Magnetic energy, superconductivity, and dark matter

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Abstract

Magnetism due to the translational, possibly oscillatory, motion of charge, as opposed to the ordering of dipoles, is not well understood, but is well described by the Darwin Lagrangian. The Coulomb interaction is used universally in atomic, molecular and solid state physics, but its natural extension when going to higher accuracy, the magnetic Darwin-Breit interaction, is not. This interaction is a velocity dependent long range interaction and as such unfamiliar to the majority of theoreticians. The $(v/c)^2$ dependence makes it at most a perturbation in few-body systems, but does not stop it from becoming potentially important as the number of particles increase. For systems where particle velocities are correlated (or coherent) over larger distances this interaction is shown to have major consequences. Based on these findings I suggest that this interaction should be investigated as possibly responsible for superconductivity and, on an interstellar scale, for the missing dark matter. Some numerical estimates and intuitive arguments are presented in support but no proofs. Instead it is my hope that the ideas presented will deserve further serious study.
Key words: Darwin Lagrangian, Magnetic interaction energy, Plasma oscillations, long range correlation, coherence length, emergent properties, Wigner crystal, superconductivity, effective mass, inductive inertia, dark matter

A man hear what he wants to hear and disregards the rest.
Paul Simon in *The Boxer*

1 Introduction

We first introduce the Darwin Lagrangian which describes the magnetic interaction energy between moving charged particles. This is a velocity dependent long range interaction which is very small for few-body systems but which can become dominating in macroscopic systems. In particular the Lagrangian predicts that the effective mass, or equivalently inductive inertia, can grow with the square of the number of particles.

The Darwin Lagrangian makes simple predictions for particles that are assumed to have the same velocity. Here we use this constraint to study the effect of the magnetic interaction energy for collectively moving charges. The crucial fact that emerges from these studies is that the effective mass of many collectively moving particles far exceeds the sum of their rest masses. In the case of superconductivity this means that the zero-point energy of coherent oscillators decreases with the number of oscillators, and this presumably leads to the superconducting phase transition. In the case of cosmic plasma filaments it leads to the conclusion that their gravitational mass can far exceed the rest mass content of the participating particles. Could this be the missing dark matter? Some numerical estimates indicate that this is a possibility.

2 The Darwin Lagrangian

The Darwin Lagrangian [1] describes the majority of electromagnetic phenomena correctly. The exception is radiation, which is neglected. The theory behind this Lagrangian is presented in a few textbooks such as Landau and Lifshitz [2, §65] and Jackson [3, Sec. 12.6]. More extensive discussions can be found in Page and Adams [4, Sec. 96], Podolsky and Kunz [5, Sec. 27], Szasz [6, Appendix], Schwinger *et al.* [7, Eq. (33.23)], or Stefanovich [8]. Basic articles of interest are Breitenberger [9], Kennedy [10], Essén [11–13]. Various applications of the Darwin Lagrangian illustrating its usefulness can be found in Kaufman [14], Stettner [15], Boyer [16, 17], Krause *et al.* [18], Essén *et al.* [19–25].

Vector potentials are not always mentioned in connection with the Darwin Lagrangian, but it can be derived by approximating the Liénard-Wiechert potentials. Landau and Lifshitz [2, §65] make a gauge transformation to the

Coulomb gauge after truncating series expansions of these. Jackson [3, Sec. 12.6] solves the vector Poisson equation obtained by neglecting the time derivative in the wave equation. Page and Adams derive it by approximating the forces [4, Sec. 96]. It can also be motivated as the best approximately relativistic action-at-a-distance Lagrangian [10, 26] and it can be shown to take retardation into account to order $(v/c)^2$.

