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Theoretical studies of shock waves in dispersive and dissipative media

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Abstract

Propagation of waves in nonlinear, dispersive and dissipative media, as described by Korteweg-de Vries-Burgers' equation (KdVB), has been studied. The focus of the investigation has been to study analytically, the structure of a shock wave that is broken down by dispersive and dissipative phenomena. To be able to use the inverse scattering transform (IST) to get analytical solutions for Korteweg-de Vries' equation (KdV), an N-wave was used as model for the initial shock. The IST is used to transform KdV, which is a nonlinear differential equation, into Marchenko's equation that is a linear Volterra integral equation. A zeroth order iteration solution, which reconstructs the initial waveform, is presented. For positive times, this solution shows a decaying shock front which slows down, leaving an oscillating tail behind. This solution is valid for moderate values of the dispersion coefficient. In order to obtain solutions for smaller values of the dispersion coefficient, asymptotic analysis is used. The corresponding asymptotic analysis for Burgers' equation, with a small dissipation coefficient, is quoted for comparison. An asymptotic analysis is also made for KdVB, in the case for which dispersion as well as dissipation is important.

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Chapter 1

Introduction

The concept of waves has been proven to be an adequate description of such fundamentally different physical phenomena as light (electromagnetic waves) and sound (acoustic waves). Propagation of electromagnetic waves in vacuum can be described in a straightforward manner by Maxwell's equations, which in this case can be rewritten as the linear wave equation. This holds also for electromagnetic waves in some fluids, e g air or water, but with different wavespeeds (depending on the media). Acoustic waves do not propagate in vacuum. Acoustic waves in non viscous fluids can, for small amplitudes, also be described by the linear wave equation. In viscous fluids, the waves are attenuated due to the dissipation of energy. For some media also dispersive effects become important, i e the propagation velocity is dependent of the wavelength. The behaviour of the waves is changed even more radically if nonlinear phenomena occurs. This usually means that energy is transferred from lower frequencies to higher within the wave. Nonlinearities make the equations much harder to handle since the different wavelengths cannot longer be treated separately, i e using the superposition principle, but on the other hand this makes everything much more interesting. Dispersion and dissipation are characteristic also for electromagnetic waves in conductive media. Under certain circumstances even nonlinear electromagnetic waves occur in dielectric[28] and magnetic media. We will here study acoustic waves and investigate how they are influenced by dissipation, dispersion and nonlinearities.

As a model for a nonlinear, dispersive and dissipative medium, a fluid composed by a liquid containing small gas bubbles has been studied. The gas bubbles are assumed to be homogeneously distributed in the liquid. It is also assumed that the typical frequencies are well below the lowest resonance frequency of the bubbles. (A description of resonance phenomena can be found in Naugolnykh &

Ostrovsky[27]) Following van Wijngaarden[33], but also considering dissipation, it can be shown that Korteweg-de Vries-Burgers' equation

$$u_t + u\, u_x - \delta\, u_{xx} + \sigma\, u_{xxx} = 0 \tag{1.1}$$

can be used to describe propagation of plane waves in this medium. It is derived from Navier-Stokes' equation, the continuity equation and the Rayleigh equation for the expansion and contraction of a gas bubble in a liquid as formulated by Lamb[22]. In this derivation it is also assumed that a small region, in relation to the wavelength, contains many bubbles and that the total amount of gas is small. Furthermore, the sound velocity is assumed constant and the behaviour of the gas bubbles isothermal. That the last assumption is plausible for most frequencies was shown by Plesset & Hsieh[29]. The liquid is also considered incompressible.

If there is no dispersion, i e $\sigma = 0$, the remaining terms of (1.1) will form Burgers' equation. If instead the dissipation is negligible, i e $\delta = 0$ they will give Korteweg-de Vries' equation. Burgers' equation is analytically solvable through the Cole-Hopf transformation. Some results from investigations of Burgers' equation will be quoted. A similar attempt to solve Korteweg-de Vries' equation leads to the inverse scattering transform. It is also here possible to obtain analytical solutions, but only for a few special cases.

In the limit that the dissipation goes to zero for Burgers' equation, it can be shown that N-waves are formed after long times independently of the initial waveforms.(See Gurbatov et al[13]) Here we will give an analytical solution of Korteweg-de Vries' equation with an N-wave as initial waveform, by using the inverse scattering transform. This solution is valid for moderate values of the dispersion coefficient σ. As a complement, asymptotic analysis for small values of σ are made for the shock region and for the shock tail region. For comparison the corresponding asymptotic analysis for Burgers' equation, by Crighton&Scott[6] for the shock region and by Enflo[8] for the shock tail region, are quoted. Performing an asymptotic analysis for Korteweg-de Vries-Burgers' equation usually leads to either the asymptotic analysis of Korteweg-de Vries' equation or the analysis of Burgers' equation. We will here also study the special case for which the values of the parameters (δ, σ) of KdVB give dissipative and dispersive effects of the same order of magnitude.

Korteweg-de Vries-Burgers' equation gives a fairly general description for nonlinear, dispersive and dissipative wave propagation. Some further applications are electromagnetic waves in ion plasma[11] and density waves in traffic flow[26]. In both these cases also shock waves occur.

Chapter 2

Derivation of Korteweg-de Vries-Burgers' Equation

(In this chapter, the only essential deviation from [33] is that the viscosity has been included in (2.1))
As starting point we will use Navier-Stokes' equation in one spatial dimension. This is as usual accompanied by the continuity equation, but instead of an ordinary state equation, the Rayleigh equation for the dynamics of a gas bubble will be used.

$$\begin{aligned} v'_{t'} + v'v'_{x'} + \frac{1}{\rho'}p'_{x'} - \frac{\nu}{\rho'}v'_{x'x'} &= 0 \quad (2.1) \\ \rho'_{t'} + (\rho' v')_{x'} &= 0 \quad (2.2) \\ \rho_l(RR_{t't'} + \frac{3}{2}R^2_{t'}) &= p_g - p_\infty \,. \quad (2.3) \end{aligned}$$

Here R is the bubble radius and l, g are indices for the liquid and the gas respectively. If the relative motion of bubbles and liquid is small p_∞ can be replaced by p', i e the average pressure in the mixture at the location of the gas bubble. Since the mass of a gas bubble is constant the isothermal assumption

$$\frac{p_g}{\rho_g} = \frac{p_0}{\rho_{g0}} \quad (2.4)$$

can be used to write

$$R = R_0 \left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}} , \quad (2.5)$$

where 0 represents the undisturbed state. Using (2.5), the equation (2.3) can be written as

$$p' = p_g - \rho_l R_0^2 \left(\left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}} \frac{d^2}{dt'^2} \left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}} + \frac{3}{2} \left(\frac{d}{dt'} \left(\frac{p_g}{p_0}\right)^{-\frac{1}{3}} \right)^2 \right) . \tag{2.6}$$

2.1 Sound velocity

In order to formulate our problem in terms of dimensionless variables, an expression for the sound velocity is needed. A derivation of this expression follows. That a unit mass of the mixture contains a constant mass of gas can be written as

$$\frac{\rho_g}{1-\beta} = \frac{\rho_{g0}}{1-\beta_0} \tag{2.7}$$

under the assumption that the mass of gas can be neglected compared to the mass of the liquid. Here β is the volume fraction of gas in the fluid. This means that the density of the medium will be

$$\rho' = (1-\beta)\,\rho_l + \beta\rho_g \,. \tag{2.8}$$

With the term $\beta\rho_g$ approximated to be zero in (2.8), the equation (2.7) can be used to obtain

$$\frac{\rho'}{\rho_l} = \frac{\frac{p_g}{p_0}}{\frac{p_g}{p_0} + \frac{\beta_0}{1-\beta_0}} \,. \tag{2.9}$$

Differentiation of (2.9) with respect to p_g gives, by using the formal equation

$$\frac{d}{d\rho'} p_g = \left(\frac{d}{dp_g} \rho' \right)^{-1} , \tag{2.10}$$

the expression

$$\frac{dp_g}{d\rho'} = \frac{p_0}{\rho_l} \frac{1-\beta_0}{\beta_0} \left(\frac{p_g}{p_0} + \frac{\beta_0}{1-\beta_0} \right)^2 . \tag{2.11}$$

For waves of moderate amplitude the ansatz

$$\frac{p_g}{p_0} = 1 + \varepsilon\xi \tag{2.12}$$

is used. Here ε is a small parameter. Substituting (2.12) into (2.11) gives

$$\frac{dp_g}{d\rho'} = \frac{p_0}{\rho_l} \frac{1-\beta_0}{\beta_0} \left(1 + \varepsilon\xi + \frac{\beta_0}{1-\beta_0} \right)^2 . \tag{2.13}$$

If only terms which are up to linear in ε are considered, this can be written as

$$\frac{dp_g}{d\rho'} \approx \frac{p_0}{\rho_l} \frac{1}{\beta_0 (1-\beta_0)} (1 + 2\varepsilon\xi (1-\beta_0)) \, . \tag{2.14}$$

From (2.6) it is obvious that $p' \to p_g$ then $R_0 \to 0$ and thus it is possible to define the sound velocity, in terms of p_g, by

$$c^2 = \frac{dp'}{d\rho'} \approx \frac{dp_g}{d\rho'} \, . \tag{2.15}$$

In the lowest order (2.14) and (2.15) leads to the relation

$$c_0^2 = \frac{p_0}{\rho_l} \frac{1}{\beta_0 (1-\beta_0)} \, , \tag{2.16}$$

where all involved entities are assumed to be constant.

2.2 Dimensionless variables

We are now ready to introduce dimensionless variables by

$$\left\{ \begin{array}{r} \tau = \frac{c_0}{\lambda} t' \\ \chi = \frac{1}{\lambda} x' \\ \varepsilon v = \frac{1}{c_0 \beta_0} v' \end{array} \right. \, , \tag{2.17}$$

where λ is a typical wavelength. Using the chain rule in the form

$$\frac{\partial \rho'}{\partial t'} = \frac{\partial \rho'}{\partial p_g} \frac{\partial p_g}{\partial t'} \tag{2.18}$$

with the equations (2.2), (2.12) and (2.13) gives, if terms which are $O\left(\varepsilon^2\right)$ are neglected, the equation

$$\frac{d\xi}{d\tau} + \varepsilon\beta_0 v \frac{d\xi}{d\chi} + (1 + (2-\beta_0)\,\varepsilon\xi) \frac{dv}{d\chi} = 0 \, . \tag{2.19}$$

It is also possible to combine (2.1), (2.6), (2.9), (2.12) and (2.16) to obtain

$$\begin{aligned} &\varepsilon \left(\frac{dv}{d\tau} + \varepsilon\beta_0 v \frac{dv}{d\chi} - \tilde{\nu} \frac{(1 + (1-\beta_0)\,\varepsilon\xi)}{(1+\varepsilon\xi)} \frac{d^2 v}{d\chi^2} + \frac{(1-\beta_0\varepsilon\xi)}{\beta_0} \frac{d\xi}{d\chi} \right) = \\ &\mu (1 - \varepsilon\beta_0\xi) \left((1+\varepsilon\xi)^{-\frac{1}{3}} \frac{d^2}{dt^2} (1+\varepsilon\xi)^{-\frac{1}{3}} + \frac{3}{2} \left(\frac{d}{dt} (1+\varepsilon\xi)^{-\frac{1}{3}} \right)^2 \right) \, , \end{aligned} \tag{2.20}$$

where we have introduced

$$\left\{ \begin{array}{r} \tilde{\nu} = \frac{c_0}{\lambda p_0} \nu \\ \mu = \frac{R_0^2}{\lambda^2 \beta_0 (1-\beta_0)} \end{array} \right. \, . \tag{2.21}$$

