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Nonlinear Wave Phenomena in a Magnetized Plasma and in a Relativistic Gas

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Contents

Preface

River wild

An old Japanese essay begins with the words,

> The flowing river never ceases and yet the water never stays the same. Foam floats upon the pools, scattering, re-forming, never lingering long. So it is with man and all his dwelling places here on earth. (Hojoki, 1212)

If translated into the words of modern science, it might read "Everything in the world is in order but flows in a chaotic river." It is not an exaggeration to say that a large number of complex phenomena termed *nonlinear* have their pure origin in fluid dynamics. This can be seen by noting that *soliton* and *chaos*, these two wide-spread words of the nonlinear physics, are the newborn babies of fluid mechanics. From blood flows in our vessels to intergalactic gaseous winds, in various natural phenomena we see fluid dynamics playing a key role, and it is therefore a must to understand these flows. It is both marvelous and miraculous that all of these natural phenomena, which have various dynamic length scales, can be essentially described by only one set of equations: the Navier-Stokes equations.

From the mathematical viewpoint, the first achievement in fluid dynamics was that of Leonhard Euler, who derived an equation describing the behaviour of perfect fluid. His theory was beautiful enough, but also too ideal to take reality -in terms of viscosity- into account. Since then, many giants such as Poisson, Saint-Venant, Gromeka and Helmholtz have tried to understand the physical nature of viscosity, and among them Claude Louis M.H. Navier arrived at the historic differential equations of real viscous flows. In a different way a British mathematician, Sir George Gabriel Stokes, also obtained essentially the same equations, which are therefore called the Navier-Stokes equations. Certainly it is the greatness of science that the equations derived almost 200 years ago are still of great interest and importance in various fields, from the building of special yachts for the Americas Cup to predicting the effect of El Niño. It is almost impossible to pin down anything which has completely nothing to do with fluid mechanics; even the joy of baseball is due to the effect of the air.

Fluids and life

It is well-known that the human body contains about 71 percent water by weight. Presumably to understand fluid dynamics is to understand what is going on every day within ourselves and around us. Complex fluid flows appear in almost all our activities such as breathing, digesting and the circulation of blood. With no knowledge of fluid mechanics, we are unable to appreciate them to the full. More or less the same can be said for other animals and particularly for fishes, whose lives are directly affected by water flows. According to the modern theory of evolution, an optimizing process has worked among many species on the earth over many years, and any living creature has its present size, shape, typical speed of motion, and functions in connection with its environment. We can enjoy swimming simply because the size of our body, around 6 feet on average if in Scandinavia, balances the viscous effect of water and the force produced by our muscles. The author recently found the reason why he was addicted to swimming. It is a sort of research activity as well as a relaxation time for him.

In contrast with fishes, which have to be in water all the time before they are arranged on dishes, a dromedary can get by for several weeks without water. When water is available, it can drink one third of its body weight quickly and save it for a long period of time. It also stores energy as a lump on its back, which has a peculiar shape and consists of fat, contrary to the general belief that it is a convenient store of water. The hump on its back is a sort of localized energy in the physical sense.

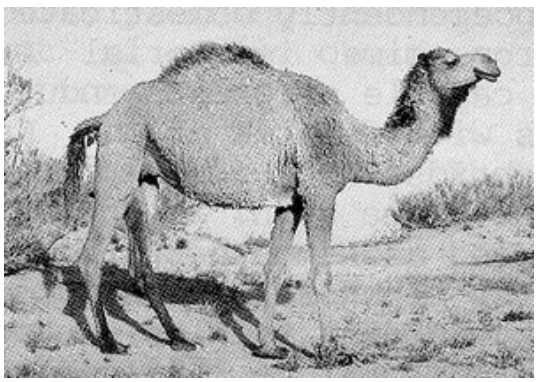

Figure 1: A dromedary with a hump on its back.

Fluids and the space

The rapid progress in technology so far in the 20th century has made it possible to go up higher and higher from the ground over the atmosphere of the earth, even to the moon! As the altitude becomes greater, the air becomes more

rarefied and then the usual approach of fluid dynamics based on the Navier-Stokes equations loses its validity. At the same time, as is well-known, there exists a magnetic field around the earth and the solar wind, which is a plasma, interacts with it, sometimes giving us the phenomena of a fantastic show called *The Northern Lights* or *Aurora*.

Figure 2: The interaction of a plasma and the magnetic field of the earth: The Northern Lights.

The degree of the rarefaction of the air is measured by a physical variable called the Knudsen number. In flows where the Knudsen number is large, which means the flow is rarefied, we must consult with Boltzmann instead of with Navier or Stokes. In their outlook, the Boltzmann equation and the Navier-Stokes equations seem totally different, but in 1917, about 50 years after the work of Boltzmann, a great Swedish scientist, David Enskog, showed that the Navier-Stokes equations could be derived as an asymptotic solution of the Boltzmann equation. Sydney Chapman also arrived at the same result independently of Enskog's work, and made further sophistication. In a different and mathematically beautiful way, Harold Grad provided the general solution of the Boltzmann equation and as a special consequence the Navier-Stokes equations were derived. It was only 100 years after they had been conceived phenomenologically, on the basis of experiment and experience, that the Navier-Stokes equations were at last derived theoretically from scientific first principles. The cooperation with the modern physics -relativity- has been made intensively and now it enables us to consider the fundamental questions of our universe and our origins.

This thesis consists of two parts which are not directly connected with each other, but are, as stated in this preface, related in a broad sense and therefore of concern to us.

N.Y.

Chapter 1

Introduction for paper 1

Since the first observation of "solitary wave" by John Scott Russell in 1834, a number of phenomena termed nonlinear waves have been one of the most important subjects in natural science. It is about 60 years later that two Dutch researchers derived the equation bearing their names, the Korteweg-de Vries equation [1], which describes the motion of long water waves. It was found that the solitary wave appeared as a special solution of the KdV equation. That is, the solitary wave became an object of mathematical analysis. After the historical discovery of the inverse scattering transform [2] for exact solution of a class of nonlinear partial differential equations, including many of physical interest, the concept of soliton has had a significant influence and consequences in various branches of mathematics, physics, and of engineering as well. In the last stage of this 20th century, soliton theory has obtained recognition as application-wise in industry. Nowadays optical-soliton [3] is a common word of the future telecommunication and its importance is needless to say.

When the term *soliton* is mentioned in the history of science, the role which numerical simulations played should not be neglected. Along with the inverse scattering transform, numerical simulations have been powerful tools to unveil the mysterious characteristics of solitons. In fact the name of soliton originated from the historical work by Norman Zabusky and Martin Kruskal in 1965 [4], which was a numerical investigation of waves in the discrete mass strings. As Zabusky himself pointed out, nonlinear science and numerical simulations have progressed mutually over the last quarter of this century. Toward the 21st century, they are getting closer in relation to each other and forming a stimulating field of science.

In recent years localized structure in the two-dimensional system described by the Davey-Stewartson (DS) 1 equations [5]

$$iA_t + A_{xx} + A_{yy} - 2|A|^2 A + 2Q_x A = 0, \tag{1.1}$$

$$Q_{xx} - Q_{yy} = 2(|A|^2)_x, \tag{1.2}$$

has been attracting a good deal of interest. The localized structure, called "dromion" [6], has many interesting characteristics. The variable A, hereafter called the main flow, is localized in two-dimensional space while the variable Q,

hereafter called the mean flow, is not. Another interesting feature is that the mean flow is driven at the boundary like a one-dimensional soliton [7]. We shall give explicit formulae describing this boundary condition later on. Dromions exist under the interaction between the main flow and the mean flows. This feature produces a stimulating question.
Can dromions be controlled arbitrarily if we handle the mean flow boundaries ?

Since the DS1 equations appear in several branches of physics, such as fluid dynamics [5] and plasma physics [8] [9], these localized structures are worth while being analyzed in detail. So far collisions of dromions [10] and time evolution of one-dromion [11] have been analyzed numerically and the results are studied in comparison with exact solutions. Many other characteristics of dromions, however, still remain veiled. It is desirable to carry out numerical simulations because the Lyapunov analysis [12] cannot be applied to investigate the stability of dromion solutions. The method is applicable only to the equation which has a Hamiltonian. In the case of the nonlinear Schrödinger equation, for example, stability of the localized structure has been fully studied by this analytical method. On the other hand, a Hamiltonian cannot be constructed for the dromion solution of the DS1 equations, since the mean flow Q is not localized and its values at $x = +\infty$ and $y = +\infty$ do not precisely coincide with those at $x = -\infty$ and $y = -\infty$ respectively. Currently we have little knowledge on the stability dromions due to this difficulty. However, from the results of several numerical works so far, the stability of dromions ensures us to conjecture the observation of localized structures in real multi-dimensional systems.

Chapter 2

The Davey-Stewartson equations

2.1 Electrostatic ion wave in a magnetized plasma

We shall show the derivation of the Davey-Stewartson equations. Although the Davey-Stewartson equations are derived from several physical systems [5] [8] [9], here we restrict our concern to a magnetized plasma. Let us consider a plasma in an applied magnetic field $\boldsymbol{B_0}$. We assume that the plasma is collisionless and described by its fluid behaviour. A further assumption is that the temperature of the electrons is so high that we may neglect the temperature of the ions. The charge of the ions is assumed to be compensated by that of electrons, that is, the plasma is neutral in charge. As long as we focus upon movement of the ions, we may assume that the electrons are always in their thermal equilibrium, thus the distribution is given by the Maxwellian.

The basic equations governing such a system are, in non-dimensional forms,

$$\frac{\partial n}{\partial t} + \nabla(n\boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\phi + a(\boldsymbol{v}\times\boldsymbol{b}), \tag{2.2}$$

$$\Delta\phi = \exp\phi - n, \tag{2.3}$$

where $n, \boldsymbol{v} = (u, v, w)$ are the density and the velocity of ions, ϕ the electrostatic potential. The vector $\boldsymbol{b} = (1, 0, 0)$ represents the direction of the magnetic field, and Δ denotes the Laplacian. The non-dimensional parameter a is given by $a = \omega_i/\Omega_i$, where $\omega_i = ZeB_0/Mc$ and $\Omega_i^2 = 4\pi n_0 Ze^2/M$. Here, n_0 is the initial unperturbed ion density, M and Z are the mass and the charge number of an ion, respectively, $-e$ is the charge of an electron, c is the speed of light, and B_0 is the magnitude of the applied magnetic field.

2.2 Reductive perturbation method

We consider time evolution of the wave packet of a perturbation $\sim \exp[i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$ on this system. We take $\boldsymbol{k}=(k_x,k_y,0)$ without loss of generality. Initial ion wave packets of perturbation are modulated by nonlinear effect. If wee see the packet on the coordinate moving at the group velocity which is determined by the linear dispersion relation of the system (2.1)-(2.3), the time variation of wave packets looks slow and hence we introduce the stretched variables [13],

$$\xi=\epsilon(x-V_{gx}t),\quad \eta=\epsilon(y-V_{gy}t),\quad \zeta=\epsilon z,\quad \tau=\epsilon^2 t, \tag{2.4}$$

where V_{gx},V_{gy} are the x and y components of the group velocity. Following the usual procedure of the reductive perturbation method, we expand the physical quantities around their stationary values as

$$n=1+\sum_{m=1}^{\infty}\epsilon^m\sum_{l=-\infty}^{\infty}n_l^{(m)}(\xi,\eta,\zeta,\tau)\exp[il(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)], \tag{2.5}$$

$$\phi=\sum_{m=1}^{\infty}\epsilon^m\sum_{l=-\infty}^{\infty}\phi_l^{(m)}(\xi,\eta,\zeta,\tau)\exp[il(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)], \tag{2.6}$$

$$\boldsymbol{v}=\sum_{m=1}^{\infty}\epsilon^m\sum_{l=-\infty}^{\infty}\begin{pmatrix}u_l^{(m)}(\xi,\eta,\zeta,\tau)\\ v_l^{(m)}(\xi,\eta,\zeta,\tau)\\ w_l^{(m)}(\xi,\eta,\zeta,\tau)\end{pmatrix}\exp[il(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)], \tag{2.7}$$

where the relations

$$n_l^{(m)*}=n_{-l}^{(m)},\quad \phi_l^{(m)*}=\phi_{-l}^{(m)},\quad \boldsymbol{v}_l^{(m)*}=\boldsymbol{v}_{-l}^{(m)}, \tag{2.8}$$

should be satisfied because of the reality condition of physical variables. Next we substitute these expanded variables into basic equations (2.1)-(2.3) and equate each coefficient of ϵ.

Detailed procedures and all the coefficients are given in ref. [8] so we write only essential results in followings.

2.3 The first order

In the first order of ϵ, we have

$$-\omega n_1^{(1)}+k_x u_1^{(1)}+k_y v_1^{(1)}=0, \tag{2.9}$$

from the equation of continuity (2.1),

$$\omega u_1^{(1)}+k_x\phi_1^{(1)}=0, \tag{2.10}$$

$$\omega v_1^{(1)}+k_y\phi_1^{(1)}-iaw_1^{(1)}=0, \tag{2.11}$$

$$\omega w_1^{(1)}+iav_1^{(1)}=0, \tag{2.12}$$

from the equations of motion. From the Poisson equation (2.3), we get

$$-n_1^{(1)} + \phi_1^{(1)} + k_x^2\phi_1^{(1)} + k_y^2\phi_1^{(1)} = 0. \tag{2.13}$$

Since our intention is to see the modulation of a plane wave with the frequency ω and the wave number $\boldsymbol{k}$, we put $n_l^{(1)}, \phi_l^{(1)}, u_l^{(1)}, v_l^{(1)}$, and $w_l^{(1)}$ equal to zero for all l except $l = 1, -1$.
Consequently we have the linear dispersion relation

$$\omega^4 - \left(a^2 + \frac{|\boldsymbol{k}|^2}{1+|\boldsymbol{k}|^2}\right)\omega^2 + a^2(\boldsymbol{b}\cdot\boldsymbol{k})^2\left(\frac{1}{1+|\boldsymbol{k}|^2}\right) = 0, \tag{2.14}$$

and

$$\phi_1^{(1)} = \frac{1}{1+|\boldsymbol{k}|^2}n_1^{(1)}, \tag{2.15}$$

$$u_1^{(1)} = c_{x11}n_1^{(1)}, \quad v_1^{(1)} = c_{y11}n_1^{(1)}, \quad w_1^{(1)} = c_{z11}n_1^{(1)}. \tag{2.16}$$

2.4 The second order

In the second order of ϵ, we have
for $l = 0$,

$$v_0^{(2)} = c_{y20}|n_1^{(1)}|^2, \quad w_0^{(2)} = 0, \tag{2.17}$$

$$\phi_0^{(2)} = n_0^{(2)} - \frac{1}{(1+|\boldsymbol{k}|^2)^2}|n_1^{(1)}|^2 \tag{2.18}$$

for $l = 1$,

$$\phi_1^{(2)} = \frac{n_1^{(2)}}{1+|\boldsymbol{k}|^2} + \frac{2jk_x}{(1+|\boldsymbol{k}|^2)^2}\frac{\partial n_1^{(1)}}{\partial\xi} + \frac{2jk_y}{(1+|\boldsymbol{k}|^2)^2}\frac{\partial n_1^{(1)}}{\partial\eta}, \tag{2.19}$$

$$u_1^{(2)} = c_{x11}n_1^{(2)} + c_{x21x}\frac{\partial n_1^{(1)}}{\partial\xi} + c_{x21y}\frac{\partial n_1^{(1)}}{\partial\eta}, \tag{2.20}$$

$$v_1^{(2)} = c_{y11}n_1^{(2)} + c_{y21x}\frac{\partial n_1^{(1)}}{\partial\xi} + c_{y21y}\frac{\partial n_1^{(1)}}{\partial\eta} + \frac{jc_{z11}}{k_y}\frac{\partial n_1^{(1)}}{\partial\zeta}, \tag{2.21}$$

$$w_1^{(2)} = c_{z11}n_1^{(2)} + c_{z21x}\frac{\partial n_1^{(1)}}{\partial\xi} + c_{z21y}\frac{\partial n_1^{(1)}}{\partial\eta} + c_{z21z}\frac{\partial n_1^{(1)}}{\partial\zeta}, \tag{2.22}$$

and for $l = 2$,

$$\phi_2^{(2)} = c_{p22}|n_1^{(1)}|^2, \quad n_2^{(2)} = c_{n22}|n_1^{(1)}|^2, \tag{2.23}$$

$$u_2^{(2)} = c_{v22x}|n_1^{(1)}|^2, \quad v_2^{(2)} = c_{v22y}|n_1^{(1)}|^2, \quad w_2^{(2)} = c_{v22z}|n_1^{(1)}|^2. \tag{2.24}$$

Equations in second order do not give the evolution of the mean current $n_0^{(2)}$, thus we need to proceed further to the third order to know the behaviour of the coupled system of he ion wave packet with the mean current.

