiltonian and the unknown, or intractable, form of the exact Hamiltonian. The new second order term that makes the approximate Hamiltonian qualitatively correct is therefore quite important.

This second order Darwin Hamiltonian has far reaching consequences for the nature of the thermal equilibrium of systems containing mobile charged particles with kinetic energy. As discussed in previous publications by the present author the magnetic interaction can be expected to play an important role in low-temperature superconductivity [17,12]. Here we calculate, using the virial theorem, the time-averages of different contributions to the energy in this Hamiltonian. We find that the magnetic energy really does lower the energy of a plasma. This, of course, means that net currents must flow in the plasma; otherwise there would not be any magnetic effects. We also estimate the size and energy of the spontaneous

magnetic structures and find that they should survive thermal fluctuations.

Finally we study the effective one-particle Hamiltonian that arises from the second order Darwin Hamiltonian when one considers the motion of one of the particles assuming given positions and momenta of all other particles. This is seen to lead to the usual result for the case of a few particle system. When macroscopic numbers contribute, on the other hand, a new effective, many-body, force is predicted. It is conjectured that it might be useful for an understanding of stellar plasmas, in particular stellar wind.

II. THE NON-RELATIVISTIC DARWIN HAMILTONIAN

A. Background, the Darwin Lagrangian

There is a well known exact relativistic Lagrangian density for charged particles interacting via the electromagnetic field. Since radiation is an unlikely phenomenon at low speeds (radiation is essentially proportional to $(v/c)^3$) one can derive an approximate Lagrangian in which the independent degrees of freedom of the electromagnetic field are neglected. This Lagrangian, which depends only on particle positions and velocities, is the Darwin Lagrangian. It can be written

$$L(\boldsymbol{r}_j, \boldsymbol{v}_j) = \sum_i \left(\frac{1}{2} m_i \boldsymbol{v}_i^2 - \frac{q_i}{2} \phi_{(i)} + \frac{q_i}{2c} \boldsymbol{v}_i \cdot \boldsymbol{A}_{(i)} \right). \tag{1}$$

Here the electromagnetic potentials at particle i are given in terms of the positions and velocities of the other particles

$$\phi_{(i)} = \sum_{j(\neq i)} \frac{q_j}{r_{ij}}, \quad \text{and} \quad \boldsymbol{A}_{(i)} = \sum_{j(\neq i)} \frac{q_j [\boldsymbol{v}_j + (\boldsymbol{v}_j \cdot \boldsymbol{e}_{ij}) \boldsymbol{e}_{ij}]}{2 c r_{ij}} \tag{2}$$

where $\boldsymbol{e}_{ij}$ is the unit vector pointing from particle i toward j and r_{ij} is the distance between them. These expressions come from approximating the exact Lienard-Wiechert potentials, thereby choosing the Coulomb gauge (in which the Coulomb interaction retains its velocity independent form). Usually a relativistic second order correction to the kinetic energy is added since Darwin originally had atomic (few body) systems in mind. In these systems magnetic effects are always relativistic effects. In macroscopic systems that we have in mind here, the largeness of Avogadro's number make magnetic effects large even non-relativistically.

It must be emphasized that the Darwin Lagrangian stands on very firm theoretical and experimental ground. It really does give an excellent description of the dynamics of interacting charged particles when radiation is negligible [3,6,9,14,18].

B. The Darwin Hamiltonian

Recently the author [12] derived the following exact expression for the non-relativistic Darwin Hamiltonian of a system of charged particles of mass m_i and charge q_i

$$H = \sum_i \left[\left(\frac{\boldsymbol{p}_i^2}{2m_i} + \frac{q_i}{2} \phi_{(i)} \right) - \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)} \right] = H_0 + I_{\mathrm{m}}, \tag{3}$$

where $\boldsymbol{A}_{(i)}$ is given by (2), and where H_0 stands for kinetic plus Coulomb energy and $I_{\rm m}$ for the magnetic energy. By means of the (tensor) operator P_{ij} defined through

$$\mathsf{P}_{ij}\boldsymbol{a} \equiv \frac{1}{2}[\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{e}_{ij})\boldsymbol{e}_{ij}] = \frac{1}{2}(\mathbf{1} + \boldsymbol{e}_{ij}\boldsymbol{e}_{ij})\boldsymbol{a}, \tag{4}$$

we can write $\boldsymbol{A}_{(i)}$ of equation (2) as follows

$$\boldsymbol{A}_{(i)} = \sum_{j(\neq i)} \mathsf{P}_{ij}\frac{\boldsymbol{v}_j}{c}\frac{q_j}{r_{ij}}. \tag{5}$$

The expression (3) for the Hamiltonian is not a closed expression, $H(\boldsymbol{r}_j, \boldsymbol{p}_j)$, since the quantities $\boldsymbol{A}_{(i)}$ depend on the velocities instead of the momenta. Since

$$\boldsymbol{v}_i = \frac{1}{m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{(i)}\right) \tag{6}$$

we get the following equation for $\boldsymbol{A}_{(i)}$ in terms of canonical momenta

$$\boldsymbol{A}_{(i)} + \sum_{j(\neq i)} \frac{\mathsf{P}_{ij}}{r_{ij}}\frac{q_j^2}{m_j c^2}\boldsymbol{A}_{(j)} = \sum_{j(\neq i)} \frac{\mathsf{P}_{ij}}{r_{ij}}\frac{q_j \boldsymbol{p}_j}{m_j c} \tag{7}$$

In order to solve this equation it is convenient to introduce matrix notation. (An exact solution of a simplified continuum version of this equation is given in section III.)