Under the assumption that $\mu = O(\varepsilon)$, if again only terms up to linear in ε are kept, (2.20) will get the form

$$\frac{dv}{d\tau} + \varepsilon\beta_0 v\frac{dv}{d\chi} - \tilde{\nu}\left(1-\beta_0\varepsilon\xi\right)\frac{d^2v}{d\chi^2} + \left(1-\beta_0\varepsilon\xi\right)\frac{d\xi}{d\chi} + \frac{\mu}{3}\frac{d^3\xi}{d\tau^2 d\chi} = 0\,. \tag{2.22}$$

2.3 Reduction of the system

The system of equations is (2.19),(2.22):

$$\left\{\begin{array}{c} \frac{dv}{d\tau} + \varepsilon\beta_0 v\frac{dv}{d\chi} - \tilde{\nu}\left(1-\beta_0\varepsilon\xi\right)\frac{d^2v}{d\chi^2} + \left(1-\beta_0\varepsilon\xi\right)\frac{d\xi}{d\chi} + \frac{\mu}{3}\frac{d^3\xi}{d\tau^2 d\chi} = 0 \\ \frac{d\xi}{d\tau} + \varepsilon\beta_0 v\frac{d\xi}{d\chi} + \left(1+\left(2-\beta_0\right)\varepsilon\xi\right)\frac{dv}{d\chi} = 0 \end{array}\right. . \tag{2.23}$$

This does not seem to be easier to solve than the original system (2.1)-(2.3). For the special case where $\epsilon = 0$, $\mu = 0$ and $\tilde{\nu} = 0$ the system (2.23), however is reduced to

$$\left\{\begin{array}{l} \frac{dv}{d\tau} + \frac{d\xi}{d\chi} = 0 \\ \frac{d\xi}{d\tau} + \frac{dv}{d\chi} = 0 \end{array}\right. , \tag{2.24}$$

with the solution

$$v = \xi = f\left(\chi - \tau\right)\,. \tag{2.25}$$

To examine solutions to (2.23) for small values of ϵ, μ and $\tilde{\nu}$ the assumption

$$v - \xi = O\left(\varepsilon\right) \tag{2.26}$$

is used. From (2.24) and (2.26) follows the expressions

$$\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\chi} = O\left(\varepsilon\right) \tag{2.27}$$

and

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\chi}\right)\left(v - \xi\right) = O\left(\varepsilon^2\right)\,. \tag{2.28}$$

From (2.27) also follows that

$$\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\chi^2}\right) = \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\chi}\right)\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\chi}\right) = O\left(\varepsilon\right)\,. \tag{2.29}$$

Addition of the two equations in (2.23) gives with (2.28)

$$\begin{aligned} 2\left(\frac{d\xi}{d\tau} + \frac{d\xi}{d\chi}\right) + \varepsilon\beta_0 v\left(\frac{dv}{d\chi} + \frac{d\xi}{d\chi}\right) - \tilde{\nu}\left(1-\varepsilon\beta_0\xi\right)\frac{d^2v}{d\chi^2} + \\ - \varepsilon\beta_0\xi\frac{d\xi}{d\chi} + \left(2-\beta_0\right)\varepsilon\xi\frac{dv}{d\chi} + \frac{\mu}{3}\frac{d^3\xi}{d\tau^2 d\chi} = 0\,, \end{aligned} \tag{2.30}$$

if terms which are $O\left(\varepsilon^2\right)$ are neglected. The expression (2.29) can be used to write the mixed derivative in (2.30) as a derivative of one variable only. If μ

and $\tilde{\nu}$ are assumed small, i e $O(\varepsilon)$, the reduced equation system can now be written as

$$\frac{d\xi}{d\tau} + \frac{d\xi}{d\chi} + \varepsilon\xi\frac{d\xi}{dx} - \frac{\tilde{\nu}}{2}\frac{d^2\xi}{dx^2} + \frac{\mu}{6}\frac{d^3\xi}{dx^3} = 0\,, \tag{2.31}$$

which is one form of Korteweg-de Vries-Burgers' equation(KdVB). This equation is also valid for v instead of ξ since we started with the assumption (2.26).

2.4 Retarded variables

A retarded variable x is introduced as

$$x = \chi - \tau \;\left(= \frac{x' - c_0 t'}{\lambda}\right)\,, \tag{2.32}$$

i e the reference system moves with the sound speed in the positive direction. This allows us to write (2.31) on the form

$$\frac{d\xi}{d\tau} + \varepsilon\xi\frac{d\xi}{dx} - \frac{\tilde{\nu}}{2}\frac{d^2\xi}{dx^2} + \frac{\mu}{6}\frac{d^3\xi}{dx^3} = 0\,. \tag{2.33}$$

Using the scaling $u = \varepsilon\xi$ and the notations for the dissipation coefficient

$$\delta = \frac{\tilde{\nu}}{2}\left(= \frac{1}{2}\frac{c_0}{\lambda p_0}\nu\right) \tag{2.34}$$

and the dispersion coefficient

$$\sigma = \frac{\mu}{6}\left(= \frac{1}{6}\frac{R_0^2}{\lambda^2\beta_0\left(1-\beta_0\right)}\right) \tag{2.35}$$

gives us the more compact form

$$u_t + uu_x - \delta u_{xx} + \sigma u_{xxx} = 0 \tag{2.36}$$

of KdVB. Thus the coefficient for the dissipative term is proportional to the viscosity of the fluid, whereas the dispersion is affected by the size of the bubbles as well as the volume fraction of gas in the fluid. We will now study how the solutions of (2.36) depend on the values of δ and σ.

Chapter 3

Burgers' equation

For non dispersive media, i e $\sigma = 0$, KdVB is reduced to Burgers' equation

$$u_t + uu_x - \delta u_{xx} = 0\,. \tag{3.1}$$

This is analytically solvable via the Cole-Hopf transform[5][14]

$$u(x,t) = 2\delta \frac{v_x}{v} = 2\delta \frac{\partial}{\partial x} \ln v\,, \tag{3.2}$$

which after some manipulations gives the heat equation

$$v_t = \delta\, v_{xx} \tag{3.3}$$

from (2.36). This equation can be solved by separation of variables and use of Fourier series. Thereafter the solution to Burgers' equation is obtained by use of (3.2). For a single frequency boundary condition, $u(0,t) = \sin(t)$, the solution can be written

$$u(x,t) = -4\,\delta \frac{\sum_{n=1}^{\infty} (-1)^n n\, e^{-n^2 \delta x} I_n(\frac{1}{2\,\delta}) \sin(n\,t)}{I_0(\frac{1}{2\delta}) + 2 \sum_{n=1}^{\infty} (-1)^n\, e^{-n^2\,\delta\,x} I_n(\frac{1}{2\,\delta}) \cos(n\,t)}\,, \tag{3.4}$$

where I_n are modified Bessel functions. This form is however difficult to handle. There are two approximate solutions to this equation which are fairly well known. They both use single Fourier series as approximations. These are for the region $x < 1$ the corrected, i e valid also for $\delta \neq 0$, Fubini solution[4]

$$u(x,t) = 2 \sum_{m=1}^{\infty} \left(\frac{J_m(m\,x)}{m\,x} - \frac{\delta}{m} \sum_{r=1}^{\infty} r\, b_r\, \left[J_{m-r}(m\,x) + J_{m+r}(m\,x) \right] \right) \sin(m\,t) \tag{3.5}$$

where[1]

$$b_r = r\,(1-x)^{-\frac{1}{2}}\,x^{-r}\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)^r \tag{3.6}$$

and for $x >> 1$ Fay's solution[9]

$$u(x,t) = 2\,\delta \sum_{n=1}^{\infty} \frac{\sin(n\,t)}{\sinh(n\,\delta\,x)}. \tag{3.7}$$

Fay's approximation happens to be an exact solution of Burger's equation but the boundary condition of the original problem is not fulfilled. In the limit when the dissipation approaches zero it can be shown that regardless of which initial condition is used, N-waves will be formed after long times [13].

3.1 Asymptotic analysis

Already for a single frequency initial condition, the solutions are a bit convoluted. To study the evolution of an shock-wave it is therefore necessary to use asymptotic analysis. This has been done with an N-wave as initial waveform for a generalized Burgers' equation

$$\frac{\partial V}{\partial T} + V\,\frac{\partial V}{\partial x} - \delta\,g\,(T)\,\frac{\partial^2 V}{\partial x^2} = 0 \tag{3.8}$$

by Crighton & Scott[6]. In this equation the factor $g\,(T)$ depends on the geometry of the problem. For wave motion with plane symmetry, which is what we are studying here, $g\,(T) = 1$. The problem was, in this article, formulated as

$$\left\{\begin{array}{l} \frac{\partial V}{\partial T} + V\frac{\partial V}{\partial x} - \delta\frac{\partial^2 V}{\partial x^2} = 0 \\ V(x,1) = \left\{\begin{array}{ll} x & |x| < 1 \\ 0 & |x| > 1 \end{array}\right. \end{array}\right. , \tag{3.9}$$

where we have inserted $g\,(T) = 1$. The dissipation coefficient δ is here assumed to be small. The choice of using $T = 1$ as the initial time is made because this gives, in the limit that δ is zero, the outer solution as

$$V_0(x,T) = \left\{\begin{array}{ll} \frac{x}{T} & |x| < T^{\frac{1}{2}} \\ 0 & |x| > T^{\frac{1}{2}} \end{array}\right. . \tag{3.10}$$

An inner variable

$$x^* = \frac{x - T^{\frac{1}{2}}}{\delta} \tag{3.11}$$

and the expansion $V = V_0^* + \delta V_1^*$ is then introduced. This gives, in the leading order of δ, the differential equation

[1] For our purposes the notation could be simplified by $c_r = rb_r$ but in [4] b_r is used also in other equations.

$$\left(V_0^* - \frac{1}{2}T^{-\frac{1}{2}}\right)\frac{\partial V_0^*}{\partial x^*} - \frac{\partial^2 V_0^*}{\partial x^{*2}} = 0\,, \tag{3.12}$$

which may be integrated to

$$\left(V_0^* - T^{-\frac{1}{2}}\right)\frac{V_0^*}{2} - \frac{\partial V_0^*}{\partial x^*} = C(T)\,. \tag{3.13}$$

This equation has the implicit solution

$$x^* = 2\int \frac{dV}{\left(V - T^{-\frac{1}{2}}\right)V - C(T)}\,. \tag{3.14}$$

It can be seen that the matching against the outer solution (3.10)

$$\begin{cases} \lim\limits_{x^*\to\infty} V_0^* = 0 \\ \lim\limits_{x^*\to-\infty} V_0^* = T^{-\frac{1}{2}} \end{cases} \tag{3.15}$$

is fulfilled if $C\,(T) \equiv 0$ is used. This gives the solution

$$V_0^* = \frac{T^{-\frac{1}{2}}}{2}\left(1 - \tanh\left(\frac{T^{-\frac{1}{2}}}{4}\left(x^* - A\,(T)\right)\right)\right)\,. \tag{3.16}$$

By integral conservation technique[6] the unknown function in (3.16) can be determined to be

$$A\,(T) = -T^{\frac{1}{2}}\ln T \tag{3.17}$$

Thus the inner solution written in the original variable is

$$V_0^* = \frac{T^{-\frac{1}{2}}}{2}\left(1 - \tanh\left(\frac{1}{4}\left(T^{-\frac{1}{2}}\frac{x - T^{\frac{1}{2}}}{\delta} + \ln T\right)\right)\right) \tag{3.18}$$

This obviously fulfils (3.15). If the first term of the argument of the tanh function dominates over the second one, the solution can be expressed on a somewhat more transparent form.