2.5 The third order

In the third order we obtain coupled equations

$$(1-V_{gx}^2)\frac{\partial^2 n_0^{(2)}}{\partial\xi^2} - 2V_{gx}V_{gy}\frac{\partial n_0^{(2)}}{\partial\xi\partial\eta} - V_{gy}^2\frac{\partial^2 n_0^{(2)}}{\partial\eta^2} + h_1\frac{\partial^2|n_1^{(1)}|^2}{\partial\xi^2} + h_2\frac{\partial^2|n_1^{(1)}|^2}{\partial\xi\partial\eta} + h_3\frac{\partial^2|n_1^{(1)}|^2}{\partial\eta^2} = 0, \tag{2.25}$$

$$(1-V_{gx}^2)\frac{\partial^2 u_0^{(2)}}{\partial\xi^2} - 2V_{gx}V_{gy}\frac{\partial u_0^{(2)}}{\partial\xi\partial\eta} - V_{gy}^2\frac{\partial^2 u_0^{(2)}}{\partial\eta^2} + h_4\frac{\partial^2|n_1^{(1)}|^2}{\partial\xi^2} + h_5\frac{\partial^2|n_1^{(1)}|^2}{\partial\xi\partial\eta} + h_6\frac{\partial^2|n_1^{(1)}|^2}{\partial\eta^2} = 0, \tag{2.26}$$

for $l=0$, and

$$i\frac{\partial n_1^{(1)}}{\partial\tau} + \alpha_1\frac{\partial^2 n_1^{(1)}}{\partial\xi^2} + \alpha_2\frac{\partial^2 n_1^{(1)}}{\partial\xi\partial\eta} + \alpha_3\frac{\partial^2 n_1^{(1)}}{\partial\eta^2} + \alpha_4\frac{\partial^2 n_1^{(1)}}{\partial\zeta^2} + \alpha_5|n_1^{(1)}|^2 n_1^{(1)} + \alpha_6(n_0^{(2)}+\alpha_7 u_0^{(2)})n_1^{(1)} = 0 \tag{2.27}$$

for $l=1$.

We introduce new dependent variables A and Q as $A = n_1^{(1)}$ and $Q = n_0^{(2)} + \alpha_7 u_0^{(2)}$, and rewrite these equations as

$$(1-V_{gx}^2)Q_{\xi\xi} - 2V_{gx}V_{gy}Q_{\xi\eta} - V_{gy}^2 Q_{\eta\eta} + h_7(|A|^2)_{\xi\xi} + h_8(|A|^2)_{\xi\eta} + h_9(|A|^2)_{\eta\eta} = 0, \tag{2.28}$$

and

$$iA_\tau + \alpha_1 A_{\xi\xi} + \alpha_2 A_{\xi\eta} + \alpha_3 A_{\eta\eta} + \alpha_4 A_{\zeta\zeta} + \alpha_5|A|^2 A + \alpha_6 QA = 0. \tag{2.29}$$

2.6 Reduction to the DS equations

In the case of perpendicular propagation to the magnetic field, the basic equation (2.3) is not exactly valid because electron inertia should not be neglected. However, the validity of considering the "perpendicular" wave propagation is discussed in ref. [8] and it is shown that we may regard the equation (2.3) as the governing equation of the system.

We can take $k_x \to 0$ in the previous section and assume that all variables are independent of ζ. Finally we obtain

$$iA_\tau + \alpha_1 A_{\xi\xi} + \alpha_3 A_{\eta\eta} + \alpha_5 |A|^2 A + \alpha_6 QA = 0, \tag{2.30}$$

$$Q_{\xi\xi} - V_{gy}^2 Q_{\eta\eta} + h_7 (|A|^2)_{\xi\xi} = 0. \tag{2.31}$$

After a suitable transformation in the coordinate space, cross derivatives have vanished. Equations (2.30) and (2.31) are the DS equations and particularly, these become DS1 equations if $\alpha_1 \alpha_3 > 0$. The region of the parameters where this DS1 condition is satisfied is also given in ref. [8].

2.7 Dromion solution

2.7.1 One-dromion

The number of physically important ,multi-dimensional soliton equations which have been solved analytically is very small. The Davey-Stewartson equations and the Kadomtsev-Petviashvili equation [14] are a few examples whose special solutions have localized structure in 2-dimension. The lump solution [15] of the KP equation, for example, is localized in 2-dimensional space, but it does not have an exponentially decaying base. On the other hand, the Davey-Stewartson 1 equations have an exact solution called "dromion", which decays exponentially in any direction in 2-D plane. Because of its unique characteristics, the dromion solution *localizes* attention of many researchers.
Let us summarize the one-dromion solution of the DS1 equations. The DS1 equations are written in the original form as

$$iA_t + A_{xx} + A_{yy} - 2|A|^2 A + 2Q_x A = 0, \tag{2.32}$$

$$Q_{xx} - Q_{yy} = 2(|A|^2)_x. \tag{2.33}$$

In equations (2.32) and (2.33), making a 45-degree rotation in the coordinate space as $x \to x + y,\ y \to x - y$, and introducing new variables, $U \equiv Q_x - |A|^2$ and $V \equiv Q_y - |A|^2$, we obtain

$$iA_t + A_{xx} + A_{yy} + (U + V)A = 0, \tag{2.34}$$

$$U_y = (|A|^2)_x,\ \ V_x = (|A|^2)_y. \tag{2.35}$$

The one dromion solution is given by [16]

$$A = \frac{G}{F}. \tag{2.36}$$

Here we have chosen

$$F = 1 + \exp(\eta_1 + \eta_1^*) + \exp(\eta_2 + \eta_2^*) + \gamma \exp(\eta_1 + \eta_1^* + \eta_2 + \eta_2^*), \tag{2.37}$$

$$G = \rho \exp(\eta_1 + \eta_2). \tag{2.38}$$

In equations (2.37) and (2.38), parameters are given by

$$\begin{aligned} |\rho| &= 2\sqrt{2k_r l_r(\gamma-1)}, \\ \eta_1 &= (k_r + ik_i)x + (\Omega_r + i\Omega_i)t, \\ \eta_2 &= (l_r + il_i)y + (\omega_r + i\omega_i)t, \\ \Omega_r &= -2k_r k_i,\ \omega_r = -2l_r l_i, \\ \Omega_i + \omega_i &= k_r^2 + k_i^2 + l_r^2 + l_i^2, \end{aligned} \tag{2.39}$$

where γ, k_r, k_i, l_r and l_i are real constants. These five are substantially free parameters of the one-dromion solution. The constant γ determines an amplitude, k_r the width of the pulse in the x-direction, and l_r in the y-direction. The quantities k_i and l_i are x and y components of velocity, respectively.

2.7.2 Boundary condition

The potentials U and V are determined simply by integrating equation (2.35). For the dromion solution, the boundary values of the potentials U and V are not zeros but are given in the form of 1-dimensional soliton by

$$U|_{y=-\infty} = \frac{8k_r^2 \exp(\eta_1 + \eta_1^*)}{[1 + \exp(\eta_1 + \eta_1^*)]^2}, \tag{2.40}$$

$$V|_{x=-\infty} = \frac{8l_r^2 \exp(\eta_2 + \eta_2^*)}{[1 + \exp(\eta_2 + \eta_2^*)]^2}. \tag{2.41}$$

One should be aware from equations (2.36)-(2.38) that the main flow A has a peak at the cross section of mean flows U and V, and that its peak decays exponentially in any direction in two dimensional plane. Figure 2.1 shows the amplitude of the main flow $|A|^2$ and Figure 2.2 shows the mean flows $U + V$. Boundary conditions (2.40) and (2.41) play an important role on handling one-dromion, which shall be discussed in paper 1.

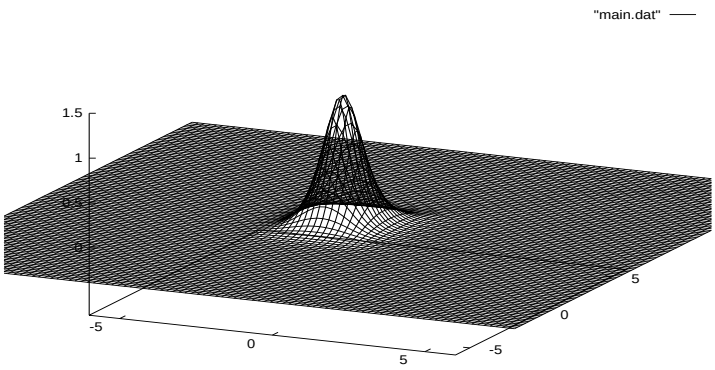

Figure 2.1: The main flow of one-dromion solution, $|A|^2$.

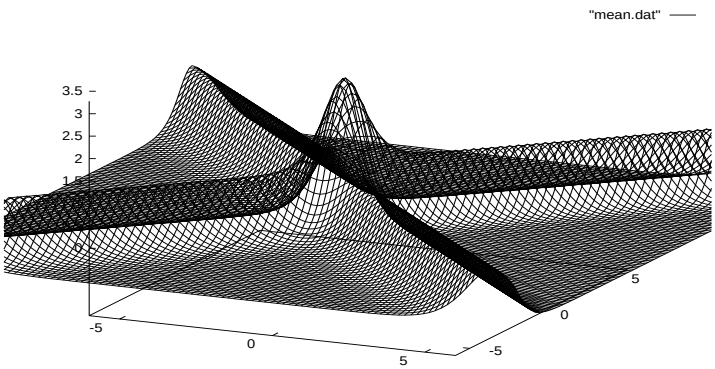

Figure 2.2: The mean flow of one-dromion solution, $U + V$.

Chapter 3

Introduction for paper 2

Since Boltzmann derived an equation which describes a dynamical system of colliding particles and showed the so-called H theorem, which consequently leads to the second law of thermodynamics, huge amount of intensive work has been done so far treating the macroscopic non-uniformities in a gas or a plasma based on the microscopic phenomena. They were all successful in the sense that concrete consistency was established from microscopic to macroscopic levels. (Except one argument on the assumption itself, which still remains open. "molecular chaos" [19] [26].)

It was not until 1940 that the relativistic generalization of the complete Boltzmann equation was derived by Lichnerowics and Marrot [20], which satisfies the Lorentz invariance of the special relativity [21] [22]. Thereafter mathematical methods of the classical(non-relativistic) kinetic theory were formally extended to analyze gas dynamic phenomena in the relativistic regime. So were the Chapman-Enskog [24] method and the Grad method [26].

A number of studies have been devoted so far to remedy the well-known difficulty in the relativistic gas dynamics [29] [31]: (a) thermal conductivity and viscosity give rise to disturbances which propagate at infinite speed, which contradicts the principle of causality [21]; (b) there exists unstable perturbation of short wavelength in the dissipation processes [32]. Moment equations attract particular attention as they, hyperbolic systems, solve these problems since they lead to finite speeds of propagation for any disturbance.

Let us return to non-relativistic flows. In 1963 Cercignani [23] improved the moment method to analyze the kinetic layer in pure shear flow and showed the first order Chapman-Enskog solution, rather than the corresponding Hermite polynomials used by Grad, to be suitable for the analysis. The Grad 13 moments method is based on expansion of the velocity distribution function in terms of Hermite polynomials. The lowest non-trivial truncation gives 13 moment equations, which for slowly varying flows lead to the Navier-Stokes equations with Fourier's expression for the heat current. However they give only the first approximations for viscosity and heat conductivity in terms of Sonine polynomials. To remedy this deficiency, Söderholm [25] generalized Cercignani's method and derived new 13 moments equations correct to first order

in the Knudsen number. The obtained moment equations have the same general form as Grad's, but with correct values of viscosity and heat conductivity. In ref. [25] the corresponding values for the relaxation times and the couplings between heat current and viscous pressure are also calculated.

In the case of relativistic flows there is a non-vanishing volume viscosity [27] also for particles with no inner degrees of freedom. The simplest truncation thus leads to 14 rather than 13 moments. The conventional 14 moment approximation [27] [28] [29] gives only the first approximation for thermodynamic coefficients such as viscosity and heat conductivity. It is our main purpose to give both finite speeds of propagation of disturbances and correct values for thermodynamic coefficients.

Chapter 4

Non-relativistic kinetic theory of gas dynamics

4.1 The Boltzmann equation and the conservation laws

Let us give a brief summary of non-relativistic gas dynamics. Consider a sample of gas consisting of molecules that have various velocities. The single particle distribution function is defined in such a way that the probable number of molecules which, at a moment t, are located with in the volume element $d\boldsymbol{x}$ in physical space and have velocities lying in the range $d\boldsymbol{c}$ is equal to

$$f(\boldsymbol{x}, \boldsymbol{c}, t) d\boldsymbol{x} d\boldsymbol{c}. \tag{4.1}$$

For this distribution function the Boltzmann equation is formulated

$$\mathcal{D} f = C(f, f). \tag{4.2}$$

The comoving time derivative in phase space is defined as

$$\mathcal{D} = \frac{\partial}{\partial t} + \boldsymbol{c} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{F} \cdot \frac{\partial}{\partial \boldsymbol{c}}, \tag{4.3}$$

where $\boldsymbol{c}$ is the molecular velocity, $\boldsymbol{F}$ the volume force. The collision operator is given by, with the standard notation [34],

$$C(f, f) = \int \int \int (f_1' f' - f_1 f) g b \, db \, d\epsilon \, d\boldsymbol{c}_1. \tag{4.4}$$

We introduce the macroscopic velocity $\boldsymbol{u}$ and the peculiar velocity $\boldsymbol{C}$ so that

$$\boldsymbol{c} = \boldsymbol{u} + \boldsymbol{C}, \tag{4.5}$$

and then the convective time derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}}. \tag{4.6}$$

The conservation of mass, momentum, and energy are written

$$\frac{Dn}{Dt} = -n\nabla \cdot \boldsymbol{u}, \tag{4.7}$$

$$\tag{4.8}$$

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{mn}(\nabla p + \nabla \cdot \boldsymbol{\Pi}) + \boldsymbol{F}, \tag{4.9}$$

$$\tag{4.10}$$

$$\frac{DT}{Dt} = -\frac{2}{3k_B n}(p\nabla \cdot \boldsymbol{u} + \mathrm{tr}(\boldsymbol{D}\boldsymbol{\Pi}) + \nabla \cdot \boldsymbol{q}), \tag{4.11}$$

where n is the number density, m the molecular mass, p the hydrostatic pressure given by $p = nk_BT$, and T is the temperature. The viscous-pressure tensor is expressed as $\boldsymbol{\Pi}$ and the heat current $\boldsymbol{q}$, and $\boldsymbol{D}$ stands for the rate of deformation $\partial U_i/\partial x_j$.