C. Matrix solution for vector potential in terms of momenta

For brevity we now introduce

$$\mathsf{T}_{ij} \equiv \frac{\mathsf{P}_{ij}}{r_{ij}}, \quad \text{and} \quad R_j \equiv \frac{q_j^2}{m_j c^2}, \tag{8}$$

where T_{ij} was originally introduced by Kaufman and Soda [19], and where R_j are the classical particle radii. Using this we introduce matrix notation as follows

$$\overleftrightarrow{\mathbf{T}} \equiv \begin{pmatrix} \mathbf{0} & \mathsf{T}_{12} & \cdots & \mathsf{T}_{1N} \\ \mathsf{T}_{21} & \mathbf{0} & \cdots & \mathsf{T}_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \mathsf{T}_{N1} & \mathsf{T}_{N2} & \cdots & \mathbf{0} \end{pmatrix}, \quad \overleftrightarrow{\mathbf{R}} \equiv \begin{pmatrix} R_1\mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & R_2\mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & R_N\mathbf{1} \end{pmatrix}, \quad \overleftrightarrow{\mathbf{U}} \equiv \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{R}}. \tag{9}$$

These are $3N \times 3N$ matrices (or $N \times N$ matrices with 3×3 matrices as elements) and the matrix $\overleftrightarrow{\mathbf{U}} = \overleftrightarrow{\mathbf{T}}\overleftrightarrow{\mathbf{R}}$ has dimensionless elements (length divided by length). We also introduce the $3N \times 1$ matrices

$$\vec{\boldsymbol{A}} \equiv \begin{pmatrix} \boldsymbol{A}_{(1)} \\ \boldsymbol{A}_{(2)} \\ \vdots \\ \boldsymbol{A}_{(N)} \end{pmatrix}, \quad \vec{\boldsymbol{A}}_0 \equiv \begin{pmatrix} c\boldsymbol{p}_1/q_1 \\ c\boldsymbol{p}_2/q_2 \\ \vdots \\ c\boldsymbol{p}_N/q_N \end{pmatrix}, \quad \vec{\boldsymbol{A}}_\lambda \equiv \begin{pmatrix} \boldsymbol{A}_{\lambda(1)} \\ \boldsymbol{A}_{\lambda(2)} \\ \vdots \\ \boldsymbol{A}_{\lambda(N)} \end{pmatrix} = -(-\overleftrightarrow{\mathbf{U}})^\lambda \vec{\boldsymbol{A}}_0, \quad \text{for } \lambda = 1, 2, \ldots. \tag{10}$$

Using this equation (7) can be written

$$\left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{U}}\right) \vec{\boldsymbol{A}} = \overleftrightarrow{\mathbf{U}} \vec{\boldsymbol{A}}_0. \tag{11}$$

If we now define $\overleftrightarrow{\mathbf{W}} \equiv [(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{U}})^{-1} \overleftrightarrow{\mathbf{U}}]$ the solution we seek can be written

$$\vec{\boldsymbol{A}}(\boldsymbol{r}_j, \boldsymbol{p}_j) = \left[\left(\overleftrightarrow{\mathbf{1}} + \overleftrightarrow{\mathbf{U}}\right)^{-1} \overleftrightarrow{\mathbf{U}}\right] \vec{\boldsymbol{A}}_0 = \overleftrightarrow{\mathbf{W}}(\boldsymbol{r}_j) \vec{\boldsymbol{A}}_0(\boldsymbol{p}_j). \tag{12}$$

The components of this column vector are thus the desired $\boldsymbol{A}_{(i)}(\boldsymbol{r}_j, \boldsymbol{p}_j)$.

Using $\vec{\boldsymbol{A}}_\lambda$ of equation (10) we can write down the formal power series solution

$$\vec{\boldsymbol{A}} = \sum_{\lambda=1}^{\infty} \vec{\boldsymbol{A}}_\lambda = -\sum_{\lambda=1}^{\infty} (-\overleftrightarrow{\mathbf{U}})^\lambda \vec{\boldsymbol{A}}_0 \tag{13}$$

to the matrix equation (11), which also, of course, implies a power series expression for $\overleftrightarrow{\mathbf{W}}$ in terms of $\overleftrightarrow{\mathbf{U}}$. The series (13) for $\overleftrightarrow{\mathbf{W}}$ will always converge if $NR_{\mathrm{e}}/R < 1$, where N is the number of particles, R_{e} the classical electron radius, and R the typical radius of the system. For sufficiently large particle densities this will not be fulfilled. The series for $\vec{\boldsymbol{A}}$ converges anyway as long as the vector $\vec{\boldsymbol{A}}_0$ has components only in the space spanned by eigenvectors of $\overleftrightarrow{\mathbf{U}}$ that have eigenvalues less than unity. There are thermodynamic arguments that this normally is the case, see subsection II E.

Trubnikov and Kosachev [11] have argued that when the series for $\overleftrightarrow{\mathbf{W}}$ does not converge it is meaningless to use truncated results in the study of plasmas. The discussion above shows that this is incorrect. Use of the quantity $\vec{\boldsymbol{A}}$ shows that the convergence of the $\overleftrightarrow{\mathbf{W}}$-series is immaterial. The conditions for the convergence of the $\vec{\boldsymbol{A}}$-series have to do with the long range correlation of momenta.