3.2 Shock tail region

The shock tail region is used to describe the development of the N-wave far away from the shock region. (See Enflo[8]) If a new variable is introduced as

$$Y = \frac{1}{2}x^*\,T^{-\frac{1}{2}}\,\left(= \frac{1}{2\,\delta}\frac{x - T^{\frac{1}{2}}}{T^{\frac{1}{2}}}\right) \tag{3.19}$$

the equation (3.16) with (3.17) gets the form

$$V_0^* = T^{-\frac{1}{2}} \left(1 + T^{\frac{1}{2}} \exp Y\right)^{-1} . \tag{3.20}$$

In the shock tail region, defined as $\left\{T = O(1), Y > \ln\left(\varepsilon^{-1}\right)\right\}$ where $\varepsilon \ll 1$, this is reduced to

$$V_0^* = T^{-1} \exp(-Y) . \tag{3.21}$$

This is, in the shock tail region, also a solution to (3.8) (with $g(T) = 1$) if the nonlinear term is neglected. In terms of the original variables, (3.21) can be written as

$$V_0^* = T^{-1} \exp\left(-\frac{1}{2\,\delta} \frac{x - T^{\frac{1}{2}}}{T^{\frac{1}{2}}}\right). \tag{3.22}$$

Chapter 4

Korteweg-de Vries' equation

If the dissipation can be neglected, i e the viscosity is zero which means that the second derivative vanishes from KdVB, the remaining equation is called Korteweg-de Vries' equation (KdV).

$$u_t + u\,u_x + \sigma\,u_{xxx} = 0 \tag{4.1}$$

KdV is analytically solvable only for a few special cases and thus most studies deal with these[1]. We will here study an initial value problem, which does not fall into this small group.

4.1 Inverse scattering transform

To solve KdV analytically the inverse scattering transform (IST) is used. (For a more comprehensive description see e g Marchenko[25], Ablowitz[2] or Lamb[21].)

$$v(x,t) = \frac{\Psi_{xx}}{\Psi} + \lambda(t) \tag{4.2}$$

If the ansatz (4.2) is rewritten as

$$\Psi_{xx} + (\lambda(t) - v(x,t))\Psi = 0\,, \tag{4.3}$$

it is recognized as the time dependent Schrödinger equation. The principle of IST, as formulated for KdV, is as follows. If $\lambda_t = 0$, (4.3) is a time independent

[1] Maybe it should be mentioned that these special cases are the soliton solutions whose peculiar properties attract "some" attention themselves.

eigenvalue problem. It can be shown that a sufficient condition for this is that the potential $v(x,t)$ fulfils Korteweg-de Vries' equation on the form

$$v_t - \frac{a}{4}\left(v_{xxx} - 6v\, v_x\right) - c\,(t)\, v_x = 0 \tag{4.4}$$

This means that if $v(x,t)$ fulfils KdV, the solutions to (4.3) are independent of the value of t used in $v(x,t)$. Especially $v(x,0)$ can be used and thereafter the solution to the KdV with this initial value is obtained from the solution to (4.3). To get the KdV equation as (4.1), we choose $c\,(t) = 0$, $a = -4\sigma$ and introduce

$$u(x,t) = -\frac{1}{6\sigma} v(x,t)\,. \tag{4.5}$$

4.2 Scattering potentials

We will now study the solutions to the Schrödinger equation (4.3). For a given localized potential, two fundamental solutions can be written as

$$\Psi_1\,(x,k) : \begin{cases} \Psi_L = e^{ikx} + R_L e^{-ikx} \\ \Psi_R = T_L e^{ikx} \end{cases} \tag{4.6}$$

$$\Psi_2\,(x,k) : \begin{cases} \Psi_L = T_R e^{-ikx} \\ \Psi_R = e^{-ikx} + R_R e^{ikx} \end{cases}, \tag{4.7}$$

where the indices L and R means to the left/right of the scattering region. Here R is the reflection coefficient while T is the transmission coefficient. Thus the first solution (4.6) represents the case of an incident wave from the left while the second solution (4.7) represents the case of an incident wave from the right. We now introduce the more general fundamental solutions $f_1(x,k)$ and $f_2(x,k)$ which fulfils

$$\begin{aligned} \lim_{x\to\infty} e^{-ikx} f_1\,(x,k) &= 1 \\ \lim_{x\to-\infty} e^{ikx} f_2\,(x,k) &= 1 \end{aligned} \tag{4.8}$$

From (4.8), we find for the fundamental solutions $f_1(x,-k)$ and $f_2(x,-k)$:

$$\begin{aligned} \lim_{x\to\infty} e^{ikx} f_1\,(x,-k) &= 1 \\ \lim_{x\to-\infty} e^{-ikx} f_2\,(x,-k) &= 1 \end{aligned} \tag{4.9}$$

Since the Schrödinger equation is linear, it is always possible to express each solution as a linear combination of a set of linearly independent solutions. Accordingly, we write

$$\begin{aligned} f_2(x,k) &= c_{11}\,(k)\, f_1(x,k) + c_{12}\,(k)\, f_1(x,-k) \\ f_1(x,k) &= c_{21}\,(k)\, f_2(x,k) + c_{22}\,(k)\, f_2(x,-k) \end{aligned}. \tag{4.10}$$

From (4.10) with (4.8) and (4.9) follows that in the limit $x \to \infty$

$$f_2(x,k) \approx c_{11}(k)\, e^{ikx} + c_{12}(k)\, e^{-ikx}\,, \tag{4.11}$$

while for $x \to -\infty$

$$f_1(x,k) \approx c_{21}(k)\, e^{ikx} + c_{22}(k)\, e^{-ikx}\,. \tag{4.12}$$

Comparing (4.11) and (4.12) with (4.6) and (4.7), the reflection and transmission coefficients can be written

$$\begin{bmatrix} R_R(k) = \frac{c_{11}(k)}{c_{12}(k)} & R_L(k) = \frac{c_{22}(k)}{c_{21}(k)} \\ \\ T_R(k) = \frac{1}{c_{12}(k)} & T_L(k) = \frac{1}{c_{21}(k)} \end{bmatrix} \tag{4.13}$$

4.3 Variation of parameters

With the fundamental solutions above as model, we now make a solution ansatz for Schrödinger's equation (4.3). This in the form, to use variation of parameters as solving method,

$$\Psi = \alpha(x)\; e^{ikx} + \beta(x)\; e^{-ikx}\,. \tag{4.14}$$

The first derivative of (4.14) with respect to x is

$$\Psi_x = (\alpha_x + ik\alpha)\; e^{ikx} + (\beta_x + ik\beta)\; e^{-ikx}\,. \tag{4.15}$$

We are free to add one constraint for $\alpha(x)$ and $\beta(x)$. By choosing

$$\alpha_x\, e^{ikx} + \beta_x\, e^{-ikx} = 0 \tag{4.16}$$

the second derivative of (4.14) will be

$$\Psi_{xx} = -k^2\Psi + ik\left(\alpha_x\, e^{ikx} - \beta_x\, e^{-ikx}\right)\,. \tag{4.17}$$

Demanding that the expressions (4.14) and (4.17) must fulfil Schrödinger's equation (4.3) gives with (4.16) an equation system for α_x and β_x

$$\begin{cases} ik\left(\alpha_x\, e^{ikx} - \beta_x\, e^{-ikx}\right) = u\Psi \\ \alpha_x\, e^{ikx} + \beta_x\, e^{-ikx} = 0 \end{cases} \tag{4.18}$$

with the solution

$$\begin{cases} \alpha_x = \frac{1}{2ik} u\Psi e^{-ikx} \\ \beta_x = -\frac{1}{2ik} u\Psi e^{ikx} \end{cases}. \tag{4.19}$$

Integration of (4.19) gives

$$\begin{cases} \alpha(x) = \frac{1}{2ik}\int_0^x u(s)\,\Psi(\alpha(s),\beta(s),s)\; e^{-iks}ds + c_1 \\ \beta(x) = -\frac{1}{2ik}\int_0^x u(s)\,\Psi(\alpha(s),\beta(s),s)\; e^{iks}ds + c_2 \end{cases} \quad . \tag{4.20}$$

From (4.8) and (4.9) we also have

$$\begin{cases} f_1(x,k) \underset{x\to\infty}{\longrightarrow} e^{ikx} \\ f_2(x,k) \underset{x\to-\infty}{\longrightarrow} e^{-ikx} \end{cases} \quad . \tag{4.21}$$

In terms of (4.14) this can be written

$$\begin{cases} f_1 : x \to \infty \sim \begin{cases} \alpha(x) \longrightarrow 1 \\ \beta(x) \longrightarrow 0 \end{cases} \\ f_2 : x \to -\infty \sim \begin{cases} \alpha(x) \longrightarrow 0 \\ \beta(x) \longrightarrow 1 \end{cases} \end{cases} \quad . \tag{4.22}$$

Combining (4.20) and (4.22) gives Volterra integral equations of second kind for the fundamental solutions.

$$\begin{aligned} f_1(x,k) &= e^{ikx} - \tfrac{1}{k}\int_x^\infty u(s)\, f_1(s,k)\sin(k(x-s))\; ds \\ f_2(x,k) &= e^{-ikx} + \tfrac{1}{k}\int_{-\infty}^x u(s)\, f_2(s,k)\sin(k(x-s))\; ds \end{aligned} \quad . \tag{4.23}$$

These could be solved with iteration methods and it is seen that the integrals converge for $\mathrm{Im}(k) > 0$, i e $f_1(x,k)$ and $f_2(x,k)$ are analytic functions in the upper half-plane. This will be used later on.

4.4 Marchenko's equation

Due to the form of (4.23) we define two new functions

$$\begin{aligned} A_R(x,k) &\equiv f_1(x,k) - e^{ikx} \\ A_L(x,k) &\equiv f_2(x,k) - e^{-ikx} \end{aligned} \quad . \tag{4.24}$$

We introduce the Fourier transform

$$\hat{A}_R(x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_R(x,k)\; e^{-iks}dk \tag{4.25}$$

and its inverse

$$A_R(x,k) = \int_{-\infty}^{\infty} \hat{A}_R(x,s)\; e^{iks}ds\,. \tag{4.26}$$

Comparing with (4.23) it is seen that

$$\begin{aligned} \hat{A}_R(x,s) &= 0 : s < x \\ \hat{A}_L(x,s) &= 0 : s > x \end{aligned} \quad , \tag{4.27}$$

which from (4.24) leads to

$$\begin{aligned} f_1(x,k) &= e^{ikx} + \int_x^{\infty} \hat{A}_R(x,s)\, e^{iks} ds \\ f_2(x,k) &= e^{-ikx} + \int_{-\infty}^{x} \hat{A}_L(x,s)\, e^{-iks} ds \end{aligned} \quad . \tag{4.28}$$

The function $u(s)$ appears in (4.23) in a similar way as $\hat{A}_R(x,s)$ and $\hat{A}_L(x,s)$ in (4.28). It can be shown that they are related by

$$\begin{aligned} u(x) &= -2\tfrac{d}{dx}\hat{A}_R(x,x) \\ u(x) &= 2\tfrac{d}{dx}\hat{A}_L(x,x) \end{aligned} \quad . \tag{4.29}$$

Since our goal is to express the solution to Korteweg-de Vries' equation in terms of solutions to Schrödinger's equation, the next step is to derive a relation between $\hat{A}_R(x,s)$ and these solutions ($f_1(x,k)$,$f_2(x,k)$). (The choice of using the right index, as opposed to the left, is somewhat arbitrary. The derivation of the expression for $\hat{A}_L(x,s)$ is done in a similar way.) With (4.13) the first equation of (4.10) can be written

$$T_R(k)\, f_2(x,k) = R_R(k)\, f_1(x,k) + f_1(x,-k)\, . \tag{4.30}$$

In order to obtain an expression that has a Fourier transform for each term we have to add and subtract terms as follows

$$\begin{aligned} &(T_R(k) - 1)\, f_2(x,k) = \\ &= R_R(k)\left(f_1(x,k) - e^{ikx}\right) + R_R(k)\, e^{ikx} + \\ &+ \left(f_1(x,-k) - e^{-ikx}\right) + \left(e^{-ikx} - f_2(x,k)\right) \\ &= R_R(k)\, A_R(x,k) + R_R(k)\, e^{ikx} + A_R(x,-k) - A_L(x,k) \end{aligned} \quad . \tag{4.31}$$

The Fourier transform of (4.31) is

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} (T_R(k) - 1)\, f_2(x,k)\, e^{iky} dk = \\ &= \int_{-\infty}^{\infty} \hat{R}_R(y+s)\, \hat{A}_R(x,s)\; ds + \hat{R}_R(x+y) + \hat{A}_R(x,y) - \hat{A}_L(x,y)\ , \end{aligned} \tag{4.32}$$

where s is the convolution variable and $\hat{R}_R(y)$ is the Fourier transform of $R_R(k)$.