4.2 The Chapman-Enskog method

In this section we introduce the normal Chapman-Enskog method. The Chapman-Enskog method is based on the expansion of the distribution function with the assumption that the Knudsen number -the ratio of the mean free path to the characteristic length scale, $\epsilon = l_f/l$- is small. The first order approximation is written

$$f = f^{(0)} + \epsilon f^{(1)} = f^{(0)}(1 + \phi), \tag{4.12}$$

where we write the deviation function as ϕ. The zeroth order solution $f^{(0)}$ is a local equilibrium distribution function, called Maxwellian,

$$f^{(0)} = n(\frac{m}{2\pi k_B T})^{\frac{3}{2}} \exp(-\mathcal{C}^2). \tag{4.13}$$

Here we have normalized the velocity as $\mathcal{C}_i = \sqrt{m/2k_BT}C_i$. The first order equation reduces to

$$-n^2 I(\phi) + \mathcal{D}f^{(0)} = 0, \tag{4.14}$$

where the linearized collision operator is

$$n^2 I(\phi) = J(f^{(0)}, f^{(0)}\phi) + J(f^{(0)}\phi, f^{(0)}). \tag{4.15}$$

The solution for ϕ is obtained to be

$$\phi = -\frac{1}{n}(\sqrt{\frac{2k_BT}{m}}A(\mathcal{C}^2)\mathcal{C}_i\frac{\partial \ln T}{\partial x_i} + 2B(\mathcal{C}^2)\,\mathcal{C}_i\overset{\circ}{\mathcal{C}}_j\,\frac{\partial u_i}{\partial x_j}). \tag{4.16}$$

Hereafter the notation $\circ$ stands for the symmetric trace-less part of a tensor.

After the coefficient functions are expanded in terms of Sonine polynomials, the heat current and the viscous-pressure tensor can be calculated

$$\boldsymbol{q} = -\lambda\nabla T, \tag{4.17}$$

$$\boldsymbol{\Pi} = -2\eta\,\overset{\circ}{\boldsymbol{D}}\,. \tag{4.18}$$

The viscosity η and the heat conductivity λ can be explicitly expressed by the coefficients of the corresponding Sonine polynomials.

4.3 The 13 moments method of Grad

4.3.1 Expansion in Hermite polynomials

Grad [26] sought solutions of the Boltzmann equation of a more general type. He expressed the distribution function as a series of the multi-dimensional Hermite polynomials with the first-order solution(containing only the first term) being the local Maxwellian.

$$f = f^{(0)} \sum_{n=0}^{\infty} \frac{1}{n!} a_i^{(n)} \mathcal{H}_i^{(n)}, \tag{4.19}$$

where $\mathcal{H}_i^{(n)}$ is Hermite polynomials of order n and $a_i^{(n)}$ are functions to be determined from the equations of change. He showed that by taking a sufficiently large number of terms, f could be determined to any desired order of approximation and converged to any solution of the Boltzmann equation.

4.3.2 13 moments approximation

The third order solution leads to an expression for f as the local Maxwellian multiplied by an expression involving the higher moments of the distribution function, namely, the heat current and the viscous-pressure as well as $n, \boldsymbol{U}, T$. The time variation of those 13 moments is obtained by the corresponding equations. Thus this simple truncation is called the 13 moments approximation. The equations for the heat current and the viscous-pressure tensor are given by

$$\frac{2m}{5k_B p} \frac{Dq_i}{Dt} + \frac{2T}{5p} \Pi_{ij,j} + \frac{1}{[\lambda]_1} q_i + T_{,i} = 0, \tag{4.20}$$

$$\frac{1}{p} \frac{D\Pi_{ij}}{Dt} + \frac{4}{5p} \overset{\circ}{q}_{i,j} + \frac{1}{[\eta]_1} \Pi_{ij} + 2 \overset{\circ}{u}_{i,j} = 0. \tag{4.21}$$

The subscript $[\ ]_1$ is attached to the viscosity and the heat conductivity since the 13 moments equations give only the first order approximation of those coefficients. Consequently, as is well-known as a crucial deficiency, they always give the Prandtl number 2/3.

4.4 Previous work

Motivated by the fact that the conventional Grad 13 moments equations do not give the correct values for viscosity and heat conductivity, Söderholm [25] derived a new set of 13 moments equations so as to remedy the deficiency.

4.4.1 Flow of two length scales

Söderholm used a weaker assumption that the length scale of the deviation function ϕ is comparable to the mean free path. As a consequence, gradients of the irreversible flows, the heat current and the viscous-pressure, are to be contained in the linearized Boltzmann equation as source terms. To solve the linearized equation, an Ansatz is formulated so that the deviation function takes

the same general form as the first order Chapman-Enskog solution. That is, ϕ is given by

$$\phi = -\frac{2m}{5pk_BT}\sqrt{\frac{2k_BT}{m}}q_i\mathcal{C}_i\hat{A}(\mathcal{C}^2) + \frac{1}{p}\Pi_{ij}\,\overset{\circ}{\mathcal{C}_i\mathcal{C}_j}\,\hat{B}(\mathcal{C}^2), \tag{4.22}$$

allowing $\boldsymbol{q}$ and $\boldsymbol{\Pi}$ to be functions of space and time. An important result is that this Ansatz precisely corresponds to Grad's if the first non-trivial truncation of the coefficient functions is done. Moreover, if $\boldsymbol{q}$ and $\boldsymbol{\Pi}$ take the special value given by the relation (4.17) and (4.18), the Ansatz reduces to the first order Chapman-Enskog solution.

4.4.2 Modified 13 moment equations

The symmetric Galerkin method is applied to the linearized Boltzmann equation, and moment equations are derived. They are written,

$$\gamma_q\frac{2m}{5k_Bp}\frac{Dq_i}{Dt} + \delta\frac{2T}{5p}\Pi_{ij,j} + \frac{1}{\lambda}q_i + T_{,i} = 0, \tag{4.23}$$

$$\gamma_\Pi\frac{1}{p}\frac{D\Pi_{ij}}{Dt} + \delta\frac{4}{5p}\,\overset{\circ}{q_{i,j}} + \frac{1}{\eta}\Pi_{ij} + 2\,\overset{\circ}{u_{i,j}} = 0. \tag{4.24}$$

As mentioned previously, if the first non-trivial truncation is done, then the coefficients $\gamma_q, \gamma_\Pi, \delta$ all take the unit value and the equation reduce to (4.20) and 4.21).

Chapter 5

Summary of papers

5.1 Paper 1

Yoshida N, Nishinari K, Satsuma J and Abe K 1998 Dromion can be remote-controlled. Journal of Physics A: Mathematical and General **31** 3325-3336
The main part of the paper is a numerical analysis of initial-boundary problems of the Davey-Stewartson 1 equations. Several types of behaviour of the one dromion, a special solution of the DS1 equations, are investigated. The equations are reformed and a new numerical scheme to solve initial-boundary problems of the DS1 equations is proposed. After the scheme is applied to computing an exact solution and checked to be an appropriate one for numerical analysis of the DS1 system, the one-dromion solution with driving the value of mean flows at the boundary is analyzed numerically.

Two types of movement of mean flows are studied and the decay ratio of the localized structure is given versus some parameters.

5.2 Paper 2

Söderholm L H and Yoshida N 1998 Moment equations based on the first order Chapman-Enskog solution -Relativistic gas-. Accepted for a conference presentation in the 21st International Symposium on Rarefied Gas Dynamics
A generalized method to derive moment equations correct to the first order in the Knudsen number is applied to relativistic gas dynamics. The linearized Boltzmann equation for a flow of two length scales is derived and moments equations are obtained by means of the symmetric Galerkin method. As an ansatz we assume that the solution for the linearized Boltzmann equation takes formally the same form as that of the first order Chapman-Enskog solution. The coefficient functions are expanded by an infinite series in powers of a dimensionless variable related to the particle energy, rather than orthogonal polynomials. In flows in the relativistic regime there is a non-vanishing volume viscosity also for particles with no inner degree of freedom. The simplest truncation thus leads to 14 moments. Discussions are given in comparison with the conventional 14 moment approximation, as for the general form and the entropy production.

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Special and strange thanks are due to the sunlight (Solljus) in Sweden. It gave me a splendid atmosphere in the summer, and during the long, cold and dark winter season it ironically provided me with a sufficient time to concentrate on study only. Without its help, I would definitely have spent much time in another way. Tack.

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Paper 1

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Dromion can be remote-controlled

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Abstract. The one-dromion solution of the Davey–Stewartson 1 equations with driving boundaries is analysed numerically. It is shown that the dromion can follow the motion of the crosspoint of the mean flows even if it is not of exact solutions. Moreover, it is found that the localized structure keeps its shape when it is exposed to the 'forced' motion of the mean flows. In the case where dromion runs on a circle path, the decay ratio of the localized structure versus the radius is given. Several types of movement of mean flows are also studied. It is expected that dromion can be controlled arbitrarily by driving mean flows at the boundaries.

1. Introduction

Recently the localized structure in the two-dimensional system described by the Davey–Stewartson (DS) 1 equations [1]

$$\mathrm{i}A_t + A_{xx} + A_{yy} - 2|A|^2A + 2Q_xA = 0 \tag{1}$$

$$Q_{xx} - Q_{yy} = 2(|A|^2)_x \tag{2}$$

has been attracting a good deal of interest. The localized structure, called 'dromion' [2], has many interesting characteristics. For example, the variable A, hereafter called the main flow, is localized in two-dimensional space while the variable Q, hereafter called the mean flow, is not. Another interesting feature is that the mean flow is driven at the boundaries like a one-dimensional soliton [3]. We shall give exact formulae describing this boundary condition in section 2. Dromion exists under the interaction between the main flow and mean flows. This feature produces a stimulating question. Can dromion be controlled arbitrarily if we handle the mean flow boundaries?

Since the DS1 equations appear in several branches of physics, such as fluid dynamics [4] and plasma physics [5], these localized structures are worth being analysed in detail. So far collisions of dromions [6] and time evolution of one-dromion [7] have been studied numerically. Many other behaviours of dromions, however, still remain veiled. The purpose of this paper is to investigate in detail how dromions behave when we drive mean flows at

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the boundary, that is, the position is forced to move in a certain way. In this situation it is expected that dromions follow the motion of the crosspoint of mean flows in view of the fact that the crosspoint plays a significant role as an attracting spot. We employ numerical computation to study the behaviour of dromions.

This paper is organized as follows. In section 2, we summarize the dromion solution of the DS1 equations. In section 3, we introduce another form of the DS1 equations and propose a new numerical scheme. Computational results are given in section 4. Concluding remarks are given in section 5.

2. The one-dromion solution

Let us summarize the one-dromion solution of the DS1 equations. In equations (1) and (2), making a 45° rotation in the coordinate space as $x \to x + y$, $y \to x - y$, and introducing new variables, $U \equiv Q_x - |A|^2$, $V \equiv Q_y - |A|^2$, we obtain

$$\mathrm{i}A_t + A_{xx} + A_{yy} + (U + V)A = 0 \tag{3}$$

$$U_y = (|A|^2)_x \qquad V_x = (|A|^2)_y. \tag{4}$$

The one-dromion solution is expressed as

$$A = \frac{\rho \exp(\eta_1 + \eta_2)}{1 + \exp(\eta_1 + \eta_1^*) + \exp(\eta_2 + \eta_2^*) + \gamma \exp(\eta_1 + \eta_1^* + \eta_2 + \eta_2^*)}. \tag{5}$$

In equation (5), parameters are given by

$$|\rho| = 2\sqrt{2k_r l_r(\gamma - 1)}$$

$$\eta_1 = (k_r + \mathrm{i}k_i)x + (\Omega_r + \mathrm{i}\Omega_i)t$$

$$\eta_2 = (l_r + \mathrm{i}l_i)y + (\omega_r + \mathrm{i}\omega_i)t$$

$$\Omega_r = -2k_r k_i \qquad \omega_r = -2l_r l_i$$

$$\Omega_i + \omega_i = k_r^2 + k_i^2 + l_r^2 + l_i^2$$

where γ, k_r, k_i, l_r and l_i are real constants. These five are substantially free parameters of the one-dromion solution. The constant γ determines an amplitude, k_r is the width of the pulse in the x-direction, and l_r in the y-direction. The quantities k_i and l_i are x and y components of velocity, respectively.

The potentials U and V are determined simply by integrating equation (4). For the dromion solution, boundary values of potentials U and V are not zeros, but are given in the form of one-dimensional soliton by

$$U|_{y=-\infty} = \frac{8k_r^2 \exp(\eta_1 + \eta_1^*)}{[1 + \exp(\eta_1 + \eta_1^*)]^2} \tag{6}$$

$$V|_{x=-\infty} = \frac{8l_r^2 \exp(\eta_2 + \eta_2^*)}{[1 + \exp(\eta_2 + \eta_2^*)]^2}. \tag{7}$$

One should be aware from equation (5) that the main flow A has a peak at the cross section of mean flows U and V, and that its peak decays exponentially in any direction in two-dimensional plane. Boundary conditions (6) and (7) play an important role on handling the one-dromion, which will be discussed in section 4.

3. Numerical scheme to solve the DS1 equations

As we see in equations (1) and (2), the time evolution of mean flow Q is not given explicitly. It is given nonlinearly and implicitly through the main flow A. This nature causes some difficulties in computation.

To make equation (2) easier to solve numerically, we split equation (2) into two first-order hyperbolic systems. We put

$$\hat{Q} = Q_x + Q_y. \tag{8}$$

Then equation (2) is written as

$$\hat{Q}_x - \hat{Q}_y = 2(|A|^2)_x \tag{9}$$

from which we obtain a conserved form

$$\hat{Q}_y - (\hat{Q} - 2|A|^2)_x = 0. \tag{10}$$

Next, by differentiating equation (8) by x, we obtain

$$(Q_x)_y + (Q_x - \hat{Q})_x = 0 \tag{11}$$

which is also in a conserved form.

Introducing an appropriate scale transformation and putting $\hat{Q} = B$, $Q_x = C$, the DS1 equations (1) and (2) are rewritten as

$$\mathrm{i}A_t + A_{xx} + A_{yy} + (2C - |A|^2)A = 0 \tag{12}$$

$$B_y - (B - |A|^2)_x = 0 \tag{13}$$

$$C_y + (C - B)_x = 0. \tag{14}$$

Although these equations look more complicated than the original ones, they are in a convenient form for numerical simulations.

Next we briefly explain the numerical method for solving the initial-boundary problem of the DS1 equations. If the main flow A is given at one moment and the outer boundary conditions $B|_{y=-\infty}$ and $C|_{y=-\infty}$ are designated.

(i) The wave equation (13) is solved from $B|_{y=-\infty}$ for the given A, to obtain the value of B on the whole area.

(ii) The wave equation (14) is solved from $C|_{y=-\infty}$ with the calculated value of B, to obtain the value of C on the whole area.

(iii) By using the value of A and the value of C calculated in procedure (ii), the time integration of equation (12) is performed, which gives the value of A at the next time step.

We adopt the second-order-upwind TVD method [8] to solve the wave equations (13) and (14). Time integration of equation (12) is performed by the Jameson–Baker method [9]. The spatial derivatives in equation (12) are calculated by the fourth-order central descritization. The grid system is generated by standard square lattices and the computational domain is chosen to be sufficiently large. On the main flow A, we fix the dumping area very near the boundaries to absorb ripples emitted from dromions [6]. We adopt the periodic boundary condition on mean flows B and C in the x-direction because their values decay rapidly to zeros. Practically this scheme needs only one boundary at $y = -\infty$.

4. Computational results

First, we simulate the exact one-dromion solution in order to see the accuracy of this scheme. In this computation, we take $k_r = l_r = \frac{4}{5}$, $k_i = l_i = \frac{1}{5}$ and $\gamma = 3$ as dromion parameters.

We checked the maximum amplitude of $|A|^2$ and the first conserved quantity of the DS1 equations,

$$I_1 = \int |A|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

The maximum fluctuation of amplitude of $|A|^2$ is 3% of the total value. These fluctuations are caused when the peak of dromion precisely hits the vertex on the computational domain. The maximum fluctuation of I_1 is $\Delta I_1/I_1 \sim 10^{-10}$. From these results, we may say that this scheme is appropriate to calculate dromion solutions.