D. Approximating the Darwin Hamiltonian

Using the matrix notation above, and denoting matrix transposition by a superscript T, the kinetic energy can be written in the form

$$T = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m_i} = \frac{1}{2} (\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0)^T \vec{\boldsymbol{A}}_0. \tag{14}$$

The interaction term can be written

$$I_{\mathrm{m}} \equiv -\sum_i \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)} = -\frac{1}{2} (\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0)^T \vec{\boldsymbol{A}} \tag{15}$$

and thus the Hamiltonian (3) on matrix form is

$$H(\boldsymbol{r}_j, \boldsymbol{p}_j) = \frac{1}{2} \left[\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0(\boldsymbol{p}_j)\right]^T \left[\vec{\boldsymbol{A}}_0(\boldsymbol{p}_j) - \vec{\boldsymbol{A}}(\boldsymbol{r}_j, \boldsymbol{p}_j)\right] + \Phi(\boldsymbol{r}_j), \tag{16}$$

where $\vec{\boldsymbol{A}}$ now is expressed in terms of $\boldsymbol{r}_j$ and $\boldsymbol{p}_j$ according to equation (12).

The problem with the expression (12) is that, for large number of particles it is not easy to get explicit results for $\vec{\boldsymbol{A}}$. The interaction term (15) can, using the series (13), be written

$$I_{\rm m} = -\sum_{\lambda=1}^{\infty} \frac{1}{2} (\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0)^T \vec{\boldsymbol{A}}_\lambda. \tag{17}$$

Darwin's original (first order) Hamiltonian corresponds to keeping only the first, $\lambda = 1$, term of this series.

The terms of this series that correspond to even λ can be rearranged using $(\overleftrightarrow{\mathbf{U}}^\lambda)^T \overleftrightarrow{\mathbf{R}} = \overleftrightarrow{\mathbf{R}} \overleftrightarrow{\mathbf{U}}^\lambda$. This makes it possible to write the interaction energy as follows

$$I_{\rm m} = \sum_{\kappa=0}^{\infty} \frac{1}{2} [-(\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0)^T \vec{\boldsymbol{A}}_{2\kappa+1} + (\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_{\kappa+1})^T \vec{\boldsymbol{A}}_{\kappa+1}] \equiv I_{\rm p} + I_{\rm d}. \tag{18}$$

The advantage of this form is that the first term in the bracket, for each κ, is a term that becomes more negative the more correlated (parallel) the particle currents ($q_i \boldsymbol{p}_i$) are, while the second term is positive definite. This corresponds to a split of the magnetic energy, $I_{\rm m}$, into paramagnetic, $I_{\rm p}$, and diamagnetic, $I_{\rm d}$, contributions. If we now keep only the $\kappa = 0$ term in this series, which we denote $I_{\rm m0} = I_{\rm p0} + I_{\rm d0}$, we retain the qualitative feature of a part that can be negative and a positive definite part. This choice, $H_0 + I_{\rm m0}$, which corresponds to keeping the first two terms of (17), leads to the second order Darwin Hamiltonian

$$\mathcal{H} = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} + \frac{q_i}{2} \phi_{(i)} - \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_{1(i)} + \frac{q_i^2}{2m_i c^2} \boldsymbol{A}_{1(i)} \cdot \boldsymbol{A}_{1(i)} \right), \tag{19}$$

where

$$\boldsymbol{A}_{1(i)} = \sum_{j(\neq i)} \mathsf{P}_{ij} \frac{\boldsymbol{p}_j}{m_j c} \frac{q_j}{r_{ij}}, \tag{20}$$

previously derived by the present author. For some purposes the more explicit expression

$$\begin{aligned} \mathcal{H} = & \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} + \sum_{i<j} \frac{q_i q_j}{r_{ij}} \\ & - \sum_{i<j} \left(\frac{\boldsymbol{p}_i \cdot \mathsf{P}_{ij} \cdot \boldsymbol{p}_j}{m_i m_j c^2} \right) \frac{q_i q_j}{r_{ij}} + \sum_i \frac{q_i^2}{m_i c^2} \frac{1}{2} \sum_{j,k(\neq i)} \left(\frac{(\boldsymbol{p}_j \cdot \mathsf{P}_{ij}) \cdot (\boldsymbol{p}_k \cdot \mathsf{P}_{ik})}{m_j m_k c^2} \frac{q_j q_k}{r_{ij} r_{ik}} \right) \end{aligned} \tag{21}$$

for $\mathcal{H} = T + \Phi + I_{\rm p0} + I_{\rm d0}$ is useful.

E. Thermodynamic argument for convergence

The crucial question is now whether the series (13) converges. Consider the $\kappa = 0$ term of $I_{\rm m}$ in (18) which is

$$I_{\rm m0} = \frac{1}{2} [-(\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_0)^T \vec{\boldsymbol{A}}_1 + (\overleftrightarrow{\mathbf{R}} \vec{\boldsymbol{A}}_1)^T \vec{\boldsymbol{A}}_1]. \tag{22}$$

To analyze this we first note that we can define a scalar product on the relevant $3N$-dimensional vectors $\vec{\boldsymbol{A}}$ through

$$(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}) \equiv (\overleftrightarrow{\mathbf{R}}\vec{\boldsymbol{A}})^T \vec{\boldsymbol{B}}. \tag{23}$$

Using this notation we can write

$$T + I_{\mathrm{m}0} = T + I_{\mathrm{p}0} + I_{\mathrm{d}0} = \frac{1}{2}[(\vec{\boldsymbol{A}}_0 \cdot \vec{\boldsymbol{A}}_0) - (\vec{\boldsymbol{A}}_0 \cdot \vec{\boldsymbol{A}}_1) + (\vec{\boldsymbol{A}}_1 \cdot \vec{\boldsymbol{A}}_1)]. \tag{24}$$

We now chose units so that $\vec{\boldsymbol{A}}_0$ has unit length (or, equivalently, $T = \frac{1}{2}$):

$$(\vec{\boldsymbol{A}}_0 \cdot \vec{\boldsymbol{A}}_0) = 1. \tag{25}$$

According to (10) $\vec{\boldsymbol{A}}_1 = \overleftrightarrow{\mathbf{U}}\vec{\boldsymbol{A}}_0$, so if we assume that the operator $\overleftrightarrow{\mathbf{U}}$ rotates $\vec{\boldsymbol{A}}_0$ an angle θ and changes its length to L we can evaluate (24) to

$$T + I_{\mathrm{m}0}(L, \theta) = \frac{1}{2}(1 - L\cos\theta + L^2). \tag{26}$$