If $x < y$ is assumed, the integration path along the real k-axis can be completed with a semicircle in the upper half-plane which does not contribute to the value of the integral if $T_R(k) - 1$ tends to zero for $k \longrightarrow \infty$ as $k^{-(1+\varepsilon)}$ or faster ($\varepsilon > 0$). Also $\hat{A}_L(x, y) = 0$ follows from (4.27). As shown earlier $f_2(x, k)$ is analytic for $\mathrm{Im}(k) > 0$. Furthermore $T_R(k)$ is analytic for $\mathrm{Im}(k) > 0$ with the exception of simple poles on the imaginary axis. We are now ready to use Cauchy's residue theorem which for this case can be written

$$\frac{1}{2\pi i} \oint_C f(z)\, dz = \sum_{\substack{\text{poles} \\ \text{inside C}}} a_{-1}^{(i)}\,, \tag{4.33}$$

where $a_{-1}^{(i)}$ are the residues of $f(z)$. For simplicity we introduce the notations

$$i\kappa_n \equiv k_n \tag{4.34}$$

for the poles of $T_R(k)$ and

$$i\gamma_n \equiv a_{-1}^{(n)} = \left(\frac{\partial c_{12}}{\partial k}\right)^{-1}\Bigg|_{k=i\kappa_n} \tag{4.35}$$

for the corresponding residues. This gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (T_R(k) - 1)\; f_2(x, k)\, e^{iky}\, dk = -\sum_n \gamma_n\, f_2(x, i\kappa_n) e^{-\kappa_n y}\,. \tag{4.36}$$

From (4.13) it is seen that the poles of $T_R(k)$ are the zeroes of $c_{12}(k)$ and thus with (4.10) follows that

$$f_2(x, i\kappa_n) = c_{11}(i\kappa_n)\; f_1(x, i\kappa_n)\,. \tag{4.37}$$

Combining (4.32) with (4.36) and using (4.37), (4.24), (4.26) and (4.27) gives Marchenko's integral equation on the form

$$\hat{A}_R(x, y) + K_R(x + y) + \int_x^{\infty} K_R(s + y)\; \hat{A}_R(x, s)\; ds = 0\,, \tag{4.38}$$

where

$$K_R(z) = \hat{R}_R(z) + \sum_n \gamma_n\, c_{11}(i\kappa_n)\; e^{-\kappa_n z}\,. \tag{4.39}$$

4.5 Time dependence

The time dependence of $\Psi(x,t)$, i e the solution to the Schrödinger equation with a solution to KdV as potential, fulfils the Gardner-Greene-Kruskal-Miura equation[10]

$$\begin{aligned} c_{11}(k,t) &= c_{11}(k,0)\; e^{8ik^3 t} \\ c_{12}(k,t) &= c_{12}(k,0) \end{aligned} \tag{4.40}$$

which can be derived by the examination of the time dependence of the reflection- and transmission coefficients in the asymptotic region. With the time dependence from (4.40) explicit, the equation (4.39) can be written

$$K_R(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_R(k,0)\; e^{i\left(kz+8k^3t\right)}\, dk + \sum_n \gamma_n\, c_{11}(i\kappa_n,0)\; e^{-\kappa_n z+8\kappa_n^3 t}\,, \tag{4.41}$$

where

$$R_R(k,0) = \frac{c_{11}(k,0)}{c_{12}(k,0)}\,. \tag{4.42}$$

This means that $K_R(z,t)$ is completely determined by $c_{11}(k,0)$ and $c_{12}(k,0)$. It is thus possible to formulate Marchenko's equation

$$\hat{A}_R(x,y;t) + K_R(x+y,t) + \int_x^{\infty} K_R(s+y,t)\; \hat{A}_R(x,s;t)\; ds = 0\,. \tag{4.43}$$

This equation is only analytically solvable for the special case $R_R(k,0) \equiv 0$ since the exponential function in the integral in (4.41) contains a cube of the integration variable. However the integral equation is linear and it is possible to solve for $\hat{A}_R(x,y;t)$ by i e iteration methods. Thereafter the solution to Korteweg-de Vries' equation is obtained as

$$u(x,t) = -2\frac{d}{dx}\hat{A}_R(x,x;t)\;. \tag{4.44}$$

Accordingly the solution to the Schrödinger equation can be used to find the solution to Korteweg-de Vries' equation.

Chapter 5

KdV with an N-wave as initial condition

We will now use the theory described in the former chapter to study Korteweg-de Vries' equation as an initial value problem with an N-wave as the initial value. We thus write our problem

$$\left\{ \begin{array}{c} u_t + uu_x + \sigma u_{xxx} = 0 \\ u\left(x,0\right) = \left\{ \begin{array}{c} x : |x| < 1 \\ 0 : |x| > 1 \end{array} \right. \end{array} \right. \tag{5.1}$$

and use the inverse scattering method to investigate how the N-wave develops and how this depends on the dispersion coefficient σ. A major part of this chapter was presented[30] at the 15^{th} International Symposium on Nonlinear Acoustics, held in Göttingen, Germany 1999.

5.1 Solving the Schrödinger equation

For our form of KdV, i e (5.1), the corresponding Schrödinger equation is (cf Karpman[19])

$$\Psi_{xx} + (\lambda + \frac{u}{6\sigma})\Psi = 0\,. \tag{5.2}$$

Due to the discontinuities in our initial condition, we must solve (5.2) in three different regions (I: $x < -1$, II: $|x| < 1$, III: $.x > 1$). We write $\lambda = k^2$ and thereby obtain the Schrödinger equations for our problem on the forms

$$\left\{ \begin{array}{c} I, III : \Psi_{xx} + k^2\Psi = 0 \\ II : \Psi_{xx} + \left(k^2 + \frac{x}{6\sigma}\right)\Psi = 0 \end{array} \right. \tag{5.3}$$

The solution to the first equation is the well known

$$\Psi_{I,III} = A_{I,III}\, e^{ikx} + B_{I,III}\, e^{-ikx}\,, \tag{5.4}$$

which makes the comparison to (4.11) and (4.12) straightforward. This is really the motivation for our choice of λ. For the middle region, we compare our Schrödinger equation with the Airy equation

$$\Psi_{zz} - z\Psi = 0\,, \tag{5.5}$$

whose solutions are called Airy functions. We make the ansatz

$$z = -\left(a\,k^2 + b\,x\right) \tag{5.6}$$

and use the chain rule to write

$$\Psi_{xx} = \left(\frac{dz}{dx}\right)^2 \Psi_{zz} = b^2\,\Psi_{zz}\,. \tag{5.7}$$

In order to obtain our Schrödinger equation in (5.3) from the Airy equation (5.5), the parameters a and b must fulfil

$$\begin{cases} a = \frac{1}{b^2} \\ b^3 = \frac{1}{6\sigma} \end{cases}. \tag{5.8}$$

We thus find that the solutions for region II can be written as linear combinations of Airy functions[1] of first and second kind on the form

$$\Psi_{II} = A_{II}\,\mathrm{Ai}\left(-\left(s^2k^2 + \frac{x}{s}\right)\right) + B_{II}\,\mathrm{Bi}\left(-\left(s^2k^2 + \frac{x}{s}\right)\right)\,, \tag{5.9}$$

where we have defined $s \equiv (6\sigma)^{\frac{1}{3}}$ to get a more compact notation. We thus have the solution

$$\Psi = \begin{cases} A_I\, e^{ikx} + B_I\, e^{-ikx} & x < -1 \\ A_{II}\,\mathrm{Ai}\left(-\left(s^2k^2 + \frac{x}{s}\right)\right) + B_{II}\,\mathrm{Bi}\left(-\left(s^2k^2 + \frac{x}{s}\right)\right) & |x| < 1 \\ A_{III}\, e^{ikx} + B_{III}\, e^{-ikx} & x > 1 \end{cases}, \tag{5.10}$$

where we also demand $\Psi \in C^1$. This demand gives two equations at each boundary point. Since the solution in each region has two coefficients, two equations are needed to uniquely determine these unknowns as functions of the coefficients in the neighbour region.

[1] It is possible to instead use linear combinations of either Bessel functions of first kind with order 1/3 and -1/3 or Bessel functions of first and second kind with order 1/3.