Our intention is to clarify whether or not dromions propagate stably under the condition that boundaries are driven to move irregularly. We apply two types of movement of boundaries, which are listed in table 1 with figure 1. The interesting one is the case in which we drive boundaries in such a way that the crosspoint of mean flows moves on a circle path. In order to realize this situation, we employ the following boundary conditions for potentials; $U|_{y=-\infty}$ varying as a function of $\sin(wt)$, and $V|_{x=-\infty}$ varying as a function

Table 1. Forced movement of boundaries described as the function of $\sin(\omega t)$ and $\cos(\omega t)$ in the power index of $\exp(\eta)$ (equations (6) and (7)).

Name	x-direction $(\eta_1 + \eta_1^*)$	y-direction $(\eta_2 + \eta_2^*)$	Path of the meanflow crosspoint
Revolution	$k_r(x + \frac{\Omega_r}{k_r}\sin(wt))$	$l_r(x + \frac{\omega_r}{l_r}\cos(wt))$	Figure (*a*)
Lissajous	$k_r(x + \frac{\Omega_r}{k_r}\sin(wt))$	$l_r(x + \frac{\omega_r}{l_r}\sin(2wt))$	Figure (*b*)

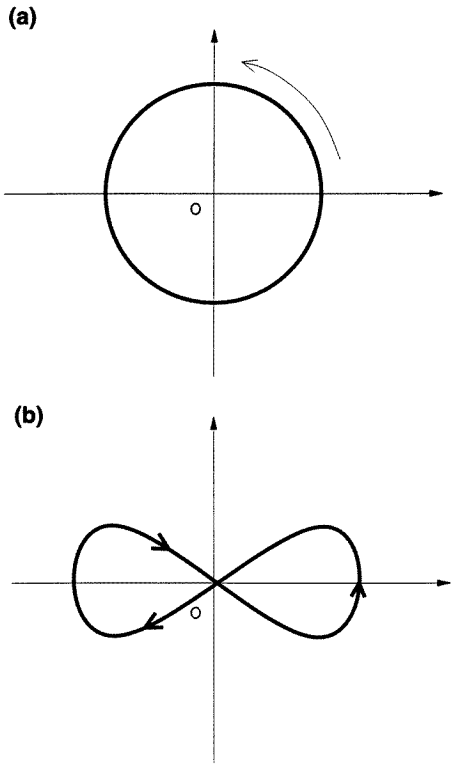

Figure 1. Sketches of the motions in table 1. The mean flow crosspoint is driven to move on these curves in the direction indicated by arrows.

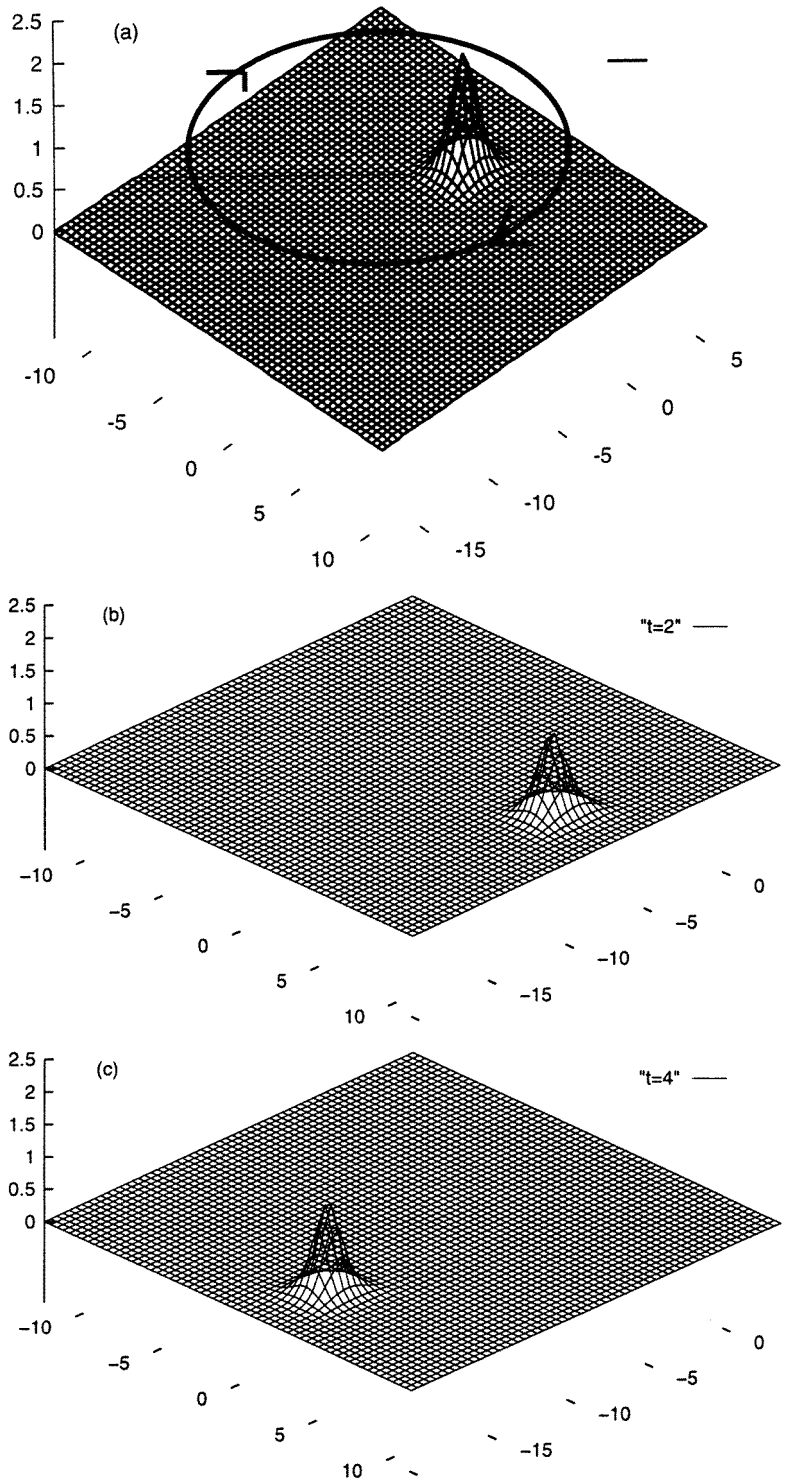

Figure 2. Solid profiles of $|A|^2$ in the case of the revolutional dromion. (*a*) $t = 0.0$: initial state, (*b*) $t = 2.0$: $\frac{1}{5}$ rotation, (*c*) $t = 4.0$: $\frac{2}{5}$ rotation, (*d*) $t = 6.0$: $\frac{3}{5}$ rotation, and (*e*) $t = 8.0$: $\frac{4}{5}$ rotation. Dromion is kept localized through the rotation.

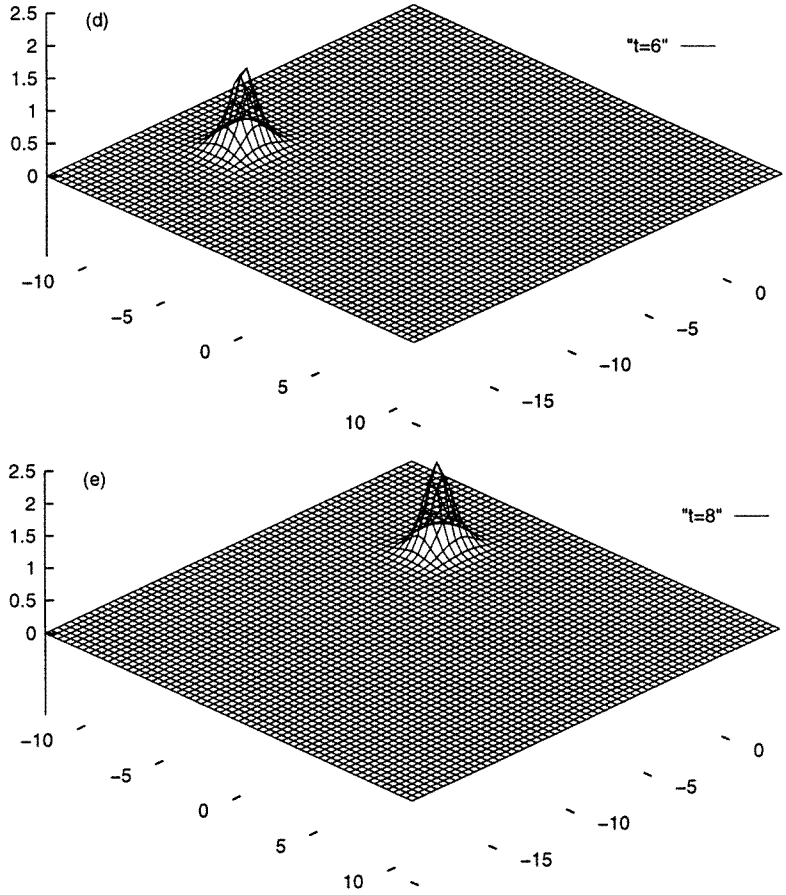

Figure 2. (Continued)

of $\cos(wt)$. That is, the power index in $\exp(\eta)$ is replaced by

$$k_r x + \Omega_r t \to k_r \left(x + \frac{\Omega_r}{k_r} \sin(wt) \right) \tag{15}$$

$$l_r y + \omega_r t \to l_r \left(y + \frac{\omega_r}{l_r} \cos(wt) \right) \tag{16}$$

in equations (6) and (7). Then the crosspoint of mean flows rotates around the original point. A simplified sketch is shown in figure 1(a). We shall now discuss the obtained results in detail. Figure 2 shows solid profiles of the main flow. At time $t = 0$, both the main flow and the mean flow are set to be exact solutions. Then the mean flow is forced to move at the boundary as explained above. In the results, surprisingly, we see that the dromion keeps its shape and propagates quasistably along the circle. In order to show how much localized volume is preserved and what amount of the original structure is emitted as ripples, the variation of the dromion volume is presented in figure 3. Dromion volume is estimated as,

$$W = \int_s |A|^2 \, \mathrm{d}x \, \mathrm{d}y$$

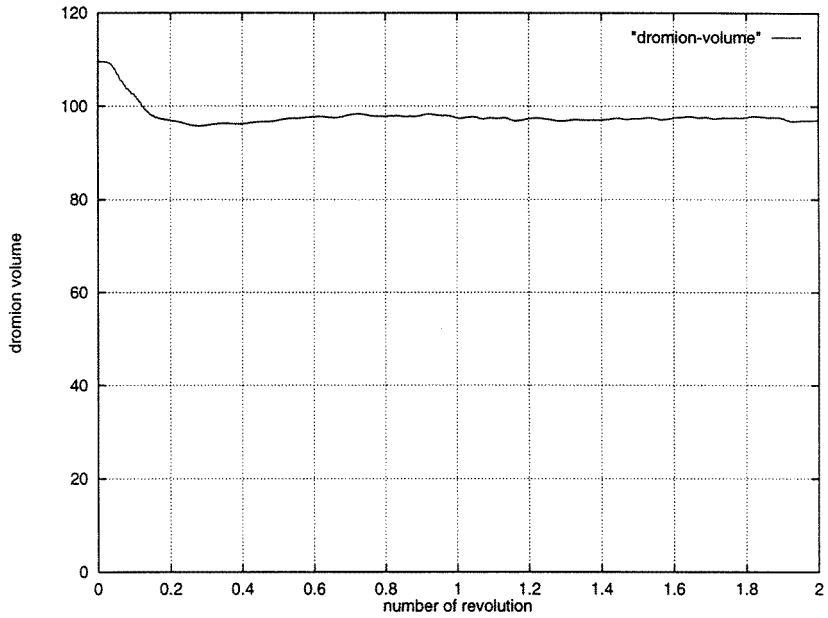

Figure 3. Time variation of the volume of the localized structure $V = \int_s |A|^2 \, \mathrm{d}x \, \mathrm{d}y$. The horizontal axis is scaled to show the time of revolution.

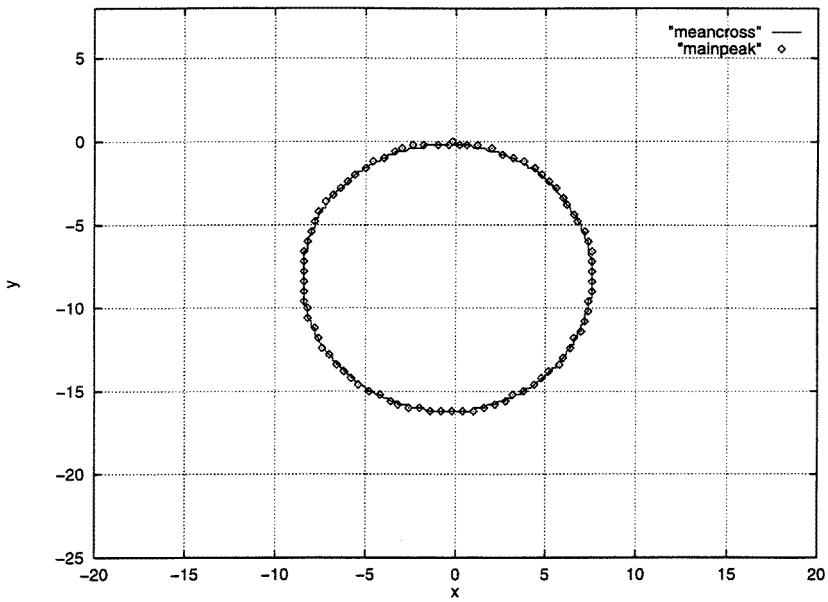

Figure 4. Pathline of the mean flow crosspoint. Dots indicate the position of the main flow peak.

where integral domain s is chosen sufficiently narrow as to contain only the localized structure at the mean flow crosspoint. In figure 3 we can see that the value decreases gradually at the beginning, but then does not change substantially. It is interesting to note

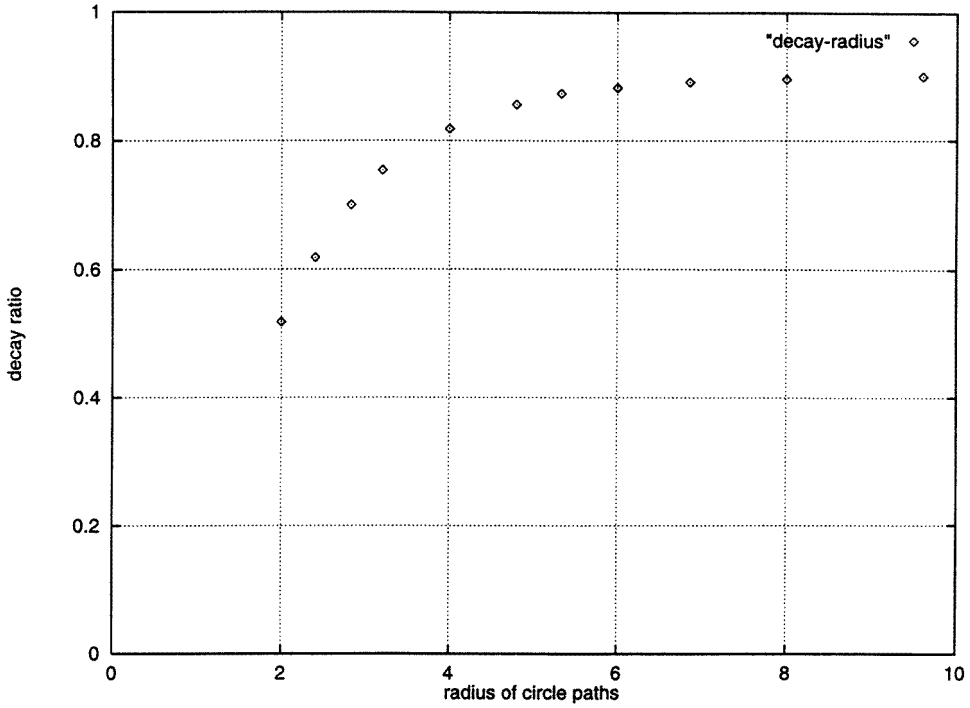

Figure 5. Decay ratio of dromion volume versus the radius of the circle. The decay ratio is normalized by the initial volume of dromion. As the radius becomes larger, the decay ratio approaches the value 1.0.

that the dromion volume remains almost steady after it loses some amount. The reason can be explained as follows. In the beginning, the dromion becomes smaller by emitting ripples, which may be caused by the accelerated motion of the mean flow crosspoint. The exact one-dromion propagates at a constant velocity on a straight path, with no acceleration. It keeps its shape and volume perfectly. In the case of revolving dromion, however, it tends to move with the mean flow crosspoint running on a circle path. When dromion is forced to move on any curve, this effect by the crosspoint acts as perturbation. Therefore the balance is broken, which is preserved completely in the case of the exact one dromion, and the dromion reduces its volume, which as a consequence flows away as ripples. Once the dromion reaches a quasistable state, it keeps its shape and propagates almost stably following the motion of the mean flow crosspoint. It should be remarked that the dromion keeps its shape even if it is exposed to the condition apart from the exact solution. We performed computations up to five times revolution and observed that the localized structure always kept its dromion-like shape. Even though the mean flows continues to give perturbation on the dromion, the volume decreases only at the beginning and the system reaches a quasistable state afterwards. Paths of the mean flow crosspoint and the peak of dromion are presented in figure 4. We see that the dromion is completely captured at the mean flow crosspoint. As discussed in the preceding papers, this feature also indicates an important role of the mean flows as an attracting force.