Since this is an energy (kinetic plus magnetic) a thermodynamic equilibrium state will tend to minimize it. The above expression is minimized for

$$L_{\min} = \frac{\cos\theta}{2} < 1 \tag{27}$$

and has the value

$$T + I_{\mathrm{m}0}(L_{\min}, \theta) = \frac{1}{2} - \frac{\cos^2\theta}{8}. \tag{28}$$

This in turn is minimized for $\theta = 0$, i.e. preferably $\vec{\boldsymbol{A}}_1$ should be parallel to $\vec{\boldsymbol{A}}_0$ and of half the length. Since we now find, that at minimum,

$$T + I_{\mathrm{m}0} = \frac{1}{2}\left(1 - \frac{1}{4}\right) \tag{29}$$

we find that the magnetic energy at most reduces the energy by 25% of the kinetic energy. This is in agreement with a different estimate in [12].

In conclusion we find that $|\overleftrightarrow{\mathbf{U}}\vec{\boldsymbol{A}}_0| \leq \frac{1}{2}|\vec{\boldsymbol{A}}_0|$ when the system is in an energy minimizing state. This indicates that $|\overleftrightarrow{\mathbf{U}}^2\vec{\boldsymbol{A}}_0|$ should be even smaller, and so on, and thus that the series (18) might converge. For a large system it will certainly not always converge, independently of state. $\vec{\boldsymbol{A}}_1$ will be large when there are long range correlations between the directions of the $q_i\boldsymbol{p}_i$. High temperature will, of course, tend to reduce such correlations and should thus aid convergence.

III. LENGTH SCALE OF MAGNETIC STRUCTURE

Here we demonstrate how one can find an exact solution for $\boldsymbol{A}_{(i)}(\boldsymbol{r}_j, \boldsymbol{p}_j)$ in a simplified case. This solution is seen to predict a length scale for magnetic structure. We then show that the approximate, second order, Hamiltonian predicts the same length scale for these structures. This provides our final argument for the usefulness of our Hamiltonian; subsequent sections discuss some applications.

We can regard equation (5), ignoring the projection part of the operator, as a solution of the (Poisson) equation

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}_v \tag{30}$$

which is well known to be solved by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\boldsymbol{J}_v(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'. \tag{31}$$

Here $\boldsymbol{J}_v$ is the current density in terms of velocity. Assuming for simplicity that there is only one kind of particle, with charge q and number density n, we can write

$$\boldsymbol{J}_v(\boldsymbol{r}) = q\boldsymbol{v}(\boldsymbol{r})n(\boldsymbol{r}). \tag{32}$$

Now using the relation (6) we get

$$\boldsymbol{J}_v(\boldsymbol{r}) = q\frac{1}{m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)n(\boldsymbol{r}) = \boldsymbol{J}_p(\boldsymbol{r}) - \frac{q^2}{mc}\boldsymbol{A}n, \tag{33}$$

where now $\boldsymbol{J}_p = q\boldsymbol{p}n/m$ is the current density in terms of momentum. Inserting this in (30) we get

$$(\nabla^2 - 4\pi\frac{q^2}{mc^2}n)\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}_p. \tag{34}$$

This equation stands in the same relation to equation (7) as (30) above did to (5). Assuming a constant density n this equation has the well known solution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\exp(-\mu|\boldsymbol{r}-\boldsymbol{r}'|)\boldsymbol{J}_p(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V', \tag{35}$$

where

$$\mu^2 \equiv 4\pi\frac{q^2}{mc^2}n \tag{36}$$

is of dimension inverse length squared (a Fourier transformed version of this result was found by Jones and Pytte [16]). We thus find that the vector potential in terms of the momentum current depends essentially only on momenta within the distance

$$R_\mathrm{m} = \frac{1}{\mu} \sim \frac{1}{\sqrt{R_\mathrm{e}n}}, \tag{37}$$

where we assume electrons so that $q^2/(mc^2) = R_{\rm e}$ is the classical electron radius.

We now consider the second order Darwin Hamiltonian, $\mathcal{H} = H_0 + I_{\rm p0} + I_{\rm d0}$, of equation (21). If R is a typical distance of interest $I_{\rm p0}$ is roughly

$$I_{\rm p0} \sim -\frac{N^2}{2}\left(\frac{v}{c}\right)^2 \frac{e^2}{R}, \tag{38}$$

where N is the number of particles. In a similar way we estimate $I_{\rm d0}$ to

$$I_{\rm d0} \sim R_{\rm e}\frac{N^3}{2}\left(\frac{v}{c}\right)^2 \frac{e^2}{R^2}. \tag{39}$$

A spontaneous magnetic structure must minimize the magnetic energy so we require that, for such a structure the sum of the above terms,

$$I_{\rm m}(R) \sim \frac{N^2}{2}\left(\frac{v}{c}\right)^2 \frac{e^2}{R}(NR_{\rm e}/R - 1), \tag{40}$$

is minimized with respect to R. This clearly means that $R = 2NR_{\rm e}$. Assuming that the number density of particles is n, i.e. $n = 3N/(4\pi R^3)$, we then find that the typical R for a magnetic structure is given by

$$R \sim \frac{1}{\sqrt{R_{\rm e} n}}. \tag{41}$$

Here n is the number density of mobile charged particles that have correlated velocities due to magnetic interaction. We see that we find the same result as in (37) above. This length scale is in agreement with that of Trubnikov and Kosachev [11], where it is called $d_{\rm c}$. One finally notes that $1/\mu$, with μ given by (36), is equivalent to the London penetration depth in superconductors.