By comparing (5.10) with (4.11) and (4.12), we can determine the coefficients c_{ij} by studying the fundamental cases. We choose to study the case that fulfils

$$\Psi \underset{x\to-\infty}{\longrightarrow} e^{-ikx} \quad , \tag{5.11}$$

i e $A_I = 0$, $B_I = 1$ (cf (4.8)). This gives by the continuity demand for $x = -1$

$$\begin{cases} A_{II} = \dfrac{e^{ik}\left\{iks\,\mathrm{Bi}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)-\mathrm{Bi}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\right\}}{\mathrm{Ai}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\mathrm{Bi}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)-\mathrm{Ai}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\mathrm{Bi}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)} \\ B_{II} = \dfrac{e^{ik}\left\{iks\,\mathrm{Ai}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)-\mathrm{Ai}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\right\}}{\mathrm{Ai}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\mathrm{Bi}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)-\mathrm{Ai}'\left(-\left[s^2k^2-\frac{1}{s}\right]\right)\mathrm{Bi}\left(-\left[s^2k^2-\frac{1}{s}\right]\right)} \end{cases} . \tag{5.12}$$

The denominator in the expressions for A_{II} and B_{II} can be simplified by the general relation for the Wronskian of Airy functions of first and second kind, see Lebedev[24],

$$\mathrm{W}\left\{\mathrm{Ai}\left(z\right),\mathrm{Bi}\left(z\right)\right\} = \mathrm{Ai}\left(z\right)\mathrm{Bi}'\left(z\right) - \mathrm{Ai}'\left(z\right)\mathrm{Bi}\left(z\right) = \frac{1}{\pi}\,. \tag{5.13}$$

This allows us to write

$$\begin{cases} A_{II} = -\pi e^{ik}\left[iks\,\mathrm{Bi}\left(-\left(s^2k^2-s^{-1}\right)\right)-\mathrm{Bi}'\left(-\left(s^2k^2-s^{-1}\right)\right)\right] \\ B_{II} = \pi e^{ik}\left[iks\,\mathrm{Ai}\left(-\left(s^2k^2-s^{-1}\right)\right)-\mathrm{Ai}'\left(-\left(s^2k^2-s^{-1}\right)\right)\right] \end{cases} . \tag{5.14}$$

Our next step is to use the continuity demand for $x = 1$ to get expressions for A_{III} and B_{III}.

$$\begin{cases} A_{III} = -\frac{1}{2}e^{-ik}\left[A_{II}\,\mathrm{Ai}\left(-\left(s^2k^2+s^{-1}\right)\right)+B_{II}\,\mathrm{Bi}\left(-\left(s^2k^2+s^{-1}\right)\right)\right] \\ \qquad\quad +\frac{1}{2iks}e^{-ik}\left[A_{II}\,\mathrm{Ai}'\left(-\left(s^2k^2+s^{-1}\right)\right)+B_{II}\,\mathrm{Bi}'\left(-\left(s^2k^2+s^{-1}\right)\right)\right] \\ B_{III} = \frac{1}{2}e^{ik}\left[A_{II}\,\mathrm{Ai}\left(-\left(s^2k^2+s^{-1}\right)\right)+B_{II}\,\mathrm{Bi}\left(-\left(s^2k^2+s^{-1}\right)\right)\right] \\ \qquad\quad +\frac{1}{2iks}e^{ik}\left[A_{II}\,\mathrm{Ai}'\left(-\left(s^2k^2+s^{-1}\right)\right)+B_{II}\,\mathrm{Bi}'\left(-\left(s^2k^2+s^{-1}\right)\right)\right] \end{cases} \tag{5.15}$$

Finally we make the identifications

$$\begin{aligned} c_{11}\left(k,t=0;\sigma\right) &= A_{III}\left(k;s\right) \\ c_{12}\left(k,t=0;\sigma\right) &= B_{III}\left(k;s\right) \end{aligned} \quad , \tag{5.16}$$

where (5.14) and (5.15) are used to express A_{III} and B_{III} as functions of k and σ. As earlier defined $s \equiv (6\sigma)^{\frac{1}{3}}$.

5.2 Solving Marchenko's equation

With (5.14), (5.15) and (5.16) we can now write Marchenko's equation on the form

$$\hat{A}_R(x,y;t,\sigma) + K_R(x+y;t,\sigma) + \int_x^\infty K_R(s+y;t,\sigma)\,\hat{A}_R(x,s;t,\sigma)\;ds = 0\,, \tag{5.17}$$

where

$$K_R(z;t,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_R(k,0;\sigma)\;e^{i\left(kz+8k^3t\right)}\,dk + \sum_n m_{Rn}(\kappa_n,\sigma)\;e^{-\kappa_n z+8\kappa_n^3 t} \tag{5.18}$$

with the notations

$$\begin{cases} R_R(k,0;\sigma) = \frac{c_{11}(k,0;\sigma)}{c_{12}(k,0;\sigma)} \\ m_{Rn}(\kappa_n,\sigma) = \left.\frac{c_{11}(k,0;\sigma)}{i\frac{\partial}{\partial k}c_{12}(k,0;\sigma)}\right|_{k=i\kappa_n} \end{cases} \tag{5.19}$$

and $k_n(\sigma) = i\kappa_n$ are the zeroes of $c_{12}(k,0;\sigma)$ at the positive imaginary axis. Due to the complexity of $c_{12}(k,0;\sigma)$, it is in our case not possible to obtain a general expression for these zeroes. To proceed the investigation we need fixed values of σ. To get the equations on the most compact form, we choose $\sigma = \frac{1}{6}$ as the first value since $s \equiv (6\sigma)^{\frac{1}{3}}$. For this value we find only one zero of $c_{12}(k,0;\sigma)$ at the positive imaginary axis.

$$\sigma = \tfrac{1}{6} \Longrightarrow \quad \kappa \approx 0.111128321 \Longrightarrow \quad m_R \approx 0.176659368 \tag{5.20}$$

This means that the summation term of $K_R(z;t,\sigma)$ is reduced to a single exponential function of time and space variables with known coefficients. Extending our investigation to smaller as well as larger values of σ we get the same result (of course the value of κ depends on σ. (See figure 5.1)), i e there is only one zero of $c_{12}(k,0;\sigma)$ in the upper half plane and this root is purely imaginary. However, this is not true then the dispersion is very weak. For dispersion coefficients slightly lower than $\sigma = 0.0049$, which gives $\kappa \approx 4$, two roots of $c_{12}(k,0;\sigma)$ are found at the positive imaginary axis and a third root shows up just below $\sigma = 0.0015$. (See figure 5.2) The integral term of (5.18) is a bit more complicated since it is not possible to perform the integration analytically, even for "simple" expressions of $R_R(k,0;\sigma)$, due to the cubic term of the integrating variable in the exponential function. The state of the art solution of this kind of problems seems to be to assume a reflectionless potential, i e $R_R(k,0;\sigma) \equiv 0$, since then "this integral does not occur" and "the inverse scattering problem can be solved exactly". If we test this assumption for our case, we get a soliton solution with an amplitude of 0.0247 (if $\sigma = \frac{1}{6}$). This soliton is an exact solution to Korteweg-de Vries' equation but it certainly does not fulfil the given

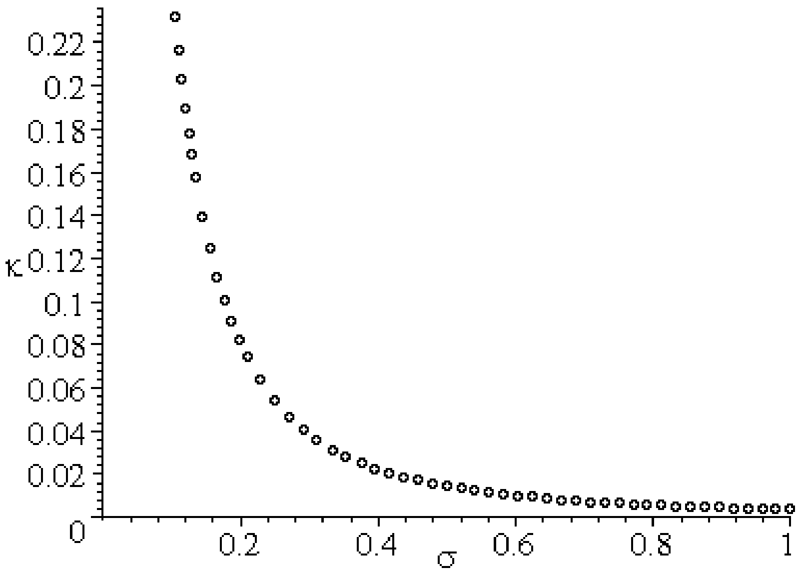

Figure 5.1. Magnitude of the zeroes of $c_{12}(k,0;\sigma)$ in the upper half plane for some different values of σ which give single roots. $(k = i\kappa)$

initial value. The amplitude of the soliton solution is $2\,\kappa^2$ and accordingly it will be larger for smaller values of σ but that does not make it look more like an N-wave. For values of σ which are small enough to give more than one root of $c_{12}(k,0;\sigma)$, each root corresponds to a soliton. In the limit $\sigma \longrightarrow 0$ (still for $R_R(k,0;\sigma) \equiv 0$), a large number of solitons will build up a triangular wave since for each soliton both amplitude and velocity is proportional to κ_n^2. A thorough investigation of single as well as multiple soliton solutions for KdV (et al) can be found in Dodd et al[7].
In our case

$$R_R(k,0;\sigma) = \exp(-2ik)$$
$$\cdot\frac{\left\{\begin{array}{l}[sk\mathrm{Ai}(\frac{1}{s}-s^2k^2)+i\mathrm{Ai}'(\frac{1}{s}-s^2k^2)][sk\mathrm{Bi}(-\frac{1}{s}-s^2k^2)+i\mathrm{Bi}'(-\frac{1}{s}-s^2k^2)]+\\ -[sk\mathrm{Ai}(-\frac{1}{s}-s^2k^2)+i\mathrm{Ai}'(-\frac{1}{s}-s^2k^2)][sk\mathrm{Bi}(\frac{1}{s}-s^2k^2)+i\mathrm{Bi}'(\frac{1}{s}-s^2k^2)]\end{array}\right\}}{\left\{\begin{array}{l}[sk\mathrm{Ai}(\frac{1}{s}-s^2k^2)+i\mathrm{Ai}'(\frac{1}{s}-s^2k^2)][sk\mathrm{Bi}(-\frac{1}{s}-s^2k^2)-i\mathrm{Bi}'(-\frac{1}{s}-s^2k^2)]+\\ -[sk\mathrm{Ai}(-\frac{1}{s}-s^2k^2)-i\mathrm{Ai}'(-\frac{1}{s}-s^2k^2)][sk\mathrm{Bi}(\frac{1}{s}-s^2k^2)+i\mathrm{Bi}'(\frac{1}{s}-s^2k^2)]\end{array}\right\}} \tag{5.21}$$

(where $s \equiv (6\sigma)^{\frac{1}{3}}$) and this makes the integral in (5.18) hard to integrate analytically even for $t = 0$. However, as seen in figure 5.3, $R_R(k,0;\sigma)$ has a symmetric real part and an anti-symmetric imaginary part and for large values of k they both go towards a zero amplitude. Accordingly this function is well

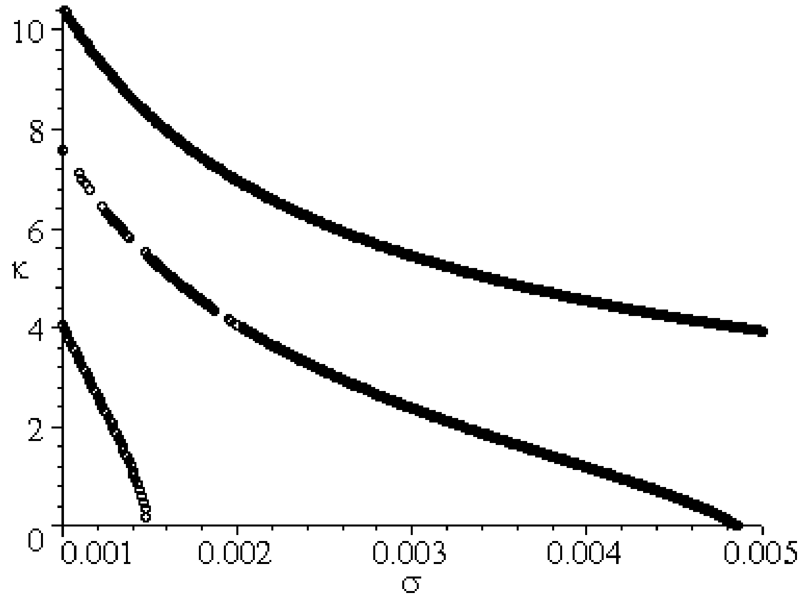

Figure 5.2. The function $c_{12}(k, 0; \sigma)$ has multiple roots at the positive imaginary axis for small values of σ. ($k = i\kappa$)

suited for numerical integration. In (5.18) the multiplication of $R_R(k, 0; \sigma)$ with the exponential function which has an imaginary argument introduces oscillations which become faster for large values of k and therefore makes the convergence better. On the other hand the presence of space and time variables in this argument means that the integration cannot be made completely numerical. We thus make the approximation

$$\hat{R}_R(z; t, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_R(k, 0; \sigma)\, e^{i\left(kz + 8k^3 t\right)} dk$$
$$\approx \frac{\Delta k}{2\pi} \sum_{p=-N}^{N} R_R(p\Delta k, 0; \sigma)\, e^{i\left(p\Delta k z + 8(p\Delta k)^3 t\right)}, \quad (5.22)$$

where N is "large enough" (and Δk "small enough"). How large "large enough" is depends on σ. For all values of σ, $R_R(0, 0; \sigma) = -1$ but for smaller values of σ the amplitude of $R_R(k, 0; \sigma)$ decline slower for growing k. This means that a large number of terms is needed for small values of σ. With the expression (5.22) for $\hat{R}_R$, we now have (5.17) in the form of a Volterra integral equation with $\hat{A}_R$ as unknown function.