It is likely that the volume of the rotating dromion has a relation to the radius of the mean flow motion. To clarify this relation, we performed several computations keeping the velocity of motion at a constant value. That is, we varied the radius r and angular velocity

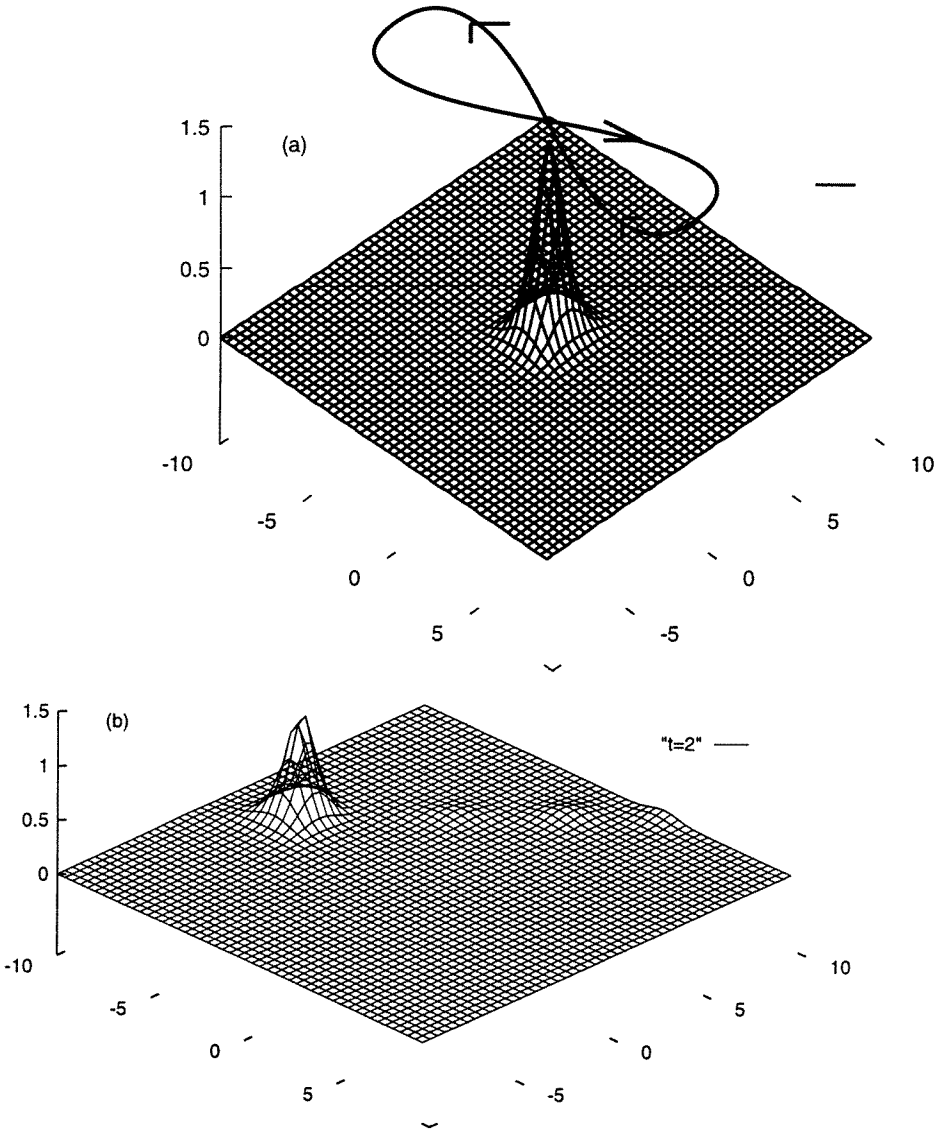

Figure 6. Solid profiles of $|A|^2$ in the case of the Lissajous motion. (a) $t = 0.0$: initial state, (b) $t = 2.0$: $\frac{1}{4}$ period, (c) $t = 4.0$: $\frac{2}{4}$ period, (d) $t = 6.0$: $\frac{3}{4}$ period, and (e) $t = 8.0$: 1 period. Dromion runs on a Lissajous curve keeping its localized structure.

ω while keeping $v = r\omega$ at a constant value. Under this condition, we calculated the decay ratio of the dromion volume after one revolution on a circle path. The relation between the decay ratio and the radius of circle path is shown in figure 5. It can be seen that the ratio of the dromion volume gets closer to the value 1.0 as the radius of the path becomes larger. This feature is reasonable because in the limit $r \to \infty$ the system is described by the exact one-dromion solution. In contrast, when the radius becomes smaller, the forced movement of the crosspoint gives a considerable effect on dromion. It emits more ripples and becomes thinner.

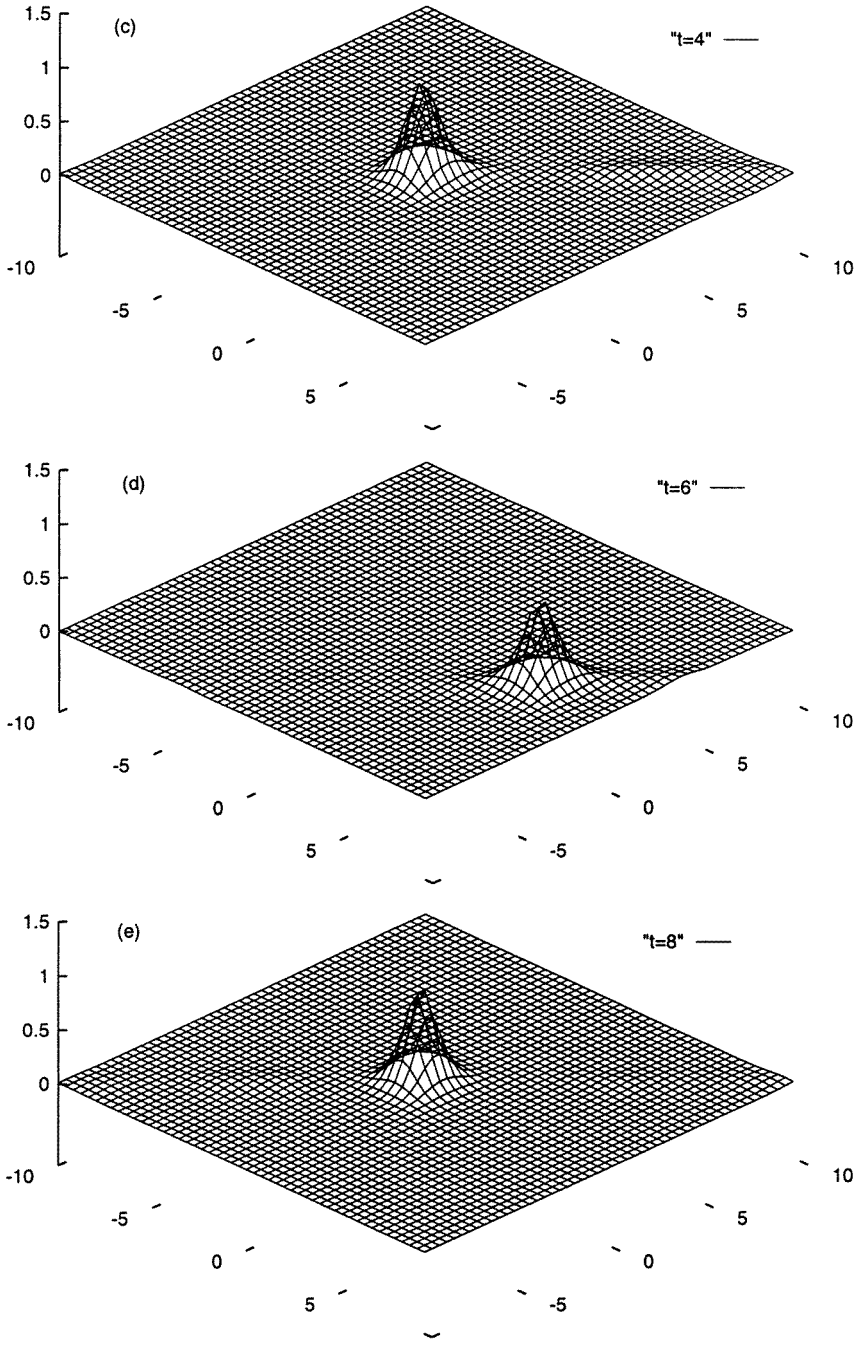

Figure 6. (Continued)

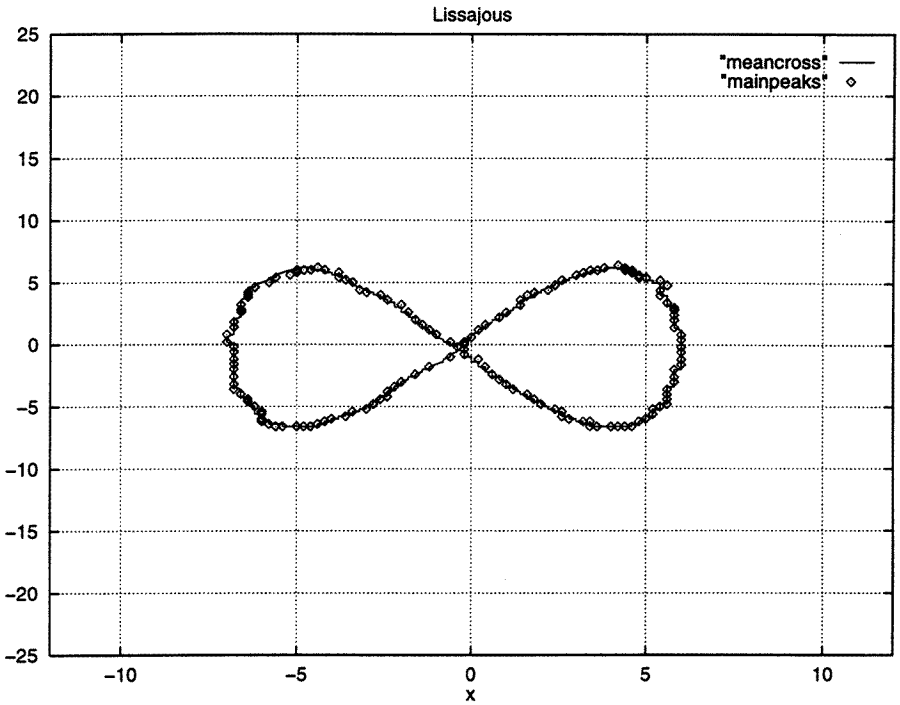

Figure 7. Pathline of the mean flow crosspoint. Dots indicate the position of the main flow peak. Both are on the Lissajous curve. The dromion perfectly follows the motion of the mean flow crosspoint.

In order to see whether the change in the orbit of mean flow crosspoint affects the behaviour of dromion or not, we performed a numerical computation in which the crosspoint moves on a Lissajous curve. This situation is realized by replacing $\cos(wt)$ with $\sin(2wt)$ in equation (16). The curve shape is presented in figure 1(b). From figure 6 and also figure 7, we see that the dromion still follows the motion of the mean flow crosspoint and keeps its shape after emitting a certain amount of ripples.

5. Concluding remarks

We give two concluding remarks.

(i) A conserved form of the DS1 equations is obtained and a new method for numerical analysis is proposed. By comparing the computational results with the exact solution, the accuracy of the new scheme is proved to be an appropriate one.

(ii) Some behaviours of dromions with driving mean flows at the boundaries are investigated by applying the scheme. Through the calculations, the stable propagation of dromions under particular conditions are observed. It is possible to remote-control dromions from the boundaries by forcing the boundary values to move appropriately.

As for the second conclusion, it has been shown in [5] that an electrostatic ion wave, which propagates perpendicularly to fixed magnetic field, is well described by the DS1 equations. Therefore it is probable that the electrostatic potential of the ion wave is localized in two-dimensional space with the shape of a dromion, and that the mean current of the ion plays a role of the mean flow. In conclusion, if it is possible to control dromions from the

boundaries, there is also the possibility of transporting the energy of the ion wave along a curved path to arbitrary places by arranging the boundary conditions. Moreover, we can estimate the remained volume of dromion in the case that the path is a simple circle.

Acknowledgments

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Paper 2

Moment equations based on first order Chapman-Enskog solution -relativistic gas-

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Abstract. A generalized method to derive moment equations correct to first order in the Knudsen number is applied to relativistic gas dynamics. The linearized Boltzmann equation for a flow of two length scales is derived and moment equations are obtained by means of the symmetric Galerkin method. As an Ansatz we assume that the deviation function in the solution for the linearized Boltzmann equation takes the same general form as that of the first order Chapman-Enskog solution. The coefficients appearing in the deviation function are expanded by an infinite series in powers of a dimensionless variable related to the particle energy, rather than orthogonal polynomials. The simplest truncation of the expansion leads to 14 moments equations and discussions on the general form and the entropy production are given in comparison with the conventional 14 moment approximation.

1. Introduction

A great deal of effort has been devoted so far to remedy the well-known difficulty in relativistic gas dynamics [1] [2]: (a) thermal conductivity and viscosity give rise to disturbances which propagate at infinite speed, which contradicts the principle of causality [3]; (b) there exists unstable perturbation of short wavelength in the dissipation processes [4]. Moment equations attract particular attention as they, hyperbolic systems, solve these problems since they lead to finite speeds of propagation for any disturbance.

Let us review non-relativistic flows. In 1963 Cercignani [5] improved the moment method to analyze the kinetic layer in pure shear flow and showed the first order Chapman-Enskog [6] solution, rather than the corresponding Hermite polynomials used by Grad [7], to be suitable for the analysis. Grad's method is based on expansion of the velocity distribution function in terms of Hermite polynomials and the lowest non-trivial truncation gives 13 moment equations, which for slowly varying flows lead to the Navier-Stokes equations with Fourier's expression for the heat current. However they give only the first approximations for viscosity and heat conductivity in terms of Sonine

polynomials. To remedy this deficiency, Söderholm [8] generalized Cercignani's method and derived new 13 moments equations correct to first order in the Knudsen number. The obtained moment equations have the same general form as Grad's, but with correct values of viscosity and heat conductivity. In ref. [8] the corresponding values for the relaxation times and the couplings between heat current and viscous pressure are also calculated.