A. The London Relation

It should be clear already from the exact expression (3) that an energy minimizing state of the system must have the momentum current parallel to the (internal) vector potential (or possibly both zero). Equation (35) is consistent with this observation. For reasonably microscopic $R_{\rm m} = 1/\mu$ the vector potential $\boldsymbol{A}(\boldsymbol{r})$ will be a superposition of nearby momentum current elements. We again get that

$$\boldsymbol{A}(\boldsymbol{r}) \approx C\boldsymbol{J}_p(\boldsymbol{r}), \tag{42}$$

where C is a positive scalar. Since the momentum current and the (ordinary) current must be parallel this leads to the phenomenological London relation between current and vector potential.

In 1981 Edwards [20] published a classical derivation of the London relation based on a variational principle. From this he concluded that there are many physical similarities in the behaviors of plasmas and of superconductors. This is in agreement with the results above.

IV. SPONTANEOUS MAGNETIC ENERGY LOWERING IN A PLASMA

A. The virial theorem

We will now make some estimates concerning the time averages of the two magnetic contributions the second order Darwin Hamiltonian (21). Hamilton's equations are, in our case,

$$\dot{\boldsymbol{r}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}, \quad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i}. \tag{43}$$

Using this we get that

$$\boldsymbol{p}_i \cdot \dot{\boldsymbol{r}}_i = \boldsymbol{p}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}, \quad \boldsymbol{r}_i \cdot \dot{\boldsymbol{p}}_i = -\boldsymbol{r}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i}, \tag{44}$$

and adding these gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}_i \cdot \boldsymbol{r}_i) = \boldsymbol{p}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} - \boldsymbol{r}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i}. \tag{45}$$

For a bound system, executing motion in a finite region, the time average of the time derivative will be zero. This is the crucial observation of the traditional virial theorem, see e.g. [21]. If we thus denote time average by an overbar we get

$$0 = \overline{\boldsymbol{p}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}} - \overline{\boldsymbol{r}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i}}. \tag{46}$$

Note that this assumes that the system is bound by the forces included in the Hamiltonian; no external walls are assumed.

Use of Euler's theorem on homogeneous functions on the expression (21) gives us

$$\sum_i \boldsymbol{p}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} = 2T + 2I_{\mathrm{p0}} + 2I_{\mathrm{d0}}, \quad \sum_i \boldsymbol{r}_i \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i} = -\Phi - I_{\mathrm{p0}} - 2I_{\mathrm{d0}}, \tag{47}$$

so the time averaged result can be written

$$0 = 2\overline{T} + \overline{\Phi} + 3\overline{I_{\mathrm{p0}}} + 4\overline{I_{\mathrm{d0}}}. \tag{48}$$

The two magnetic contributions both contain c^{-2} so we can write this

$$0 = \left(2\overline{T} + \overline{\Phi}\right) + \frac{1}{c^2}\left(3\overline{I'_{\mathrm{p0}}} + 4\overline{I'_{\mathrm{d0}}}\right). \tag{49}$$

If we formally regard c as an independent parameter here, so that letting $c \to \infty$ means turning magnetism off, and vice versa, we find that the two groups of terms must be zero separately. This is equivalent to assuming that the magnetic effects lead to purely additive independent contributions to the energy (first order perturbations). We then have both that

$$0 = 2\overline{T} + \overline{\Phi}, \tag{50}$$

the classical virial theorem, and that

$$0 = 3\overline{I_{\rm p0}} + 4\overline{I_{\rm d0}}, \tag{51}$$

separately. Since the kinetic energy, T, and the diamagnetic energy, $I_{\rm d0}$, are both positive definite we express the time average of the Hamiltonian, $\overline{\mathcal{H}} = \overline{H_0} + \overline{I_{\rm p0}} + \overline{I_{\rm d0}}$, in terms of these. Use of the above virial results then gives

$$\overline{\mathcal{H}} = -\overline{T} - \frac{1}{3}\overline{I_{\rm d0}}, \tag{52}$$

where, $\overline{H_0} = -\overline{T}$, as usual. We thus conclude that the virial theorem, together with the assumption that magnetism is a first order perturbation, predicts that *magnetic interactions lower the energy.*

This is a highly non-trivial result; external magnetic fields are well-known not to change the energy of a classical system of charged particles [22,23]. Trubnikov and Kosachev [11] as well as Liboff and Lie [24] have published calculations that indicate that the contribution to the energy from internal magnetic interaction is positive. This type of calculations are quite tricky, and if at any point in the calculation momentum, $\boldsymbol{p}_i$, and mass times velocity, $m_i\boldsymbol{v}_i$, are confused the results may be wrong. Witalis [25] have published calculations supporting the above negative energy result.

One might think that the virial calculation does not apply to a plasma since it will not, in general, be self-confined. Note, however, that there is no assumption that the motions are random or thermal. If there is some way to arrange a bound motion, by appropriate choice of initial conditions the theorem will apply to that motion. Witalis [25] has suggested that an ionic ring current might have such properties.

An interesting observation regarding the above calculations: if the traditional first order Darwin Hamiltonian, containing only $I_{\rm p0}$, had been used, the result for the magnetic correction to the energy would have been zero. This makes statistical mechanics based on the first order Darwin Hamiltonian [26] dubious. In this respect therefore there is a dramatic difference between Darwin's first order and Essén's second order Hamiltonian.