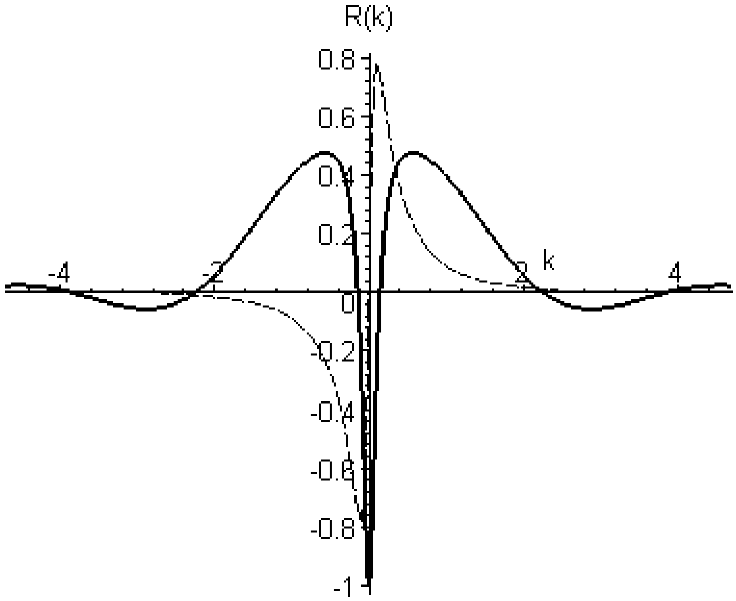

Figure 5.3. $R_R\left(k, 0; \frac{1}{6}\right)$ with symmetric real part (solid line) and anti-symmetric imaginary part (dashed line)

5.3 Iteration solution

Since the number of terms in the approximation of K_R is large, we will use an iteration solution for the integral equations to keep the amount of calculations which have to be done at a reasonable level. The zeroth step of the iteration solution is to put $\hat{A}_R(x, s; t, \sigma) = 0$ in (5.17). We thus write this solution as

$$\hat{A}_R(x, y; t, \sigma) = -K_R(x + y; t, \sigma) \ . \tag{5.23}$$

By (4.44) the corresponding solution for Korteweg-de Vries' equation is

$$u(x, t) = -2\frac{d}{dx}\hat{A}_R(x, x; t, \sigma) = 2\frac{d}{dx}K_R(2x; t, \sigma) \ . \tag{5.24}$$

As a first test of our approximation, we examine how well the solution gives back the initial waveform. In figure 5.4 it is seen that the zeroth order approximation matches the initial condition fairly well with $N = 20385$ and $\Delta k = 0.01$ used in (5.22). The overshooting at the discontinuities is known as the Gibbs phenomenon[12]. The width of the overshoot regions can be reduced by using more terms in the series but the amplitude of the overshoots will increase and converge to a value which in the limit of an infinite Fourier series is almost 9 percent of the height of the discontinuity. The series converges at all points

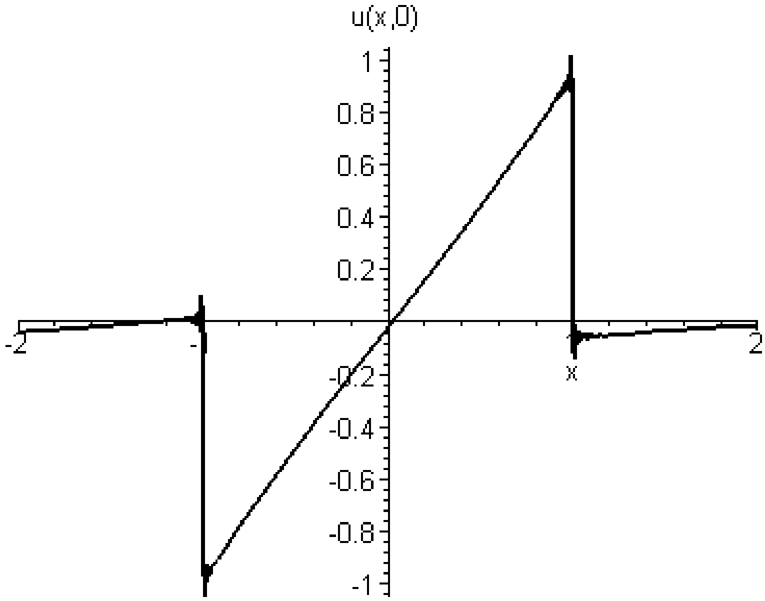

Figure 5.4. The initial waveform is reconstructed by the zeroth order iteration solution for KdV with $\sigma = \frac{1}{6}$ at $t = 0$.

except at the discontinuity. At the discontinuity the difference between the limit values from the both sides is larger, than the actual height of the discontinuity, by the factor $\frac{2\,\mathrm{Si}(\pi)}{\pi}$. The slightly negative slope of the line for $x < -1$ is due to the error of the approximation in the iteration solution. For the first order iteration solution it becomes slightly positive and for the second order solution again negative and so on. The zeroth order iteration solution is only suitable for values of the dispersion coefficient σ which are not too small. One reason for this is that that part of the kernel of the integral equation, which for reflectionless potentials gives solitons as exact solutions, is a sum of exponential functions in the zeroth order iteration solution. Another reason is that the amplitude of the reflection coefficient, which is to be integrated over the entire real axis by a numerical approximation, declines more slowly for smaller values of the dispersion coefficient. Thus a larger number of terms is needed to achieve the same accuracy. For positive times (see figure 5.5) it is seen how the front of the N-wave is smoothened out and slowed down, leaving an oscillating tail behind. (Note that we are using a retarded spatial variable.) As could be expected, the shock wave is broken down faster for larger values of the dispersion constant σ. This behaviour, i e the damped wave front and an oscillating tail has been shown for numerical solutions where the first derivative of an Gaussian curve was used as initial value by Berezin[3]. Experiments by Kuznetsov et al[20] where a triangle wave (the "upper half" of the N-wave)

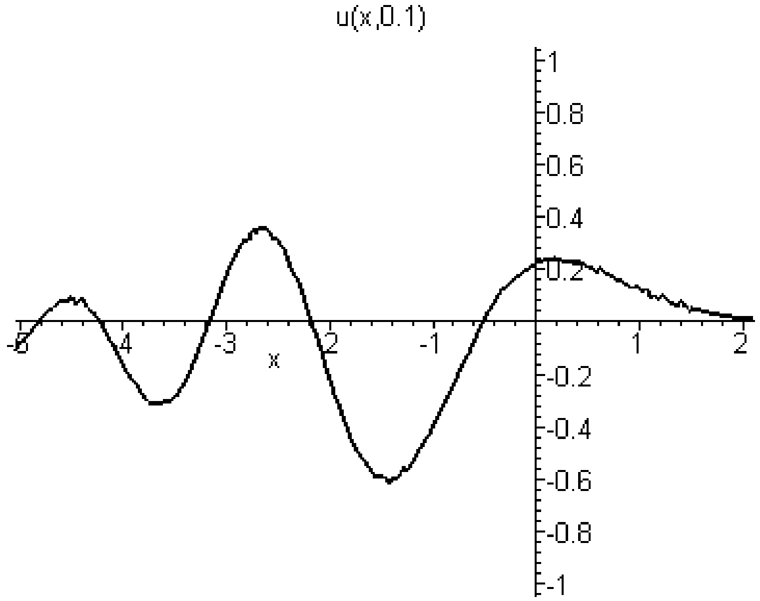

Figure 5.5. Zeroth order iteration solution for KdV with $\sigma = \frac{1}{6}$ at $t = 0.1$.

was used as initial pulse also gave similar results. Kameda et al[17][18] have performed experimental and numerical studies of shock waves of Heaviside type in bubbly fluids. Even in this case a smooth wave front followed by oscillations with decreasing amplitude was found. They have also shown that the amplitude as well as the period of the oscillations are effected if the distribution of bubbles in the liquid is nonuniform. For small times, see figure 5.6, we see that the peak amplitude of the shock wave front first rises but then declines as time increases. Oscillations starts near the shock and spreads quickly in the negative x-direction.

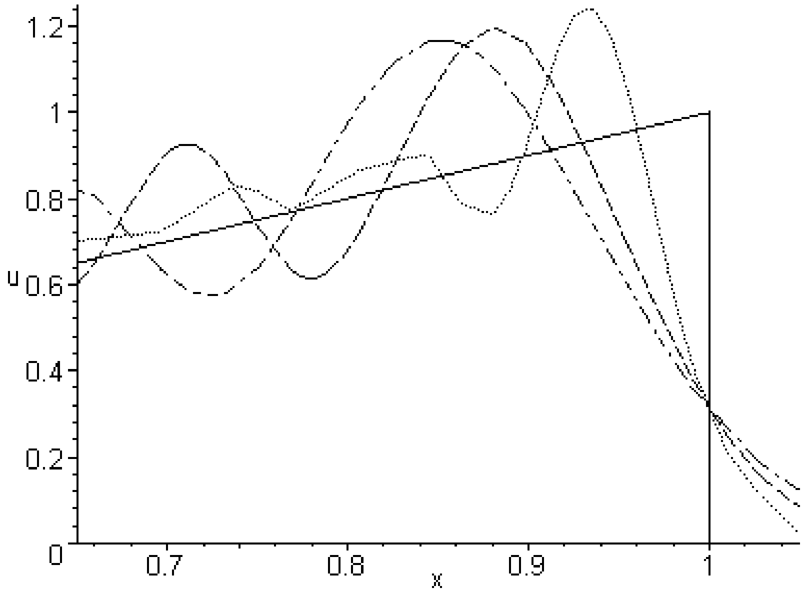

Figure 5.6. The zeroth order iteration solution for KdV with $\sigma = 0.25$ at $t = \{10^{-5}, 0.5 * 10^{-4}, 10^{-4}\}$ shows how oscillations starts near the shock and spread towards left.