The Darwin Lagrangian for N charged particles, of mass m_a and charge e_a, can be written

$$L_D = \sum_{a=1}^{N} \left[\frac{m_a}{2} \boldsymbol{v}_a^2 - \frac{e_a}{2} \phi_a(\boldsymbol{r}_a) + \frac{e_a}{2c} \boldsymbol{v}_a \cdot \boldsymbol{A}_a(\boldsymbol{r}_a) \right], \tag{1}$$

where,

$$\phi_a(\boldsymbol{r}_a) = \sum_{b(\neq a)}^{N} \frac{e_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|}, \tag{2}$$

and,

$$\boldsymbol{A}_a(\boldsymbol{r}_a) = \sum_{b(\neq a)}^{N} \frac{e_b}{2c} \frac{[\boldsymbol{v}_b + (\boldsymbol{v}_b \cdot \hat{\boldsymbol{e}}_{ab})\hat{\boldsymbol{e}}_{ab}]}{|\boldsymbol{r}_a - \boldsymbol{r}_b|}. \tag{3}$$

Here $\hat{\boldsymbol{e}}_{ab} = (\boldsymbol{r}_a - \boldsymbol{r}_b)/|\boldsymbol{r}_a - \boldsymbol{r}_b|$, and relativistic corrections to the kinetic energy are neglected. In many circumstances one can neglect the magnetic interaction energies since the Coulomb electric interaction dominates strongly, especially in few-body systems. As will be seen below, however, when there are macroscopic numbers of correlated charged particles this is no longer permissible. It is noteworthy that macroscopic numbers of correlated charged particles is the rule rather than an exception in plasmas, conductors, and superconductors.

3 Plasma oscillations

One can use (1) to calculate how a charge density of electrons oscillates relative to a fixed background of positive charge. For collective motion of N electrons with velocity $\boldsymbol{v} = \dot{x}\hat{\boldsymbol{e}}_x$ the kinetic energy is simply $T = N m_e \dot{x}^2/2$. If one further assumes that the particles have fixed distributions in space apart form the relative translational motion one can get (nearly) analytical results for the remaining two terms, for simple geometries in the continuum limit. If we denote the displacement of the negative charges by x the total Coulomb potential energy is well approximated by,

$$\Phi(x) = \sum_{a=1}^{N_{tot}} \frac{e_a}{2} \phi_a(\boldsymbol{r}_a) = \Phi(0) + \frac{1}{2} \left(\frac{d^2\Phi}{dx^2} \right)_{x=0} x^2, \tag{4}$$

in the limit of small x. Here $\Phi(0)$ is a large negative constant that does not contribute to the dynamics; the positive background only provides the restoring force in the oscillation. The assumption that the electrons ($m_a = m_e$, $e_a = -e$) move collectively along the x-direction simplifies the magnetic contribution, the third term in (1). One finds

$$U_D = \sum_{a=1}^{N} \frac{e}{2c} \boldsymbol{v}_a \cdot \boldsymbol{A}_a(\boldsymbol{r}_a) = \left(\frac{e^2}{2c^2} \sum_{a=1}^{N-1} \sum_{b=a+1}^{N} \frac{1+\cos^2\theta_{ab}}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} \right) \dot{x}^2, \tag{5}$$

where $\cos\theta_{ab} = \hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{e}}_{ab}$. For a charge density of electrons with fixed geometry this is simply a constant times $\dot{x}^2$. We thus find that the Darwin Lagrangian for the system becomes

$$L_D = N\left(\frac{1}{2} m_{\rm eff} \dot{x}^2 - \frac{1}{2}\kappa x^2\right). \tag{6}$$

Here $m_{\rm eff}$ is m_e plus a contribution from (5).

Calculations of the constants $m_{\rm eff}$ and κ can be done by elementary methods. The result will be a formula for the square of the oscillation frequency $\omega^2 = \kappa/m_{\rm eff}$. This was done for a sphere of radius R in [19] with the result

$$\omega^2 = \frac{\frac{Ne^2}{R^3}}{m_e\left(1 + \frac{4}{5}\frac{r_e N}{R}\right)}. \tag{7}$$