B. Energy and stability of magnetic structures

For few-body systems it is a natural consequence of the relativistic nature of magnetism that magnetic effects are proportional to $(v/c)^2$. For macroscopic systems, on the other hand, the largeness of Avogadro's number together with the slow, $1/r$, decrease of $\boldsymbol{A}$ with distance, results in a totally different situation. If one used a Hamiltonian without the diamagnetic term, $\sim \boldsymbol{A}^2$, or, alternatively calculated $\boldsymbol{A}$ in (3) ignoring the screening of equation (35), the magnetic energy has a tendency to diverge. In subsection II E, formula (29), we found that magnetic energy is minimized when it is roughly 25% of the kinetic energy. This is also in agreement with a different calculation in Essén [12] (section XI). We can also find this result by inserting the minimizing $R = 2R_{\rm e}N$ into the estimate (40). This gives

$$I_{\rm m} \sim -\frac{1}{8}Nmv^2, \tag{53}$$

which means that $I_{\rm m} \sim -T/4$. It is instructive to note how c^{-2} cancels in this calculation because of its appearance in $R_{\rm e}$. Note that the number N here is the number of particles in a magnetic structure. This number is given by $N \sim nR_{\rm m}^3 = n^{-1/2}R_{\rm e}^{-3/2}$, so it increases with decreasing density.

An expression for magnetic energy inspired by the 'exact' result (35) should be something like

$$I_{\rm m} = -\sum_{i<j}\left(\frac{\boldsymbol{p}_i \cdot \mathsf{P}_{ij} \cdot \boldsymbol{p}_j}{m_i m_j c^2}\right)\frac{q_i q_j \exp(-r_{ij}/R_{\rm m})}{r_{ij}} \tag{54}$$

where we have replaced the Coulomb-like factor in $I_{\rm p0}$ of equation (21) with a Yukawa-type exponentially screened factor. It should give a reasonable estimate for constant density. An estimate for the magnetic energy of a spontaneous magnetic structure based on this expression is

$$I_{\rm m} \sim -\frac{1}{2}N^2\left(\frac{v}{c}\right)^2\frac{e^2\exp(-1)}{R_{\rm m}}. \tag{55}$$

From equation (37) we have $R_{\rm m} = 1/\sqrt{R_{\rm e}n}$. If we put $n = N/R_{\rm m}^3$ into this we get $R_{\rm m} = NR_{\rm e}$, and this inserted into the above estimate for $I_{\rm m}$ gives

$$I_{\rm m} \sim -\frac{1}{2\exp(1)}N\left(\frac{v}{c}\right)^2\frac{e^2}{R_{\rm e}} \sim -\frac{1}{6}Nmv^2. \tag{56}$$

The agreement of this result with (53) is reassuring.

Statistical mechanical considerations tell us that the phase space probability density is given by

$$p(\boldsymbol{r}_j, \boldsymbol{p}_j) \sim \exp(-\mathcal{H}/kT) = \exp[-\{H_0(\boldsymbol{r}_j, \boldsymbol{p}_j) + I_{\rm m}(\boldsymbol{r}_j, \boldsymbol{p}_j)\}/kT]. \tag{57}$$

Here k is the Boltzmann constant and T in the denominator is the absolute temperature. We have found that the contribution of $I_{\rm m}$ to the energy is negative and of the same order of magnitude as the other contributions and that it increases quadratically with increasing speed of the particles, i.e. linearly with increasing temperature. Statistical mechanics thus predicts that magnetic structures in phase space should not to be sensitive to thermal disruption. Note that this also means that currents must flow in the thermal equilibrium plasma due to anisotropic velocity distributions (contrary to conventional wisdom [22]).

V. EFFECTIVE ONE-PARTICLE HAMILTONIAN

If we wish to study the equations of motion for particle i assuming given positions and momenta for the remaining particles we need to separate out all terms of $\mathcal{H}$ that contain the position and momentum coordinates of particle i. We introduce the notation

$$\mathcal{H}^{(i)} = \frac{\boldsymbol{p}_i^2}{2m_i} + \frac{q_i}{2}\phi_{(i)} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{1(i)} + \frac{q_i^2}{2m_i c^2}\boldsymbol{A}_{1(i)} \cdot \boldsymbol{A}_{1(i)}, \tag{58}$$

so that (19) can be written $\mathcal{H} = \sum_i \mathcal{H}^{(i)}$. If we, further denote by $\mathcal{H}_i^{(j)}$ those terms in $\mathcal{H}^{(j)}$ that contain position and momentum coordinates of particle i we can rewrite $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}^{(i)} + \sum_{j(\neq i)} \mathcal{H}_i^{(j)} + \mathcal{H}^{[i]} = \mathcal{H}_{\rm eff}^{(i)} + \mathcal{H}^{[i]} \tag{59}$$

where $\mathcal{H}^{[i]}$ are those terms of $\mathcal{H}$ that do not refer to particle i. Then $\mathcal{H}_{\rm eff}^{(i)}$ is the effective Hamiltonian of particle i. This Hamiltonian can then be compared to the Hamiltonian

$$\mathcal{H}_{\rm ext} = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2 + q\phi(\boldsymbol{r},t) \tag{60}$$

of a charged particle in an external electromagnetic field.

A. The weak field case

We now calculate $\mathcal{H}_{\rm eff}^{(i)}$. First, if we ignore the last, diamagnetic terms of the other particles, we have

$$\sum_{j(\neq i)} \mathcal{H}_i^{(j)} \approx \frac{q_i}{2}\phi_{(i)} - \frac{q_i}{2m_i c}\boldsymbol{p}_i \cdot \boldsymbol{A}_{1(i)} \tag{61}$$

We thus get the effective one-particle Hamiltonian

$$\mathcal{H}_{\rm eff}^{(i)} \approx \frac{1}{2m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}_{1(i)}\right)^2 + q_i\phi_{(i)}. \tag{62}$$

i.e. essentially equation (60). We thus get the expected agreement in this case. Note that the new second order term, for particle i, is essential for the result (62); without it one would not have a complete square for the combined kinetic and magnetic energies.