Chapter 6

Asymptotic analysis of KdV

Since the analytical solution to Korteweg-de Vries' equation is somewhat complicated, we will here study the asymptotic behaviour of solutions to KdV. Even for this case an N-wave will be used as initial value, but for convenience we choose our time variable to have $T = 1$ as starting point. We therefore rewrite our problem (5.1) as

$$\begin{cases} \frac{\partial V}{\partial T} + V\frac{\partial V}{\partial x} + \sigma\frac{\partial^3 V}{\partial x^3} = 0 \\ V(x,1) = \begin{cases} x & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases} . \tag{6.1}$$

The dispersion coefficient σ is assumed to be small. In the limit that the dispersion coefficient σ is zero, we get the outer solution

$$V_0(x,T) = \begin{cases} \frac{x}{T} & |x| < T^{\frac{1}{2}} \\ 0 & |x| > T^{\frac{1}{2}} \end{cases} . \tag{6.2}$$

First we investigate the discontinuity at $x = T^{\frac{1}{2}}$. We introduce the inner variable

$$x^* = \frac{x - T^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} \tag{6.3}$$

and the expansion $V = V_0^* + \sigma^{\frac{1}{2}} V_{\frac{1}{2}}^*$. This gives, in the leading order of σ, the differential equation

$$\left(V_0^* - \frac{1}{2}T^{-\frac{1}{2}}\right)\frac{\partial V_0^*}{\partial x^*} + \frac{\partial^3 V_0^*}{\partial x^{*3}} = 0\,, \tag{6.4}$$

which is readily integrated to

$$\left(V_0^* - T^{-\frac{1}{2}}\right)\frac{V_0^*}{2} + \frac{\partial^2 V_0^*}{\partial x^{*2}} = A(T)\,. \tag{6.5}$$

This equation has the implicit solution

$$x^* = \int \frac{dV}{\sqrt{B\,(T) + 2A\,(T)\,V + \frac{T^{-\frac{1}{2}}}{2}V^2 - \frac{1}{3}V^3}}\,. \tag{6.6}$$

To achieve the matching against the outer solution (6.2) the condition

$$\lim_{x^*\to\infty} V_0^* = 0 \tag{6.7}$$

must be fulfilled. This means that $V = 0$ should be a singular point of the integral in (6.6), i e zero is a double root of the polynomial under the square root. Since two coefficients are already known, the only allowed form is

$$x^* = \int \frac{dV}{\sqrt{\frac{T^{-\frac{1}{2}}}{2}V^2 - \frac{1}{3}V^3}}\,, \tag{6.8}$$

which we can rewrite as

$$x^* = \int \frac{dV}{|V|\sqrt{\frac{T^{-\frac{1}{2}}}{2} - \frac{1}{3}V}}\,. \tag{6.9}$$

After integration we obtain

$$V_0^* = \frac{3}{2}T^{-\frac{1}{2}}\left(1 - \tanh^2\left(2^{-\frac{3}{2}}T^{-\frac{1}{4}}\left(x^* + C\,(T)\right)\right)\right) \tag{6.10}$$

Unfortunately, this does not fulfil the matching condition

$$\lim_{x^*\to-\infty} V_0^* = T^{-\frac{1}{2}}\,. \tag{6.11}$$

We return to (6.6) to see if we can find a solution for $x^* < 0$ which we could combine with (6.10) valid for $x^* > 0$ .If we rewrite (6.6) on the form

$$x^* = \int \frac{dV}{\sqrt{-\frac{1}{3}\left(V - a\,(T)\right)\left(V - b\,(T)\right)\left(V - c\,(T)\right)}}\,, \tag{6.12}$$

where

$$c\,(T) = \frac{3}{2}T^{-\frac{1}{2}} - 2\,b\,(T)\ , \tag{6.13}$$

we get the general solution

$$V_0^* = c(T) + (1 - a(T))\,\mathrm{sn}^2\left(\sqrt{\frac{2}{3}(c(T) - b(T))}\,(x^* + d(T)),\, \sqrt{\frac{c(T) - a(T)}{c(T) - b(T)}}\right), \tag{6.14}$$

where $\mathrm{sn}(z,k)$ is one of the Jacobi elliptic functions. This function is periodic[1] unless $k = 1$ and z is real ($\mathrm{sn}(x,1) = \tanh(x), x \in \Re$). In our case, this means that matching conditions only can be fulfilled if $a(T) \equiv b(T)$ and $c(T) > b(T)$. Combining this with (6.13), we see that matching is only possible if $b(T) < \frac{1}{2}T^{-\frac{1}{2}}$ and thus it is not possible to accomplish (6.11).

To understand why this matching fails, we try to get nearer the shock. Crighton & Scott[6] have made an analysis of what they call "the embryo shock region" for Burgers' equation. We will now make the corresponding investigation for Korteweg-de Vries' equation. To study this small region we introduce the scalings

$$\begin{cases} \hat{x} = \frac{x-1}{\sigma} \\ \hat{T} = \frac{T-1}{\sigma} \\ V\left(\hat{x}, \hat{T}; \sigma\right) = \hat{V}_0(x,T) + o(1) \end{cases}, \tag{6.15}$$

where $V\left(\hat{x}, \hat{T}; \varepsilon\right) = O(1)$ and σ is small. Substituting this into (6.1) we get

$$\begin{cases} \frac{\partial \hat{V}_0}{\partial \hat{T}} + \hat{V}_0 \frac{\partial \hat{V}_0}{\partial \hat{x}} + \frac{\partial^3 \hat{V}_0}{\partial \hat{x}^3} = 0 \\ \hat{V}_0(\hat{x}, 0) = 1 - \mathrm{H}(\hat{x}) \end{cases}, \tag{6.16}$$

where H is the Heaviside step function. The differential equation (6.16) cannot be solved by using the inverse scattering transform since IST demands that the potential, which here would be $\hat{V}_0(\hat{x}, 0) = 1 - \mathrm{H}(\hat{x})$, must be localized. We therefore try a "quasi-localized" potential in the form of

$$\hat{V}_0(\hat{x}, 0) = \begin{cases} e^{\alpha \hat{x}} & \hat{x} < 0 \\ 0 & \hat{x} > 0 \end{cases}, \tag{6.17}$$

where we let $\alpha \longrightarrow 0$. This gives the solution to the Schrödinger equation in the region $\hat{x} < 0$ as

$$\Psi(\hat{x}) = c_1 \mathrm{J}_{\frac{2ik}{\alpha}}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{e^{\frac{1}{2}\alpha\hat{x}}}{\alpha}\right) + c_2 \mathrm{Y}_{\frac{2ik}{\alpha}}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{e^{\frac{1}{2}\alpha\hat{x}}}{\alpha}\right). \tag{6.18}$$

To get a solution for $\alpha \longrightarrow 0$, asymptotic expansions for Bessel functions of infinitely imaginary order and infinite real argument must be found. Even though this sounds fascinating, it is probably not a good idea.

[1] Actually $\mathrm{sn}(z,k)$ is a doubly periodic function in the complex plane but here we are only interested in the behaviour for real arguments.

However it is interesting to notice the self similarity of KdV, i e the scaled KdV equation is a new KdV equation. Comparing with numerical solutions with similar conditions (e g [11] and [17]) we see that the solutions shows a fast but smooth decay for $\hat{x} > 0$ and for $\hat{x} < 0$ the solution oscillates around $\hat{V} = 1$ with the amplitude of the oscillations being largest near $\hat{x} = 0$ and declining for smaller values of $\hat{x}$.

We instead try to combine the outer solution (6.2) for $x^* < 0$ with the inner solution written as (6.10) for $x^* > 0$. This can be done by calculating the value of $C(T)$ in (6.10) which makes

$$\lim_{x^* \to 0} V_0^* = T^{-\frac{1}{2}} . \tag{6.19}$$

The result is

$$C(T) = 2^{\frac{3}{2}} T^{\frac{1}{4}} \operatorname{arctanh}\left(3^{-\frac{1}{2}}\right) , \tag{6.20}$$

i e the inner solution is

$$V_0^* = \frac{3}{2} T^{-\frac{1}{2}} \left(1 - \tanh^2\left(2^{-\frac{3}{2}} T^{-\frac{1}{4}} x^* + \operatorname{arctanh}\left(3^{-\frac{1}{2}}\right)\right)\right) \tag{6.21}$$

for $x^* > 0$. We combine this with the outer solution and rewrite it in terms of $\{x, T\}$.

$$V_0 = \begin{cases} \frac{x}{T} & x < T^{\frac{1}{2}} \\ \frac{3}{2} T^{-\frac{1}{2}} \left(1 - \tanh^2\left(2^{-\frac{3}{2}} T^{-\frac{1}{4}} \frac{x - T^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} + \operatorname{arctanh}\left(3^{-\frac{1}{2}}\right)\right)\right) & x > T^{\frac{1}{2}} \end{cases} . \tag{6.22}$$

This solution is continuous but not differentiable at $x = T^{\frac{1}{2}}$ and the combination with the outer solution means that we do not get any information about how the oscillations develop for $x < T^{\frac{1}{2}}$.

Investigating the discontinuity at $x = -T^{\frac{1}{2}}$, starting with (6.2), we find a similar behaviour as the analysis above for $x = T^{\frac{1}{2}}$. Whether it is a similarity or a difference that also at this point the matching can be done to the right, but not to the left, is not obvious. Here we use the inner variable

$$x^+ = \frac{x + T^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} \tag{6.23}$$

(cf (6.3)). Proceeding in the same way as (6.4)-(6.10), we find the inner solution

$$V_0^+ = \frac{1}{2} T^{-\frac{1}{2}} \left(1 - 3\tanh^2\left(2^{-\frac{3}{2}} T^{-\frac{1}{4}} \left(x^+ + C(T)\right)\right)\right), \tag{6.24}$$

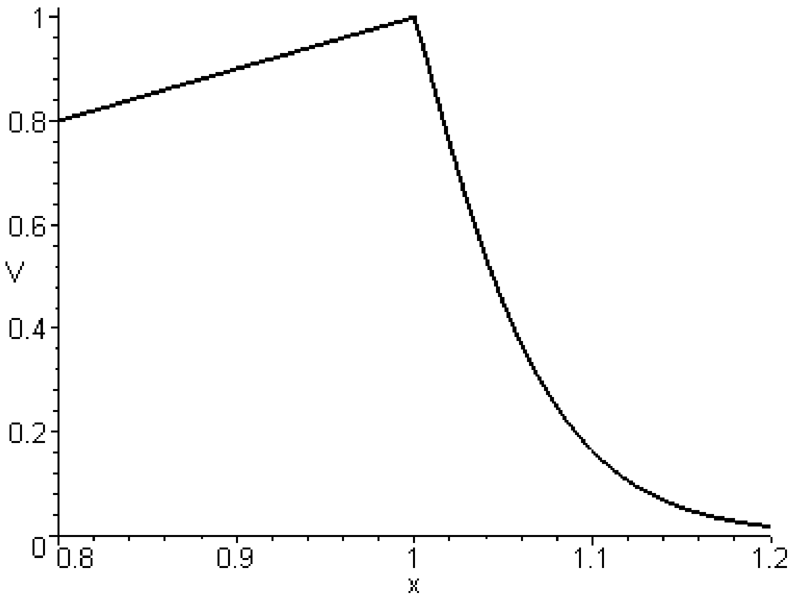

Figure 6.1. The asymptotic solution ($\sigma = 0.001, T = 1.001$) shows the decline for $x > T^{\frac{1}{2}}$ but fails to describe the oscillations for $x < T^{\frac{1}{2}}$.

which fulfils

$$\lim_{x^+ \to \infty} V_0^+ = -T^{-\frac{1}{2}} \tag{6.25}$$

but not

$$\lim_{x^+ \to -\infty} V_0^+ = 0\,. \tag{6.26}$$

As with (6.19), we now choose to let (6.24) fulfil

$$\lim_{x^+ \to 0} V_0^+ = 0\,, \tag{6.27}$$

which can be done by

$$C\,(T) = 2^{\frac{3}{2}} T^{\frac{1}{4}} \operatorname{arctanh}\left(3^{-\frac{1}{2}}\right), \tag{6.28}$$

which happens to be identical to (6.20). Thus we have the solution

$$V_0 = \begin{cases} 0 & x < -T^{\frac{1}{2}} \\ \frac{1}{2} T^{-\frac{1}{2}} \left(1 - 3\tanh^2\left(2^{-\frac{3}{2}} T^{-\frac{1}{4}} \frac{x+T^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} + \operatorname{arctanh}\left(3^{-\frac{1}{2}}\right)\right)\right) & x > -T^{\frac{1}{2}} \end{cases} \tag{6.29}$$

near $x = -T^{\frac{1}{2}}$. Also here we have combined our inner solution with the outer solution (6.2), which for $x < -T^{\frac{1}{2}}$ is identically zero. This means that here, as well as for $x = T^{\frac{1}{2}}$, the asymptotic analysis fails to describe the development of oscillations to the left of the discontinuities.