In this paper we apply a similar method to relativistic gas dynamics. In the case of relativistic flows there is a non-vanishing volume viscosity [9] also for particles with no inner degrees of freedom. The simplest truncation thus leads to 14 rather than 13 moments. The full first order Chapman-Enskog solution is used as a basis to analyze a flow of two length scales and the resulting moment equations are of the same form as those obtained by the 14 moment approximation [9] [10]. It will be shown that our resulting equations give correct values of viscosity and heat conductivity as is in the non-relativistic case and thus gives the proper expression of the entropy production. This paper is organized as follows. We shall give descriptions of the basic elements of relativistic kinetic theory in section 2. The conservation laws and the fluid dynamic equations are given briefly in section 3. After following the normal Chapman-Enskog procedure in section 4, we consider a flow of two length scales and derive the linearized Boltzmann equation in section 5. In section 6, moment equations are derived by means of the variational method, and they are compared with those of conventional 14 moment approximations. The entropy production is explicitly given in section 7. Concluding remarks are given in section 8.

2. Elements of relativistic kinetic theory

We draw an outline of the formal structure of relativistic kinetic theory. Properties of special relativity are taken into account and the relativistic kinetic equation is postulated which gives the rate of change of the distribution function. We consider a simple system consisting of particles in the relativistic regime, having mass m with momenta $\boldsymbol{p}$ and energies cp^0, where p^0 is given by

$$p^0 = \sqrt{\boldsymbol{p}^2 + m^2c^2}. \tag{1}$$

Hereafter c stands for the speed of light.

Following the standard notation of relativity theory, $x = x^\mu = (ct, \boldsymbol{x})$ denotes a time-space point and $p^\mu = (p^0, \boldsymbol{p})$ four-momenta. We introduce a function which gives the four-momenta distribution

$$f(x, p) = f(x^\mu, p^\mu). \tag{2}$$

The relativistic Boltzmann equation which gives the rate of change of the single particle distribution function is

$$p^\mu \partial_\mu f(x,p) = C(f,f), \tag{3}$$

where $C(f,f)$ is the collision integral term representing changes due to particle interactions. By using distribution function, the local density and the particle flow are defined as

$$n(\boldsymbol{x},t) = \int d^3p\, f(x,p), \tag{4}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \int d^3p\, \boldsymbol{u} f(x,p), \tag{5}$$

where

$$\boldsymbol{u} = \frac{c\boldsymbol{p}}{p^0}. \tag{6}$$

Equations (4) and (5) may be combined to the particle four-flow $N^\mu(x) = (cn(\boldsymbol{x},t), \boldsymbol{j}(\boldsymbol{x},t))$

$$N^\mu(x) = c\int \frac{d^3p}{p^0} p^\mu f(x,p). \tag{7}$$

This is a natural way of taking moments of the distribution function in the relativistic kinetic theory, because the factor d^3p/p^0 is a Lorentz scalar.

The energy-momentum tensor is given by

$$T^{\mu\nu} = c\int \frac{d^3p}{p^0} p^\mu p^\nu f(x,p), \tag{8}$$

which is a symmetric tensor. Next, it is desirable to define a vector field which is essentially the hydrodynamic velocity of continuous media. In relativistic gas dynamics it is defined as a time-like four-vector U^μ of length c

$$U^\mu(x) U_\mu(x) = c^2. \tag{9}$$

Throughout this paper we follow the definition made by Eckart [11]. Eckart relates the hydrodynamic velocity to the particle four-flow N^μ. With a normalizing factor, it is expressed as

$$U^\mu = \frac{cN^\mu}{\sqrt{N^\nu N_\nu}}. \tag{10}$$

An important and useful tensor, hereafter called the spatial projector, is defined by the hydrodynamic four-velocity and the metric tensor $g^{\mu\nu}(=\mathrm{diag}(1,-1,-1,-1))$,

$$\Delta^{\mu\nu}(x) = g^{\mu\nu} - \frac{1}{c^2} U^\mu(x) U^\nu(x). \tag{11}$$

The projector annihilates U_μ

$$\Delta^{\mu\nu}(x)U_\mu(x) = 0. \tag{12}$$

This property means that in the rest frame there is no particle current, see (10).

By this projector, the heat flow is defined as the energy current in the rest frame

$$q^\mu = U_\nu T^{\nu\sigma}\Delta^\mu_\sigma, \tag{13}$$

and the pressure tensor as

$$P^{\mu\nu} = \Delta^\mu_\sigma T^{\sigma\tau}\Delta^\nu_\tau = -p\Delta^{\mu\nu} + \Pi^{\mu\nu}. \tag{14}$$

Here we have made a decomposition with respect to the reversible part which is essentially the hydrostatic pressure p, and the irreversible part. The irreversible part $\Pi^{\mu\nu}$ is called the viscous-pressure tensor.

3. Conservation Laws and Fluid Dynamic Equations

Starting from the conservation laws we derive the fluid dynamic equations. The dynamics of a system with physical quantities, which are the particle density, the hydrodynamic four-velocity and the energy density, is determined by these equations.

3.1. Conservation laws

The conservation laws of the total particle number and energy-momentum are given by

$$\partial_\mu N^\mu(x) = 0, \tag{15}$$

$$\partial_\nu T^{\mu\nu}(x) = 0, \tag{16}$$

respectively. The total particle density conservation law (15) gives the rate of change of the number density,

$$Dn = -n\nabla_\mu U^\mu, \tag{17}$$

where the convective time derivative D and the spatial gradient operator ∇_μ are defined as

$$D = U^\mu\partial_\mu, \quad \nabla_\mu = \Delta_{\mu\nu}\partial^\nu. \tag{18}$$

The equation of motion is obtained from the energy-momentum conservation law. Contracted with the projector, it takes the form of

$$\frac{1}{c^2}hnDU^\mu = \nabla^\mu p - \Delta^\mu_\nu\nabla_\sigma\Pi^{\nu\sigma} + (hn)^{-1}\Pi^{\mu\nu}\nabla_\nu p \\ - \frac{1}{c^2}(\Delta^\mu_\nu Dq^\nu + q^\mu\nabla_\nu U^\nu + q^\nu\nabla_\nu U^\mu), \tag{19}$$

where h is the enthalpy per particle. It should be mentioned that all the terms in the right-hand side of (19) are projected, to have their spatial components only. In a similar way, (16) is contracted with U_μ and the equation of energy is obtained.

$$Den = -hn\nabla_\mu U^\mu + \Pi^{\mu\nu}\nabla_\nu U_\mu - \nabla_\mu q^\mu + \frac{2}{c^2}q^\mu DU_\mu, \tag{20}$$

where en is the energy density, with e the average energy per particle.

4. The Chapman-Enskog method

4.1. The first order solution

The Chapman-Enskog method is based on a successive approximation to the distribution function. Suppose that the solution of the relativistic Boltzmann equation is given by

$$f = f^{(0)} + \epsilon f^{(1)} = f^{(0)}(1 + \phi(x, p)), \tag{21}$$

following the assumption that the spatial non-uniformities are so small that they can be measured by a small parameter ϵ. Physically this parameter is interpreted as the Knudsen number defined as the ratio of the mean free path to the characteristic length scale. The zeroth order solution $f^{(0)}$ is the Jüttner function; the relativistic version of the Maxwellian distribution,

$$f^{(0)}(x, p) = \frac{1}{2\pi\bar{h}} \exp\left(\frac{\mu(x) - p^\mu U_\mu(x)}{k_B T(x)}\right). \tag{22}$$

The Planck's constant is written as $\bar{h}$ and the local constant parameter μ can be interpreted as the Gibbs function per particle. The deviation function ϕ is determined by the first order equation in the Chapman-Enskog method which is linear in ϕ

$$\begin{aligned} &c^{-2}\left((\frac{4}{3} - \gamma)(p^\mu U_\mu)^2 + [(\gamma - 1)h - \gamma k_B T]\, p^\mu U_\mu - \frac{1}{3}m^2c^4\right) X \\ &- (p^\mu U_\mu - h)p_\nu X_q^\nu + p^\mu p^\nu \overset{\circ}{X}_{\mu\nu} = k_B T\mathcal{L}[\phi], \end{aligned} \tag{23}$$

where γ is the ratio of heat capacities c_p/c_v and $\mathcal{L}[\phi]$ stands for the linearized collision operator, which is given by

$$\mathcal{L}[\phi] = \frac{1}{2}\int \frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p_1'^0}\frac{d^3p'}{p'^0} f_1^{(0)}(\phi - \phi_1 - \phi' - \phi_1')W. \tag{24}$$

We have expressed the thermodynamic forces as

$$X = -\nabla^\mu U_\mu, \tag{25}$$

$$X_q^\nu = -\frac{T}{h}\nabla^\nu\left(\frac{\mu}{T}\right) \quad (= \nabla^\mu \log T - \frac{k_B T}{h}\nabla^\mu \log p), \tag{26}$$

$$\overset{\circ}{X}^{\mu\nu} = (\Delta_\sigma^\mu\Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau})\nabla^\sigma U^\tau, \tag{27}$$

respectively. The conventional notation $\circ$ over a tensor stands for the traceless part of the tensor. We introduce a cross-section $\sigma(T)$ which is a natural generalization of that in the non-relativistic theory, of dimension of an area, and the deviation function is obtained to be

$$\phi = \frac{1}{cn\sigma(T)}(AX - cB_\mu X_q^\mu + C^{\mu\nu}\overset{\circ}{X}_{\mu\nu}). \tag{28}$$

The coefficient functions, which have been chosen to be dimensionless, can be constructed from the momentum vector p^μ, the metric tensor $g^{\mu\nu}$ and the thermodynamic variables n, T, U^μ. Here we shall make use of two dimensionless parameters

$$z = \frac{mc^2}{k_B T}, \quad \tau = \frac{p^\mu U_\mu}{k_B T}. \tag{29}$$

The parameter z is the inverse of temperature of the system and τ is the energy of a particle in the rest frame. The limit $z \to \infty$ is the the non-relativistic limit, while the limit $z \to 0$ corresponds to the ultra-relativistic case. The coefficient A is a function of τ and z, and independent of the density. (This can be clearly shown by following the procedure of the Chapman-Enskog method, but here we make a brief comment that the dependency on the density has already been extracted as the factor $1/n$ in (28).) Thus we have

$$A = A(\tau, z). \tag{30}$$

The same step is applied to the vector function B_μ and the tensor function $C^{\mu\nu}$, which are naturally constructed from the symmetrized space-like part of a vector or a tensor relevant to p^μ. They are given by

$$B_\mu = B(\tau, z)\Delta_{\mu\nu}\pi^\nu = B(\tau, z)\pi_{\bar{\mu}} \quad (\pi_\mu = \frac{c}{k_B T}p_\mu), \tag{31}$$

$$C^{\mu\nu} = C(\tau, z)(\Delta^\mu_\sigma\Delta^\nu_\rho - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\rho})\pi^\sigma\pi^\rho = C(\tau, z)\,\overset{\circ}{\overline{\pi^\mu\pi^\nu}}, \tag{32}$$

where the symmetrized space-like and traceless part of a vector and a tensor has been defined respectively in each formula. We shall put $^{-}$ to indicate the space-like part of a vector or a tensor. Substitution of (28) into the linearized equation (23) gives equations which determine the coefficients A, B, and C. The coefficients of each of the thermodynamic forces are equated separately,

$$Q = \frac{c}{nk_B T\sigma(T)}\mathcal{L}[A(\tau, z)], \tag{33}$$

$$(\tau - \hat{h})\pi_{\bar{\mu}} = \frac{c}{nk_B T\sigma(T)}\mathcal{L}[B(\tau, z)\pi_{\bar{\mu}}], \tag{34}$$

$$\overset{\circ}{\overline{\pi^\mu\pi^\nu}} = \frac{c}{nk_B T\sigma(T)}\mathcal{L}[C(\tau, z)\,\overset{\circ}{\overline{\pi^\mu\pi^\nu}}], \tag{35}$$

where we have put

$$Q = (\frac{4}{3} - \gamma)\tau^2 + \left[(\gamma - 1)\hat{h} - \gamma\right]\tau - \frac{1}{3}z^2. \tag{36}$$

4.2. Irreversible flows

Firstly we introduce a dimensionless inner product of two functions $F(p)$ and $G(p)$ as the integral over momentum space

$$\begin{aligned}(F, G) &= \frac{k_B T}{cn}\int \frac{d^3p}{p^0} F(p)G(p)f^{(0)} \\ &= \frac{1}{4\pi z^2 K_2(z)}(\frac{c}{k_B T})^2 \int \frac{d^3p}{p^0} F(p)G(p)\, e^{-\tau},\end{aligned} \tag{37}$$

where $K_2(z)$ is the modified Bessel function of the second kind, see Appendix. The first order approximation (21) is inserted with the deviation function (28) into the definitions of the irreversible flows and we obtain, respectively,

$$\Pi \quad = -\eta_v \nabla^\mu U_\mu \ (= \eta_v X), \tag{38}$$

$$q^\nu \quad = \lambda\left(-\frac{T^2}{h}\nabla^\nu(\frac{\mu}{T})\right) \ (= \lambda T X_q^\nu), \tag{39}$$

$$\overset{\circ}{\Pi}{}^{\mu\nu} = 2\eta\ \overline{\overset{\circ}{\nabla^\mu U^\nu}} \ \ (= 2\eta\ \overset{\circ}{X}{}^{\mu\nu}), \tag{40}$$

where the trace product of $\Pi^{\mu\nu}$ is extracted as $\Pi = -\frac{1}{3}\Pi^\mu_\mu$ †. The last expression in each equation helps us to see correspondence to the non-relativistic case. The volume viscosity and the shear viscosity are defined formally in the same manner as in the non-relativistic case. For the heat conductivity, equation (39) gives relativistic Fourier's law. The viscosity and the heat conductivity here are given by,

$$\eta_v = -\frac{1}{3}\frac{k_B T}{c\sigma(T)}(\overline{\pi^\mu \pi_\mu}, A) = \frac{k_B T}{c\sigma(T)}(Q, A) \tag{41}$$

$$\lambda = -\frac{1}{3}\frac{ck_B}{\sigma(T)}(\pi^{\bar{\mu}}(\tau - \hat{h}), B\pi_{\bar{\mu}}) = -\frac{1}{3}\frac{ck_B}{\sigma(T)}((\tau - \hat{h}), (z^2 - \tau^2)B), \tag{42}$$

$$\eta = \frac{1}{10}\frac{k_B T}{c\sigma(T)}(\overline{\overset{\circ}{\pi^\mu \pi^\nu}}, C\ \overline{\overset{\circ}{\pi_\mu \pi_\nu}}) = \frac{1}{15}\frac{k_B T}{c\sigma(T)}((z^2 - \tau^2), (z^2 - \tau^2)C). \tag{43}$$

Let us now suppose that the exact solution $A(\tau, z)$ may be expressible as an infinite series in powers of the variable τ, that is, we suppose that we can approximate $A(\tau, z)$

† If we adopt the metric signature $(-1, +1, +1, +1)$, an extra minus sign in the expression for the heat flow is introduced, so that it looks more familiar.

to any desired degree of accuracy by a polynomial of the form

$$A^{(p)}(\tau,z) = \sum_{s=0}^{p} A^{(p)s}(z)\tau^s$$
$$= A^{(p)0}(z) + A^{(p)1}(z)\tau^1 + A^{(p)2}(z)\tau^2 + ... + A^{(p)p}(z)\tau^p. \quad (44)$$

In the non-relativistic Chapman-Enskog procedure coefficient functions are expanded by Sonine polynomials and their orthogonality property brings about a great simplification in the accurate evaluation of transport properties. In the relativistic kinetic theory, however, the complicated form of the Jüttner distribution function does not lead us to such a simplification. Several authors [12][13] tried to construct orthogonal polynomials to expand the distribution function itself or for the calculation of transport coefficients, but the complexity of such polynomials reduces the advantage of using their special properties. Following the method of de Groot *et al*, we simply adopt the expansion in powers of τ. The same assumption is made for both $B(\tau,z)$ and $C(\tau,z)$ and we have

$$B^{(p)}(\tau,z) = \sum_{s=0}^{p} B^{(p)s}(z)\tau^s, \quad C^{(p)}(\tau,z) = \sum_{s=0}^{p} C^{(p)s}(z)\tau^s. \quad (45)$$

5. Flow of two length scales

The first order Chapman-Enskog solution gives a good approximation to a flow of length scale much larger than the mean free path, but it is not satisfactory for flows which have perturbations of a length scale of the mean free path. For a better analysis of those flows, we make a weaker assumption than the one adopted in the first order Chapman-Enskog procedure.