B. The remaining terms

We now study the contribution to $\mathcal{H}_{\rm eff}^{(i)}$ from the diamagnetic terms of the other particles. We thus consider

$$I_{\rm d}^i \equiv \sum_{j(\neq i)} \frac{q_j^2}{2m_j c^2}\boldsymbol{A}_{1(j)} \cdot \boldsymbol{A}_{1(j)} = \tag{63}$$

$$\sum_{j(\neq i)} \frac{q_j^2}{2m_j c^2}\left(\frac{q_i\boldsymbol{p}_i}{m_i c}\cdot\frac{\mathsf{P}_{ij}}{r_{ij}} + \sum_{k(\neq i,j)}\frac{q_k\boldsymbol{p}_k}{m_k c}\cdot\frac{\mathsf{P}_{kj}}{r_{kj}}\right)\cdot\left(\frac{q_i\boldsymbol{p}_i}{m_i c}\cdot\frac{\mathsf{P}_{ij}}{r_{ij}} + \sum_{l(\neq i,j)}\frac{q_l\boldsymbol{p}_l}{m_l c}\cdot\frac{\mathsf{P}_{lj}}{r_{lj}}\right)$$

If we now define

$$\boldsymbol{A}_{1(j)}^{[i]} \equiv \sum_{k(\neq i,j)}\frac{q_k\boldsymbol{p}_k}{m_k c}\cdot\frac{\mathsf{P}_{kj}}{r_{kj}} = \boldsymbol{A}_{1(j)} - \frac{q_i\boldsymbol{p}_i}{m_i c}\cdot\frac{\mathsf{P}_{ij}}{r_{ij}} \tag{64}$$

we can rewrite (63) as

$$I_{\rm d}^i = \sum_{j(\neq i)} \frac{(\boldsymbol{p}_i \cdot \mathsf{P}_{ij})^2}{2m_i} \frac{q_i^2 q_j^2}{m_i m_j c^4} \frac{1}{r_{ij}^2} + \frac{q_i}{m_i c} \boldsymbol{p}_i \cdot \sum_{j(\neq i)} \frac{q_j}{m_j c} \left(\frac{q_j}{c} \boldsymbol{A}_{1(j)}^{[i]} \right) \cdot \frac{\mathsf{P}_{ij}}{r_{ij}} + \sum_{j(\neq i)} \frac{q_j^2}{2m_j c^2} \boldsymbol{A}_{1(j)}^{[i]} \cdot \boldsymbol{A}_{1(j)}^{[i]} \tag{65}$$

Here the last sum does not contain $\boldsymbol{r}_i$ or $\boldsymbol{p}_i$ and thus belongs to $\mathcal{H}^{[i]}$. The first two, however, should contribute to the effective one-particle Hamiltonian for particle i.

C. A divergent self-interaction

We first consider the first of these contributions. We can write it

$$\sum_{j(\neq i)} \frac{(\boldsymbol{p}_i \cdot \mathsf{P}_{ij})^2}{2m_i} \frac{q_i^2 q_j^2}{m_i m_j c^4} \frac{1}{r_{ij}^2} = \frac{\boldsymbol{p}_i^2}{2m_i} \sum_{j(\neq i)} \frac{q_i^2 q_j^2}{m_i m_j c^4} \frac{1}{r_{ij}^2} (1 + 3\cos^2 \theta_{ij}) \equiv \frac{\boldsymbol{p}_i^2}{2m_i} F(\boldsymbol{r}_i, \boldsymbol{p}_i) \tag{66}$$

where θ_{ij} is the angle between $\boldsymbol{p}_i$ and $\boldsymbol{e}_{ij}$. Here $q_i^2/(m_i c^2)$ are classical particle radii. Since these are largest for electrons we assume that only these contribute. To estimate the factor F we further ignore the directional dependence and assume that particle i is at the center of a spherical distribution of electrons of constant number density n_e inside a radius R. We then get

$$F \sim \left(\frac{e^2}{mc^2} \right)^2 \int \frac{n_e}{r^2} {\rm d}V = 4\pi \left(\frac{e^2}{mc^2} \right)^2 n_e R = 3 \left(\frac{R_{\rm e}}{R} \right)^2 N. \tag{67}$$

F will thus become of unit order of magnitude when $R^2 \sim N R_{\rm e}^2$. Since the Thompson scattering cross section for photons is $\sim R_{\rm e}^2$ this means that for such R-values the plasma is opaque. For R-values of the order of magnitude of a magnetic structure $R = R_{\rm m} \sim N R_{\rm e}$ we see that $F \sim 1/N$ and essentially negligible. The fact that F can become large in a large plasma is thus seen to be an artifact of our approximation. As formula (35) shows all results depending of integrations beyond the $R_{\rm m}$ length scale are spurious. In what follows we ignore this term.