Here $r_e = e^2/(m_e c^2)$ ia the classical electron radius. $\omega(R)$ is plotted in Fig. 1. In the limit of few particles, or negligible Nr_e/R, this gives the plasma oscillation frequency as normally given in the literature,

$$\omega_p^2 = \frac{4\pi}{3}\frac{e^2 n_0}{m_e} \tag{8}$$

where $n_0 = N/V$ is the number density inside the sphere. In the opposite limit of macroscopic numbers of electrons N one obtains

$$\omega_\infty^2 = \frac{5c^2}{4R^2}. \tag{9}$$

This seems to be the frequency of a longitudinal electromagnetic wave in the sphere. A similar calculation for a (two-dimensional) square of side length L gives a similar result,

$$\omega^2 = \frac{\frac{2e^2N}{L^3}K_s}{m_e\left(1 + \frac{3}{4}\frac{Nr_e}{L}C_s\right)}, \tag{10}$$

where $C_s = (4/3)[1 - \sqrt{2} - 3\ln(\sqrt{2} - 1)]$ and $K_s = 16(2 - \sqrt{2})$ [1].

[1] H. Essén and A. B. Nordmark (2019), unpublished.

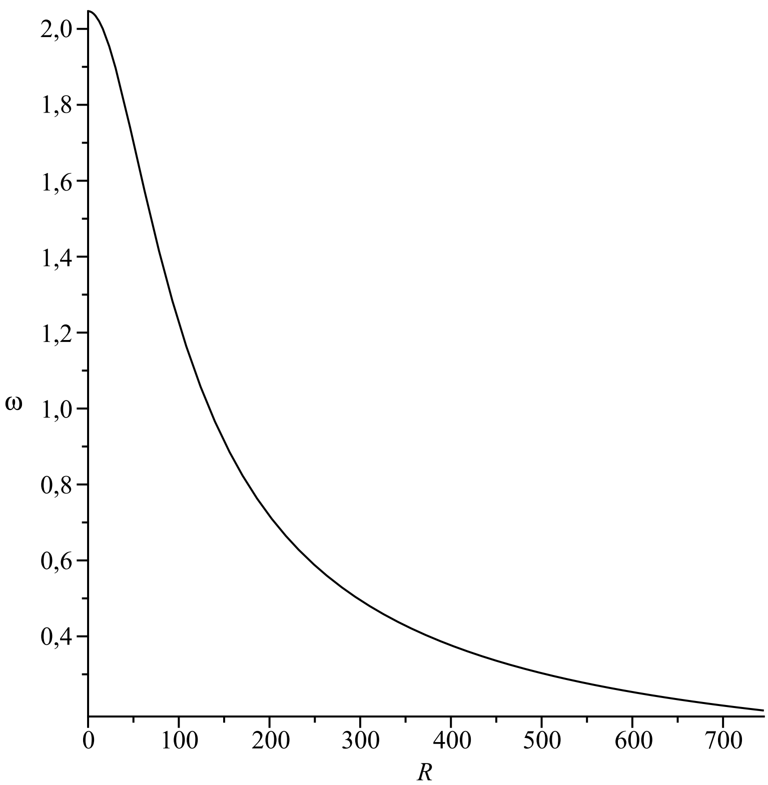

Figure 1: The frequency ω of Eq. (7) as a function of radius R. Atomic units are used ($e = m_e = \hbar = 1$, $c = 137$) and the density is assumed to be one electron per sphere of one Bohr radius a_0. The formula plotted is $\omega(R) = \sqrt{\frac{4\pi}{3\left(1+\frac{16\pi}{15(137)^2}R^2\right)}}$ and R is in atomic units (Bohr radii). The frequency is reduced by one order of magnitude at $R = 750\,a_0$.

4 Superconductivity

In the early history of superconductivity it was conjectured that a transition of the electrons at the Fermi surface to a Wigner crystal [27] was responsible for the phase transition. Since no new interaction comes into play this did not seem correct, even if the Wigner crystal idea is still investigated [28, 29]. When one takes the magnetic interaction energy into account, however, the zero point energy $E_0 = \hbar\omega/2$ and oscillation frequency of the (pairs of) electrons go down considerably if they oscillate coherently with coherence length R, as indicated in Fig. 1. It is interesting to note that Vasiliev [30, 31] finds that superconductivity is caused by ordering of the zero point oscillations. Frenkel [32] advanced the theory that the increased inductive inertia of correlated conduction electrons explains superconductivity, and the present author presented estimates indicating that the Darwin energy is important in superconductors [33]. In Fig. 1 it is seen that the zero point energy goes down by one order of magnitude in 750 Bohr-radii, assuming one electron per cubic Bohr-radius. In general coherence lengths in superconductors is one or two orders of magnitude larger [34], so the numbers are quite reasonable. The isotope effect agrees well with the assumption that lattice oscillations destroy the coherence.