5.1. The linearized Boltzmann equation

We still assume that the zeroth order flow has a large length scale, but allow the first order flow to have a length scale of the order of the mean free path. Accordingly we put

$$f = f^{(0)}(\epsilon x, p) + \epsilon f^{(1)}(x,p). \quad (46)$$

Inserting (46) into the decomposed (with respect to the convective time derivative and the gradient operator) relativistic Boltzmann equation and equating the order of each terms, we have

$$c^{-2}p^\mu U_\mu D^{(1)} f^{(0)}(\epsilon x,p) + c^{-2}p^\mu U_\mu D^{(0)} f^{(1)}(x,p) + f^{(0)}\mathcal{L}[\phi]$$
$$= -p^\mu \nabla_\mu f^{(0)}(\epsilon x,p) - p^\mu \nabla_\mu f^{(1)}(x,p). \quad (47)$$

On account of the different length scales of $f^{(0)}$ and $f^{(1)}$, we find

$$c^{-2}p^\mu U_\mu D^{(0)} f^{(1)}(x,p) + p^\mu \nabla_\mu f^{(1)}(x,p) = f^{(0)}(c^{-2}p^\mu U_\mu D^{(0)}\phi + p^\mu \nabla_\mu \phi). \quad (48)$$

The notation $D^{(n)}$ indicates the approximation of the time derivative in each order in the Chapman-Enskog method. It follows the assumption that the time and space dependence of the distribution function is given by its dependence on the independent variables n, U^μ, T and their spatial gradients.

Equation (48) is inserted into the first order equation (47), to give

$$\frac{1}{f^{(0)}}[c^{-2}p^\mu U_\mu D^{(1)}f^{(0)}(\epsilon x,p)+p^\mu\nabla_\mu f^{(0)}(\epsilon x,p)]+\mathcal{L}[\phi]$$
$$=-c^{-2}p^\mu U_\mu D^{(0)}\phi-p^\mu\nabla_\mu\phi. \tag{49}$$

5.2. Ordering

Now that we have derived the linearized equation, a remaining work is to neglect terms in both $D^{(1)}f^{(0)}(\epsilon x,p)$ and $\nabla_\mu f^{(0)}(\epsilon x,p)$ except ones of order ϵ^1. We shall make use of the balance equations (17), (19), and (20) with the formula

$$D^{(1)}f^{(0)}=\frac{\partial f^{(0)}}{\partial n}D^{(1)}n+\frac{\partial f^{(0)}}{\partial T}D^{(1)}T+\frac{\partial f^{(0)}}{\partial U^\mu}D^{(1)}U^\mu. \tag{50}$$

Terms of order ϵ^1 in balance equations are of our interest. Let us recall our basic assumption; macroscopic variables such as p and U^μ must have the form of $p(\epsilon x), U^\mu(\epsilon x)$. Thus it follows that the gradient of these macroscopic variables are of order ϵ^1. To begin with we consider the equation of continuity, which has been already obtained as

$$Dn=-n\nabla_\mu U^\mu. \tag{51}$$

The term in the right-hand side is of order ϵ^1, so it is to be included. The next one is more complicated. In the equation of motion

$$\frac{1}{c^2}hnDU^\mu=\nabla^\mu p-\Delta^\mu_\nu\nabla_\sigma\Pi^{\nu\sigma}+(hn)^{-1}\Pi^{\mu\nu}\nabla_\nu p$$
$$-\frac{1}{c^2}(\Delta^\mu_\nu Dq^\nu+q^\mu\nabla_\nu U^\nu+q^\nu\nabla_\nu U^\mu), \tag{52}$$

since the order of the irreversible flows are ϵ^1, we find that the third term in the right-hand side is of order ϵ^2 and so are the fifth and sixth terms, therefore they should be omitted. The same estimation is applied to the equation of energy (20). Finally we obtain the extra terms in addition to the so-called Euler terms. For the normal Chapman-Enskog method with one length scale, the derivative is given by

$$\frac{1}{f^{(0)}}(D^1f^{(0)})_E=-\frac{n}{k_BT}\frac{\partial\mu}{\partial n}\nabla_\mu U^\mu$$
$$-\frac{1}{c_v}\left[T\frac{\partial}{\partial T}(\frac{\mu}{T})+\frac{p^\nu U_\nu}{T}\right]\nabla_\mu U^\mu$$
$$-\frac{1}{k_BT}\frac{c^2}{hn}p_\mu\nabla^\mu p, \tag{53}$$

$$\tag{54}$$

but for our case, as the length scale of the irreversible flows is the order of the mean free path, there are also the following terms

$$\frac{1}{f^{(0)}}(D^{(1)}f^{(0)})_r = -\frac{1}{pc_v}\left[T\frac{\partial}{\partial T}(\frac{\mu}{T}) + \frac{p^\nu U_\nu}{T}\right]\nabla_\mu q^\mu + \frac{1}{k_B T}\frac{c^2}{hn}p_\mu\left(\Delta^\mu_\nu \nabla_\sigma \Pi^{\nu\sigma} + \frac{1}{c^2}\Delta^\mu_\nu D q^\nu\right). \tag{55}$$

We have split terms into the Euler terms (denoted E) and the extra terms (denoted r) for further discussions.

6. 14 moment equations

The present section deals with the derivation of moment equations. The same Ansatz as used by Söderholm [8] is formulated for the solution of the linearized relativistic Boltzmann equation. Then the symmetric Galerkin method is directly applied to the linearized equation, which consequently gives moment equations. In the last part we shall compare our resulting equations with those of 14 moment approximation by Israel [1], Marle [10], and de Groot *et al* [9].

6.1. Solution of the linearized equation

The first order Chapman-Enskog solution (28) may formally be written as

$$\phi = \frac{1}{nk_B T}\left(\hat{A}\Pi + \frac{3}{c}\hat{B}_\mu q^\mu + 5\hat{C}^{\mu\nu}\overset{\circ}{\Pi}_{\mu\nu}\right), \tag{56}$$

where $\hat{A}, \hat{B}_\mu, \hat{C}^{\mu\nu}$ are normalized coefficient functions. They are defined as

$$\hat{A} = \frac{1}{(Q,A)}A, \tag{57}$$

$$\hat{B}_\mu = \frac{1}{((\tau-\hat{h}),(z^2-\tau^2)B)}B_\mu, \tag{58}$$

$$\hat{C}^{\mu\nu} = \frac{1}{((z^2-\tau^2),(z^2-\tau^2)C)}C^{\mu\nu}, \tag{59}$$

so that $\Pi, q^\mu, \overset{\circ}{\Pi}_{\mu\nu}$ have their usual meaning as the heat flow and the viscous-pressure.

Here we propose a new Ansatz to solve the linearized Boltzmann equation (49). We use (56) but allow Π, q^μ, and $\overset{\circ}{\Pi}^{\mu\nu}$ to be functions determined by the equation. One should notice that if we replace $\hat{A}, \hat{B}_\mu, \hat{C}^{\mu\nu}$ with the polynomials which are truncated after the first non-vanishing contribution to the irreversible flow,

$$\begin{aligned}
\hat{A} &\to (A^0(z) + A^1(z)\tau + A^2(z)\tau^2)/(\alpha^2 A^2(z)),\\
\hat{B}_\mu &\to (B^0(z) + B^1(z)\tau)\pi_{\bar{\mu}}/(-\beta^1 B^1(z)),\\
\hat{C}^{\mu\nu} &\to C^0(z)\,\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\,/(\gamma^0 C^0(z)),
\end{aligned} \tag{60}$$

this corresponds to the 14 moments approximation obtained in earlier works [1] [9] [10]. ($\alpha^2, \beta^1, \gamma^0$ are regarded as mere coefficients here, but will be defined in Appendix.) On the other hand, if Π, q^μ, and $\overset{\circ}{\Pi}{}^{\mu\nu}$ take the particular forms given by (38)-(40), our Ansatz reduces to the first order Chapman-Enskog solution.

6.2. Variational method

Using the equations (33)-(35) satisfied by A, B, C, we can express the left-hand side of (49) for our Ansatz as, with the result of (55),

$$\begin{aligned}
&\frac{1}{f^{(0)}}\left(c^{-2}p^\mu U_\mu D^{(1)} f^{(0)}(\epsilon x, p) + p^\mu \nabla_\mu f^{(0)}(\epsilon x, p)\right) + \mathcal{L}[\phi] \\
&= -\frac{k_B T}{c^2}\left[Q(X - \frac{1}{\eta_v}\Pi) - c(X_q^\mu - \frac{1}{\lambda T} q^\mu)(\tau - \hat{h})\pi_{\bar{\mu}} + (\overset{\circ}{X}{}_{\mu\nu} - \frac{1}{2\eta}\overset{\circ}{\Pi}{}_{\mu\nu})\,\overset{\circ}{\pi^\mu \pi^\nu}\right] \\
&+ \theta \nabla_\mu q^\mu + \psi \pi_{\bar{\nu}}(\nabla_\sigma \Pi^{\nu\sigma} + \frac{1}{c^2} D q^\nu),
\end{aligned} \tag{61}$$

where

$$\theta = \frac{1}{p c_v}\frac{p^\mu U_\mu}{c^2}\left[T\frac{\partial}{\partial T}(\frac{\mu}{T}) + \frac{p^\nu U_\nu}{T}\right], \tag{62}$$

$$\psi = \frac{c}{hn}\frac{p^\mu U_\mu}{c^2} = \frac{\tau}{nc\hat{h}}. \tag{63}$$

The remaining elements are $c^{-2}p^\mu U_\mu D^{(0)}\phi$ and $p^\mu \nabla_\mu \phi$ in the right-hand side of (49). Since only the derivatives of Π, q^μ, and $\overset{\circ}{\Pi}{}^{\mu\nu}$ contribute, we obtain

$$\begin{aligned}
&c^{-2}p^\rho U_\rho D^{(0)}\phi + p^\sigma \nabla_\sigma \phi = \frac{c^{-2}p^\rho U_\rho}{n k_B T}(\hat{A} D\Pi + \frac{3}{c}\hat{B}_\mu D q^\mu + 5\hat{C}^{\mu\nu} D\,\overset{\circ}{\Pi}{}_{\mu\nu}) \\
&+ \frac{p^\sigma}{n k_B T}(\hat{A}\nabla_\sigma \Pi + \frac{3}{c}\hat{B}_\mu \nabla_\sigma q^\mu + 5\hat{C}^{\mu\nu}\nabla_\sigma\,\overset{\circ}{\Pi}{}_{\mu\nu}).
\end{aligned} \tag{64}$$

As a consequence, the linearized Boltzmann equation with source terms (49) can be expressed as the sum of the terms in (61) and (64). Following the normal procedure of the symmetric Galerkin method, we demand that the inner product of the sum of terms in (61) with our basis equals that of the sum of terms in (64).

6.3. Moment equations

6.3.1. Viscous-Pressure: Trace The inner product with A gives

$$\begin{aligned}
&-\frac{k_B T}{c^2}(X - \frac{1}{\eta_v}\Pi)(Q, A) - \nabla_\mu q^\mu(\theta, A) \\
&+ \frac{1}{nc^2} D\Pi(\tau\hat{A}, A) + \frac{3}{nc^2}\nabla_\sigma q^\mu(\pi^\sigma \hat{B}_\mu, A) = 0.
\end{aligned} \tag{65}$$

From this equation, the relaxation equation for Π is obtained

$$\frac{1}{nk_BT}(\varphi_p D\Pi + \varphi_i \nabla_\mu q^\mu) + \frac{1}{\eta_v}\Pi + \nabla_\mu U^\mu = 0, \tag{66}$$

where coefficients φ are defined in the following manner.

$$(\tau\hat{A}, A) = \varphi_p(Q, A). \tag{67}$$

Recalling the definition of $\hat{A}$ in (57), we see that the coefficient φ_p is given by

$$\varphi_p = (\tau\hat{A}, \hat{A}). \tag{68}$$

Similarly, the other coefficient is given by

$$\varphi_i = -3nc^2((z^2 - \tau^2)\hat{B}, \hat{A}). \tag{69}$$

When truncation after the first non-trivial term is done, the coefficient functions A and $\hat{A}$ are explicitly given by

$$A = A^0(z) + A^1(z)\tau + A^2(z)\tau^2, \tag{70}$$

$$\hat{A} = \frac{1}{\alpha^2 A^2}(A^0(z) + A^1(z)\tau + A^2(z)\tau^2). \tag{71}$$

It should be mentioned that when the first non-trivial truncation is done our resulting equation corresponds to the result of 14 moment approximation by de Groot *et al.* However, our result (66) gives the correct value of the volume viscosity in contrast with the conventional 14 moment approximation which gives only the first approximation for thermodynamic coefficients.

6.3.2. Heat Current In the same way, taking the inner product with B_σ leads to the relaxation equation

$$\frac{1}{nk_BT}(\frac{\vartheta_i}{c^2}\widetilde{\mathcal{D}}q^\nu + \vartheta_v \Delta^\nu_\sigma \nabla_\rho \overset{\circ}{\Pi}{}^{\sigma\rho} + \vartheta_p \nabla^\nu \Pi) + \frac{1}{\lambda T}q^\nu + \frac{T}{h}\nabla^\nu(\frac{\mu}{T}) = 0, \tag{72}$$

equivalently,

$$\begin{aligned}&\frac{1}{nk_BT}(\frac{\vartheta_i}{c^2}[Dq^\nu + \frac{q^\mu}{c^2}U^\nu DU_\mu] + \vartheta_v \Delta^\nu_\sigma \nabla_\rho \overset{\circ}{\Pi}{}^{\sigma\rho} + \vartheta_p \nabla^\nu \Pi)\\ &+ \frac{1}{\lambda T}q^\nu + \frac{T}{h}\nabla^\nu(\frac{\mu}{T}) = 0.\end{aligned} \tag{73}$$

Here we have adopted an operator $\widetilde{\mathcal{D}}$ which acts as

$$\widetilde{\mathcal{D}}q^\nu = \Delta^\nu_\lambda q^\lambda{}_{,\kappa} U^\kappa \ \ (= \Delta^\nu_\lambda Dq^\lambda). \tag{74}$$

Our assumption should be respected here again. Let us recall the ordering procedure. The order of the term $q^\mu DU_\mu$ is $\sim O(\epsilon^2)$, thus this term should be excluded from the

relaxation equation above. The same can be applied to the extra term from $\Delta^\nu_\sigma \nabla_\rho \overset{\circ}{\Pi}{}^{\sigma\rho}$. Finally we have the relaxation equation in the same general form as de Groot's,

$$\frac{1}{nk_BT}(\frac{\vartheta_i}{c^2}Dq^\nu + \vartheta_v \nabla_\rho \overset{\circ}{\Pi}{}^{\nu\rho} + \vartheta_p \nabla^\nu \Pi) + \frac{1}{\lambda T}q^\nu + \frac{T}{h}\nabla^\nu(\frac{\mu}{T}) = 0. \tag{75}$$

Nevertheless it is worth while noticing that certainly the form of (72) is preferable since the space-like vector components are consistent in (72), while it is not in (75).