D. The physical contribution

The second term in (65), on the other hand, contributes in a simple way to $\mathcal{H}_{\rm eff}^{(i)}$. If we define

$$\boldsymbol{A}_{(i)}^v \equiv \sum_{j(\neq i)} \frac{q_j}{m_j c} \left(\boldsymbol{p}_j - \frac{q_j}{c} \boldsymbol{A}_{1(j)}^{[i]} \right) \cdot \frac{\mathsf{P}_{ij}}{r_{ij}} = \boldsymbol{A}_{1(i)} - \boldsymbol{A}_{(i)}^A, \tag{68}$$

we get

$$\mathcal{H}_{\rm eff}^{(i)} = \frac{\boldsymbol{p}_i^2}{2m_i} + q_i \phi_{(i)} - \frac{q_i}{m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)}^v + \frac{q_i^2}{2m_i c^2} \boldsymbol{A}_{1(i)} \cdot \boldsymbol{A}_{1(i)}. \tag{69}$$

Equivalently, according to the definitions of (68), we find

$$\mathcal{H}_{\rm eff}^{(i)} = \frac{1}{2m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}^v_{(i)}\right)^2 + q_i\phi_{(i)} + \frac{q_i^2}{m_i c^2}\left(\boldsymbol{A}^v_{(i)}\cdot\boldsymbol{A}^A_{(i)} + \frac{1}{2}\boldsymbol{A}^A_{(i)}\cdot\boldsymbol{A}^A_{(i)}\right) \tag{70}$$

for the effective one-particle Hamiltonian.

A few points are worth mentioning here. Firstly it is possible, by means of a slightly different definition of $\boldsymbol{A}^{[i]}_{1(j)}$, to find the algebraic form (70) without discarding the divergent self interaction (66). It can, so to speak, be absorbed in the definition of the vector potential. The are, however, no definitions that will bring the Hamiltonian to the form (62). One also notes that for a two particle system $\boldsymbol{A}^{[i]}_{1(j)} = \boldsymbol{A}^{[1]}_{1(2)} = \boldsymbol{0}$, so $\boldsymbol{A}^A_{(i)} = \boldsymbol{0}$, and the extra terms of (70) vanish when there are only two particles.

The notation $\boldsymbol{A}^v_{(i)}$ is chosen since this is a vector potential defined in terms of $(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})$, a quantity that corresponds to $m\boldsymbol{v}$ with normal interpretations of the quantities. Similarly the notation $\boldsymbol{A}^A_{(i)}$ is chosen since this quantity is a potential defined in terms of the vector potential. The explicit expression for $\boldsymbol{A}^A_{(i)}$ is found to be

$$\boldsymbol{A}^A_{(i)} = \sum_{j(\neq i)}\frac{q_j}{m_j c}\left(\frac{q_j}{c}\boldsymbol{A}^{[i]}_{1(j)}\right)\cdot\frac{\mathsf{P}_{ij}}{r_{ij}} = \sum_{j(\neq i)}\frac{q_j^2}{m_j c^2}\left(\sum_{k(\neq i,j)}\frac{q_k\boldsymbol{p}_k}{m_k c}\cdot\frac{\mathsf{P}_{kj}}{r_{kj}}\right)\cdot\frac{\mathsf{P}_{ij}}{r_{ij}} \tag{71}$$

For an energy minimizing state this vector will thus tend to be parallel to $\boldsymbol{p}_i$ and also $\boldsymbol{A}^v_{(i)}$ so that the first extra term of (70) will be mostly positive.

E. The final result

In conclusion we have now found that the Hamiltonian (70) of a particle moving among other charged particles with given motions can be written

$$\mathcal{H}_{\rm eff} = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 + q\phi + \frac{q^2}{mc^2}V_A \tag{72}$$

where ϕ and

$$V_A(\boldsymbol{r},t) \equiv \boldsymbol{A}\cdot\boldsymbol{A}^A + \frac{1}{2}\boldsymbol{A}^A\cdot\boldsymbol{A}^A \tag{73}$$

are scalar fields, and $\boldsymbol{A}$ and $\boldsymbol{A}^A$ are vector fields. If this is correct it would mean that, apart from coupling to electric and magnetic fields as usual, the particle also couples to the, essentially repulsive, scalar field V_A with coupling constant $q^2/(mc^2)$.

The integrations that determine the field V_A should, of course not be extended much beyond the distance scale $R_{\rm m}$. This means that in the interior of a plasma the repulsive force will mainly contribute to the pressure but not accelerate the particles. At a surface it, however, seems to be mainly a repulsive force.

If one considers a charged particle in the neighborhood of a plasma (for example a star) one knows that there will not normally be any net electric fields. One also note that a (time-independent) magnetic field will not do work on the particle. The new term of equation (72)

will, however, do work on such a particle and accelerate it away from a current distribution (a star is predicted to have such a distribution according to the discussion in section IV). It is thus tempting to believe that this term should be taken seriously and that it will provide a simple and natural explanation of the astrophysical phenomena manifested by stellar wind and comet tails. Until more quantitative studies have been completed these beliefs must be considered preliminary.

VI. CONCLUSIONS

Above we have investigated the derivation, the credibility, and some consequences of the second order Darwin Hamiltonian. This Hamiltonian, which is an estimate of the conserved phase space energy of a system of charged particles, to the extent that such a concept is meaningful, takes magnetic effects into account. In view of the great generality and simplicity of energy and statistical mechanical arguments such a Hamiltonian should be useful in the investigation of how magnetic phenomena arise.

The main achievements of this paper are the following:

(1) The proof that the magnetic energy can be split into paramagnetic (potentially energy lowering) terms and diamagnetic (positive definite) terms. The fact that the second order Darwin Hamiltonian takes into account the leading term of both types makes it considerably better than its predecessor, the simplified (first order) Darwin Hamiltonian.

(2) The thermodynamic and other arguments that the second order Darwin Hamiltonian predicts magnetic structures of the correct size. Especially the agreement with predictions of the simplified exact solution (35).

(3) The proof based on the virial theorem that magnetic energy lowers the energy of a plasma in the sense that the time average of the magnetic terms contribute a negative net result to the time average of the energy. The thermodynamic estimates which show that magnetic structures are stable to thermal disruption.

(4) The prediction of new effective many-body forces on a charge in a plasma through the effective one-particle Hamiltonian. Magnetohydrodynamics also predicts such forces but this may give a more direct way of understanding them.

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