5 Dark matter

The decay time of currents is $\tau \sim \mathcal{L}/\mathcal{R}$ where $\mathcal{L}$ is inductance and $\mathcal{R}$ resistance. As emphasized by Kulsrud [35] these times are enormous in astrophysical plasmas. The currents producing astrophysical magnetic fields will only decay on a time scale comparable to the age of the universe. These plasmas are thus effectively superconducting. The effective mass m_{eff} of Eq. (6) is a measure of the inductance, or inductive inertia. Simple estimates show that this mass is in general much larger than the rest mass. That this is the case for conduction electrons in a metal was noted already in 1936 by Darwin [36] and several times later [23, 37].

It is tempting to speculate that dark matter is in fact due to magnetic energy in interstellar plasmas. Here we make some simple estimates. The Darwin magnetic energy, the first term of Eq. (6), $U_D = N m_{\text{eff}} \dot{x}^2/2$, will contribute $M_D = U_D/c^2$ to gravitational mass in the universe. Consider a cube of side length L. If we assume that the number of protons in this cube is N and that L also is a typical distance between them we find from Eq. (5) that

$$\frac{U_D}{c^2} = M_D \sim \frac{1}{4}\frac{e^2}{c^2}\frac{N^2}{L}\beta^2 \tag{11}$$

where $\beta = |\dot{x}|/c$. This magnetic mass should be compared to the total proton mass $M_p = Nm_p$. The ratio is

$$\frac{M_D}{M_p} \sim \frac{(e^2/c^2)(N/L)}{4m_p}\beta^2. \tag{12}$$

Putting in the numerical values gives

$$\frac{M_D}{M_p} \sim (3.83 \cdot 10^{-19}\mathrm{m})(N/L)\beta^2. \tag{13}$$

The number of protons is $N = n_p L^3$ where n_p is the proton number density. This gives

$$\frac{M_D}{M_p} \sim (3.83 \cdot 10^{-19}\mathrm{m})\, n_p L^2 \beta^2. \tag{14}$$

To get some numbers we assume that $n_p = 4.0\,\mathrm{m}^{-3}$ and that the ratio M_D/M_p is 10 (magnetic mass is 10 times proton mass). This gives

$$10 \sim (3.83 \cdot 10^{-19})\, 4.0(L^2/\mathrm{m}^2)\beta^2. \tag{15}$$

The side length of the cube over which velocity must be correlated is then

$$L \sim 2.5 \cdot 10^9 \beta^{-1}\mathrm{m}. \tag{16}$$

assuming that the speed is $c/100$, so that $\beta = 10^{-2}$, we find that $L \sim 2.5 \cdot 10^{11}$ m. This is somewhat more that one astronomical unit (AU$\approx 1.5 \cdot 10^{11}$ m), a tiny distance in the interstellar perspective. So, with a density of 4 protons per cubic meter and a correlated speed of 1% of the speed of light over a distance of order of magnitude one AU one finds that the gravitational mass M_D of the magnetic energy is ten times the total proton rest mass. This suggests to the me that dark matter may, in fact, reside in magnetic energy and the effective mass of the cosmic magnetic fields.

6 Conclusions

Since Darwin's 1936 paper [36] it should have been clear that investigations of conduction electrons in metals that do not take into account the magnetic interaction energy are meaningless. No amount of mathematical wizardry will make this interaction go away. It is also a natural candidate for emergent properties in larger systems, such as superconductivity, while remaining a perturbation in few body systems.

The insight that large plasmas with coherent velocities have energies that are many orders of magnitude larger than that corresponding to the rest mass of the constituent particles should be investigated as a possible candidate for dark matter. Recently Nicastro *et al.* [38] found that missing baryons are believed to reside in large-scale filaments in the warm-hot intergalactic medium. Perhaps the rest of the missing dark matter is also there in the form of magnetic energy?

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