In the equations above, coefficients have been defined as follows:

$$\vartheta_i = \vartheta_{i1} + \vartheta_{i2}, \tag{76}$$

$$\vartheta_p = \vartheta_{p1} - \vartheta_{i2}, \tag{77}$$

$$\vartheta_v = \vartheta_{v1} + \vartheta_{i2}, \tag{78}$$

where

$$\vartheta_{i1} = 3(\tau\hat{B}, (z^2 - \tau^2)\hat{B}), \tag{79}$$

$$\vartheta_{i2} = (\frac{\tau}{\hat{h}}, (z^2 - \tau^2)\hat{B}), \tag{80}$$

$$\vartheta_{p1} = (\hat{A}, (z^2 - \tau^2)\hat{B}), \tag{81}$$

$$\vartheta_{v1} = 5((z^2 - \tau^2)\hat{C}, (z^2 - \tau^2)\hat{B}). \tag{82}$$

Let us notice $z^2 - \tau^2 = \pi^\mu \pi_\mu = (\frac{c}{k_BT})^2 \boldsymbol{p}^2$.

6.3.3. Viscous-Pressure: Traceless part The inner product with $C_{\sigma\tau}$ gives the relaxation equation for the viscous-pressure tensor,

$$\frac{1}{nk_BT}(\varrho_v \widetilde{\mathcal{D}}\, \overset{\circ}{\Pi}{}^{\mu\nu} + \varrho_i \overline{\overset{\circ}{\nabla^\mu q^\nu}}) + \frac{1}{2\eta}\, \overset{\circ}{\Pi}{}^{\mu\nu} - \overline{\overset{\circ}{\nabla^\mu U^\nu}} = 0, \tag{83}$$

where we again adopted the operator

$$\widetilde{\mathcal{D}}\, \overset{\circ}{\Pi}{}^{\mu\nu} = \Delta^\mu_\sigma \Delta^\nu_\rho \overset{\circ}{\Pi}{}^{\sigma\rho}{}_{,\kappa}\, U^\kappa \quad (= \Delta^\mu_\sigma \Delta^\nu_\rho D\, \overset{\circ}{\Pi}{}^{\sigma\rho}). \tag{84}$$

Coefficients are given by

$$\varrho_v = 5(\tau(z^2 - \tau^2)\hat{C}, (z^2 - \tau^2)\hat{C}), \tag{85}$$

$$\varrho_i = 3((z^2 - \tau^2)\hat{B}, (z^2 - \tau^2)\hat{C}). \tag{86}$$

6.4. Comparison with 14 moments approximation

In ref. [9] de Groot *et al* derived relaxation equations for the viscous-pressure tensor and the heat flow by truncating polynomial expansions after the first non-trivial terms. Since they give the rate of change of 14 moments of the Jüttner distribution function,

they are called 14 moments approximation. They are written, with Eckart's definition for the hydrodynamic four-velocity,

$$\frac{1}{[\eta_v]_1}\Pi + \frac{1}{nk_BT}(\alpha' D\Pi - \beta'\nabla_\mu q^\mu) + \nabla_\mu U^\mu = 0, \tag{87}$$

$$\frac{1}{T[\lambda]_1}q^\nu + \frac{1}{nk_BT}(\frac{1}{c^2}\beta'' Dq^\nu + \alpha''\nabla^\nu\Pi + \gamma''\nabla_\rho \overset{\circ}{\Pi}{}^{\nu\rho}) + \frac{T}{h}\nabla^\nu(\frac{\mu}{T}) = 0. \tag{88}$$

$$\frac{1}{2[\eta]_1}\overset{\circ}{\Pi}{}^{\mu\nu} + \frac{1}{nk_BT}\left(\gamma''' D\overset{\circ}{\Pi}{}^{\mu\nu} - \beta'''\overset{\circ}{\nabla^\mu q^\nu}\right) - \overset{\circ}{\nabla^\mu U^\nu} = 0, \tag{89}$$

Coefficients α, β, γ are expressed in terms of moments of the Jüttner distribution function with the help of the modified Bessel function of the second kind. Explicit forms are written in the same reference [9]. As is shown by laborious manipulations, if we truncate the expansion after the first non-trivial term, our resulting equation precisely correspond to equations (87)-(89) in general forms, i.e. $\varphi_p = \alpha'$, $\varphi_i = -\beta'$. However, it should be emphasized that in equations (66), (72), and (83) the viscosity and the heat conductivity always take the correct values, while in the 14 moment approximation they are only in the lowest order approximation, as is made explicit by the notation $[\ ]_1$. Thus we may say that it is possible to obtain from our results moment equations approximate to any desired degree of accuracy with correct values for thermodynamic coefficients.

6.5. Comparison with the non-relativistic case

Now we shall show the correspondence of our results in the non-relativistic limit to the equations in the previous work by Söderholm. In ref. [8] new 13 moments equations are derived:

$$\gamma_q\frac{2m}{5k_Bp}\frac{Dq_i}{Dt} + \delta\frac{2T}{5p}\Pi_{ij,j} + \frac{1}{\lambda}q_i + T_{,i} = 0, \tag{90}$$

$$\gamma_\Pi\frac{1}{p}\frac{D\Pi_{ij}}{Dt} + \delta\frac{4}{5p}\overset{\circ}{q_{i,j}} + \frac{1}{\eta}\Pi_{ij} + 2\overset{\circ}{u_{i,j}} = 0, \tag{91}$$

with coupling coefficients $\gamma_q, \gamma_\Pi, \delta$ given by the inner product of the coefficient functions which appear in the deviation function. The notation $, j$ stands for the spatial gradient $\frac{\partial}{\partial x_j}$.

Let us return to our result. In the non-relativistic limit volume viscosity disappears and we may consider the relaxation equations (72) and (83), from which the volume viscous term is excluded.

$$\frac{1}{nk_BT}(\frac{\vartheta_i}{c^2}\tilde{\mathcal{D}}q^\nu + \vartheta_v\Delta^\nu_\sigma\nabla_\rho\overset{\circ}{\Pi}{}^{\sigma\rho}) + \frac{1}{\lambda T}q^\nu + \frac{T}{h}\nabla^\nu(\frac{\mu}{T}) = 0, \tag{92}$$

$$\frac{1}{nk_BT}(\varrho_v\widetilde{\mathcal{D}}\overset{\circ}{\Pi}^{\mu\nu}+\varrho_i\overline{\overset{\circ}{\nabla^\mu q^\nu}})+\frac{1}{2\eta}\overset{\circ}{\Pi}^{\mu\nu}-\overline{\overset{\circ}{\nabla^\mu U^\nu}}=0. \tag{93}$$

We show that these equations reduce to equations (90) and (91) in the following way. Firstly we see that the coefficient ϑ_{i2} defined in (80) vanishes, by a simple estimation in the non-relativistic limit, τ approaches to mc^2/k_BT and $z^2-\tau^2$ becomes cp^2/k_BT which is essentially the square of the particle velocity. That is, we have

$$(\frac{mc^2}{k_BT},\hat{B}\frac{cp^2}{k_BT})\sim(u,\hat{B}u)=0, \tag{94}$$

where we have used the orthogonal property of the basis function with the velocity u. Thus we have, for coupling coefficients in (92), pure contributions from ϑ_{i1} and ϑ_{v1}. One can easily find that in the non-relativistic limit these coefficients coincide with the values of γ_q and δ in (90) respectively. (See also [8].) Similarly, the coefficients ϱ_v and ϱ_i become γ_Π and δ in the non-relativistic limit. Further it is worth noticing that in the rest frame

$$\tilde{\mathcal{D}}=\frac{\partial}{\partial t},\quad \nabla_\mu=(0,\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}), \tag{95}$$

which are the usual time derivative and the spatial gradient. Consequently we may say that in the non-relativistic limit our resulting equations (92) and (93) precisely reduce to the equations derived by Söderholm.

7. Entropy production

In this last section we derive the expression for the entropy production from our Ansatz and the equations obtained so far. In relativistic kinetic theory the entropy production is given by (see de Groot *et al* [9])

$$\sigma(x)=-k_Bc\int\frac{d^3p}{p^0}[\log(2\pi\bar{h})^3f(x,p)]p^\mu\partial_\mu f(x,p). \tag{96}$$

By substituting $f=f^{(0)}(1+\phi)$, we have, up to the second order in ϕ,

$$\sigma=-k_Bc\int\frac{d^3p}{p^0}[p^\mu\partial_\mu\log(2\pi\bar{h})^3f^{(0)}]f^{(0)}\phi-\frac{1}{2}k_Bc\,\partial_\mu\int\frac{d^3p}{p^0}p^\mu f^{(0)}\phi^2. \tag{97}$$

Hereafter we write σ_1 for the first term in the right-hand side and σ_2 for the second term. Since we made use of the viscous-pressure and the heat flow themselves as functions of which the deviation function consists, calculations are straight forward.

$$\begin{aligned}T\sigma_1 =& -k_BTc\int\frac{d^3p}{p^0}\frac{k_BT}{c}[\pi^\mu\partial_\mu\log(2\pi\bar{h})^3f^{(0)}]f^{(0)}\phi\\ =& -(k_BT)^2\int\frac{d^3p}{p^0}\{(\tau-\hat{h})\pi^\mu\frac{\nabla_\mu T}{T}+\pi^\mu\frac{\nabla_\mu p}{p}-\frac{1}{c}\pi^\mu\pi^\nu\overline{\overset{\circ}{\nabla_\mu U_\nu}}\end{aligned}$$

$$+\frac{1}{3c}(\tau^2-z^2)\nabla^\mu U_\mu+\frac{\tau}{c}[(\tau-\hat{h}+1)\frac{DT}{T}+\frac{Dn}{n}-\frac{1}{c}\pi^\mu DU_\mu]\}f^{(0)}\phi, \tag{98}$$

and

$$\begin{aligned} T\sigma_2 = & -\frac{1}{2}k_BTc\int\frac{d^3p}{p^0}\frac{k_BT}{c}(c^{-1}\tau D+\pi^{\bar{\mu}}\nabla_\mu)f^{(0)}\phi^2 \\ = & -\frac{1}{2n^2}\int\frac{d^3p}{p^0}(c^{-1}\tau D+\pi^{\bar{\mu}}\nabla_\mu)f^{(0)} \\ & \times(\hat{A}\Pi+\frac{3}{c}\hat{B}_\mu q^\mu+5\hat{C}^{\mu\nu}\overset{\circ}{\Pi}_{\mu\nu})(\hat{A}\Pi+\frac{3}{c}\hat{B}^\mu q_\mu+5\hat{C}_{\mu\nu}\overset{\circ}{\Pi}^{\mu\nu}). \end{aligned} \tag{99}$$

Terms associated with the trace of the viscous-pressure tensor are summarized and expressed as

$$\begin{aligned} T\sigma|_\Pi &= (Q,\hat{A})[-\nabla^\mu U_\mu-\frac{1}{nk_BT}(\varphi_pD\Pi+\varphi_i\nabla_\mu q^\mu)]\Pi \\ &= (Q,\hat{A})\frac{\Pi^2}{\eta_v}=\frac{\Pi^2}{\eta_v}. \end{aligned} \tag{100}$$

The last line is obtained with the help of the relaxation equation (66). The inner product $(Q,\hat{A})$ takes the unit value obviously from the definition of $\hat{A}$.
In the same way the other terms are calculated and the entropy production is expressed in the positive definite form,

$$T\sigma=\frac{\Pi^2}{\eta_v}-\frac{q^\mu q_\mu}{\lambda T}+\frac{\overset{\circ}{\Pi}^{\mu\nu}\overset{\circ}{\Pi}_{\mu\nu}}{2\eta}\geq 0. \tag{101}$$

8. Concluding Remarks

We summarize two concluding remarks:

(i) The linearized Boltzmann equation for a relativistic gas flow of two length scales is derived. By applying the symmetric Galerkin method to the linearized equation, a set of moment equations based on the full first order Chapman-Enskog solution are obtained. The resulting moment equations are correct in first order in the Knudsen number.

(ii) The obtained moment equations are compared with the result of the conventional 14 moment approximation. It has been shown that both take the same general form but our resulting moment equations are well formed in view of consistency of space-time decomposition. Moreover they give correct values for the viscosity and heat conductivity, which consequently lead to the correct estimation for the entropy production.

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Appendix A. Moments of the Jüttner function

In order to calculate coefficients $\varphi, \vartheta, \varrho$, we need to calculate the moment integrals of the Jüttner distribution function

$$(1, \pi^{(n)}) = \frac{1}{4\pi z^2 K_2(z)} \left(\frac{c}{k_B T}\right)^{n+2} \int \frac{d^3p}{p^0}\, p^{(n)} \exp(\frac{-p_\mu U^\mu}{k_B T}), \tag{A1}$$

where $K_2(z)$ is the modified Bessel function of the second kind, and the symbol (n) denotes the set of symmetrized tensor indices $\mu_1\mu_2...\mu_n$.
Here we shall show only elements of our interest.

$$a^n = (1, \tau^n) = \frac{1}{z^2 K_2(z)} \int_z^\infty d\tau\, \sqrt{\tau^2 - z^2}\tau^n e^{-\tau}, \tag{A2}$$

$$b^n = -(\pi^{\bar{\mu}}, \tau^n \pi_{\bar{\mu}}) = \frac{-1}{z^2 K_2(z)} \int_z^\infty d\tau\, (\tau^2 - z^2)^{3/2}\tau^n e^{-\tau}, \tag{A3}$$

$$c^n = \frac{3}{2}(\overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \tau^n \overset{\circ}{\pi_\mu \pi_\nu}) = \frac{1}{z^2 K_2(z)} \int_z^\infty d\tau\, (\tau^2 - z^2)^{5/2}\tau^n e^{-\tau}. \tag{A4}$$

The modified Bessel functions satisfy the recurrence relation [14]

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z), \tag{A5}$$

where K_n is defined in the integral form as

$$K_n(z) = \frac{2^{n-1}(n-1)!}{(2n-2)!} \frac{1}{z^n} \int_z^\infty d\tau\, (\tau^2 - z^2)^{n-3/2}\tau e^{-\tau}. \tag{A6}$$

Use is made of this recurrence relation and all a^n, b^n, c^n can be obtained as follows:

$$a^{n+1} = (n+2)a^n + z^2 a^{n-1} - (n-1)z^2 a^{n-2}, \tag{A7}$$

$$b^n = a^{n+2} + z^2 a^n, \tag{A8}$$

$$c^n = a^{n+4} - 2z^2 a^{n+2} + z^4 a^n. \tag{A9}$$

We need only the first five values of a^n below.

$$a^0 = \frac{1}{z^2}(\hat{h} - 4), \tag{A10}$$

$$a^1 = 1, \tag{A11}$$

$$a^2 = \hat{h} - 1, \tag{A12}$$

$$a^3 = 3\hat{h} + z^2, \tag{A13}$$

$$a^4 = (15 + z^2)\hat{h} + 2z^2, \tag{A14}$$

where the reduced enthalpy is given by

$$\hat{h} = \frac{h}{k_B T} = z\frac{K_3(z)}{K_2(z)}. \tag{A15}$$

Now we shall define another group of coefficients. To obtain the coefficient functions in equations (44)-(45), the variational method is applied and then we need to solve a set of linear algebraic equations, see [9] for detailed procedures. We encounter the following coefficients:

$$\alpha^r = (\tau^r, Q) = (\frac{4}{3} - \gamma)a^{r+2} + [(\gamma - 1)\hat{h} - \gamma]a^{r+1} - \frac{1}{3}z^2 a^r, \tag{A16}$$

$$\beta^r = -(\tau^r \pi^{\bar{\mu}}, (\tau - \hat{h})\pi_{\bar{\mu}}) = b^{r+1} - \hat{h}b^r, \tag{A17}$$

$$\gamma^r = (\tau^r \overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \overset{\circ}{\overline{\pi_\mu \pi_\nu}}) = \frac{2}{3}c^r. \tag{A18}$$

These coefficients are to be used when calculating the viscosity and the heat conductivity.

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