6.1 Peak region

Even though the solution obtained by using Marchenko's equation is valid for moderate values of σ, while the asymptotic analysis is valid for small σ, it is interesting to notice that the amplitude of the shock front rises before it starts to decline. We therefore investigates if it is possible to use the location of the peak of the shock front $x_0(T)$, to analyse the early stages of the break down of the shock wave. Also here we study the Korteweg-de Vries' equation (6.1) with the outer solution (6.2). We define $x_0(T)$ to be the location of the peak of the shock front, i e $x_0(T)$ is the value of x that fulfils

$$\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0\,. \tag{6.30}$$

An inner variable can now be defined as

$$\hat{x} = \frac{x - x_0(T)}{\sigma^{\frac{1}{2}}} \tag{6.31}$$

If $x_0(T) = T^{\frac{1}{2}}$, this is the same as (6.3). With the expansion $V = \hat{V}_0 + \sigma^{\frac{1}{2}}\hat{V}_1$, the new differential equation in the leading order of σ will be

$$\frac{\partial^3 \hat{V}_0}{\partial \hat{x}^3} - \frac{dx_0}{dT}\frac{\partial \hat{V}_0}{\partial \hat{x}} + \left.\frac{\partial^2 \hat{V}_0}{\partial \hat{x}^2}\right|_{\hat{x}=0} \hat{x}\hat{V}_0 = 0\,. \tag{6.32}$$

where $\frac{dx_0}{dT}$ and $\left.\frac{\partial^2 \hat{V}_0}{\partial \hat{x}^2}\right|_{\hat{x}=0}$ are parameters which may be functions of T. The values of these parameters are not known but to define a peak, we must have

$$\left.\frac{\partial^2 \hat{V}_0}{\partial \hat{x}^2}\right|_{\hat{x}=0} < 0. \tag{6.33}$$

To get some idea about the value of $\frac{dx_0}{dT}$, we look at the previously defined outer solution (6.2). For this outer solution the shock front is located at $x_0(T) = T^{\frac{1}{2}}$, which would give

$$\frac{dx_0}{dT} = \frac{1}{2}T^{-\frac{1}{2}}\,. \tag{6.34}$$

Since we are looking for a solution that could replace the outer solution (6.2) at the peak, this is probably not the correct value. However, it seems plausible to assume that

$$\frac{dx_0}{dT} > 0 \tag{6.35}$$

We thus have the ordinary differential equation

$$v'''(\hat{x}) - a_1\, v'(\hat{x}) - a_2\, x\, v(\hat{x}) = 0 \tag{6.36}$$

where a_1 and a_2 are positive. Even though it is linear, it does not have any Liouvillian ("closed form") solutions (cf Singer & Ulmer[31][32]). It is possible to transform it to a nonlinear second order differential equation, with variable coefficients, but that equation does not seem to fall into any of the classes of differential equations with known solution methods. Still, (6.36) is linear and thereby it is possible to assign a series solution on the form

$$v(\hat{x}) = \sum_{j=0}^{\infty} c_j \hat{x}^j \tag{6.37}$$

Substituting (6.37) into (6.36), we find that the coefficients c_j must fulfil

$$c_{j+3} = \frac{(j+1)\, a_1\, c_{j+1} + a_2\, c_{j-1}}{(j+1)\,(j+2)\,(j+3)} \tag{6.38}$$

where $c_0 = v(0)$ (the amplitude of the peak) , $c_1 = 0$ (according to (6.30)) and $c_2 = -a_2$ (by the notation used to write (6.32) as (6.36)). ($c_j \equiv 0 \forall j < 0$) This means that all odd number coefficients are zero, which makes our series solution symmetric. The inner solution (6.10) with $C(T) = 0$ would give the parameters

$$\begin{cases} c_0 = \frac{3}{2} T^{-\frac{1}{2}} \\ a_1 = \frac{1}{2} T^{-\frac{1}{2}} \\ a_2 = \frac{3}{8} T^{-1} \end{cases} . \tag{6.39}$$

This makes (6.37) with (6.38) an accurate description of the soliton-like function (6.10) with $C(T) = 0$ near $x = T^{\frac{1}{2}}$.

6.2 Shock tail region

The solutions (6.22) and (6.29) from the asymptotic analysis above, is valid only near the shocks. To investigate the shock tail region for KdV, we consider the variable (3.19) used for Burgers' equation and the difference between the inner variables (3.11) and (6.3). We therefore try the scaling

$$Y = \frac{1}{2} \frac{1}{\sigma^{\frac{1}{2}}} \frac{x - T^{\frac{1}{2}}}{T^{\frac{1}{2}}} . \tag{6.40}$$

(Note that σ is small) With this substituted into (6.1) we get, in the leading order of σ, the equation

$$-\frac{1}{4T} \frac{\partial V^*}{\partial Y} + \frac{1}{2} T^{-\frac{1}{2}}\, V^* \frac{\partial V^*}{\partial Y} + \frac{1}{8} T^{-\frac{3}{2}} \frac{\partial^3 V^*}{\partial Y^3} = 0 \,. \tag{6.41}$$

If we assume that $V^* = O(\varepsilon)$ where $\varepsilon \ll 1$, which is fulfilled if the value of Y is large enough, this is reduced to

$$-\frac{1}{4T}\frac{\partial V^*}{\partial Y} + \frac{1}{8}T^{-\frac{3}{2}}\frac{\partial^3 V^*}{\partial Y^3} = 0 \tag{6.42}$$

in the leading order of $\{\sigma, \varepsilon\}$. (Actually we define the shock tail region as the set of Y:s with large enough values) The general solution of (6.42) is

$$V^*(Y,T) = C_1(T) + C_2(T)\, e^{\sqrt{2}T^{\frac{1}{4}}Y} + C_3(T)\, e^{-\sqrt{2}T^{\frac{1}{4}}Y}. \tag{6.43}$$

Demanding that

$$V^*(Y,T) \underset{Y\to\infty}{\longrightarrow} 0 \tag{6.44}$$

we must have $C_1(T) = 0$ as well as $C_2(T) = 0$ and thus

$$V^*(Y,T) = C_3(T)\, e^{-\sqrt{2}T^{\frac{1}{4}}Y} = C_3(T)\exp\left(-\frac{1}{\sqrt{2}\sigma}\frac{x - T^{\frac{1}{2}}}{T^{\frac{1}{4}}}\right). \tag{6.45}$$

Comparing with (6.21), we see that $C_3(T) \propto T^{-\frac{1}{2}}$.
($C_3(T) = 6T^{-\frac{1}{2}}\exp\left(-\sqrt{2}\operatorname{arctanh}\left(\frac{1}{\sqrt{3}}\right)\right) \approx 2.36\, T^{-\frac{1}{2}}$)

Chapter 7

Asymptotic analysis of KdVB

Since we do not have an analytical solution to Korteweg-de Vries-Burgers' equation, we will here study the asymptotic behaviour of solutions to KdVB. Even here an N-wave will be used as initial value. (Note that $T = 1$ is used as the initial time.)

$$\left\{ \begin{array}{c} \frac{\partial V}{\partial T} + V\,\frac{\partial V}{\partial x} - \delta\,\frac{\partial^2 V}{\partial x^2} + \sigma\,\frac{\partial^3 V}{\partial x^3} = 0 \\ V(x,1) = \left\{ \begin{array}{ll} x & |x| < 1 \\ 0 & |x| > 1 \end{array} \right. \end{array} \right. . \tag{7.1}$$

Here we have two coefficients (δ and σ) which we consider to be small. This is however not enough information. We must also determine how large the coefficients are with respect to each other since if one of them gives a larger contribution than the other we will only get one of the solutions for either Burgers' or Korteweg-de Vries' equation as analysed above. In the limit that both coefficients are zero, we get the outer solution

$$V_0(x,T) = \left\{ \begin{array}{ll} \frac{x}{T} & |x| < T^{\frac{1}{2}} \\ 0 & |x| > T^{\frac{1}{2}} \end{array} \right. . \tag{7.2}$$

We find that in order to get a differential equation for the asymptotic analysis which contains dissipation as well as dispersion, we must have $\sigma = o\left(\varepsilon^2\right)$ if $\delta = o\left(\varepsilon\right)$ where ε is small. We introduce the inner variable $x^* = \frac{x - T^{\frac{1}{2}}}{\varepsilon}$ and the expansion $V = V_0^* + \varepsilon V_1^*$. This gives, in the leading order of ε, the differential equation

$$\left(V_0^* - \frac{1}{2}T^{-\frac{1}{2}}\right)\frac{\partial V_0^*}{\partial x^*} - a\frac{\partial^2 V_0^*}{\partial x^{*2}} + b\frac{\partial^3 V_0^*}{\partial x^{*3}} = 0\,, \tag{7.3}$$

where we have chosen to use $\delta = a\varepsilon$ and $\sigma = b\varepsilon^2$ with $\{a, b\} = o(1)$. With $\widetilde{V} = V_0^* - \frac{1}{2}T^{-\frac{1}{2}}$, this can be written on the form

$$\widetilde{V}\frac{\partial \widetilde{V}}{\partial x^*} - a\frac{\partial^2 \widetilde{V}}{\partial x^{*2}} + b\frac{\partial^3 \widetilde{V}}{\partial x^{*3}} = 0\,. \tag{7.4}$$

Since all terms may be described as partial derivatives with respect to x^*, we can integrate it to

$$\frac{1}{2}\widetilde{V}^2 - a\frac{\partial \widetilde{V}}{\partial x^*} + b\frac{\partial^2 \widetilde{V}}{\partial x^{*2}} = C(T)\,. \tag{7.5}$$

Despite that this is an ordinary differential equation with T as a parameter, it is hard to find a solution. To illustrate that this equation is not easily solved we note that Leach[23], then trying to find first integrals for the modified Emden equation

$$q'' + \alpha\,(x)\; q' + q^n = 0 \tag{7.6}$$

(for $\alpha\,(x) \neq \frac{k}{x}$), states that "The case $n = 2$ is seen to be particularly difficult to solve.".

However, the equation (7.5) can be rewritten with an implicit solution because the dependence of x^* is only through $\widetilde{V}$ and its derivatives

$$x^* = \int \frac{d\widetilde{V}}{p\left(\widetilde{V}\right)}\,, \tag{7.7}$$

but here $p\left(\widetilde{V}\right)$ is the solution to

$$p(\widetilde{V})\left(\frac{\partial p}{\partial \widetilde{V}} - \frac{a}{b}\right) + \frac{1}{2b}(\widetilde{V}^2 + 2C\,(T)) = 0\,, \tag{7.8}$$

which is one form of Abel's differential equation. This equation is not, as far as we know, more easily solved than (7.5) and thus we probably do not want to integrate this solution according to (7.7) even if we could find it. We therefore turn our attention back to (7.5) but we will try to solve it using Abel's identity.

7.1 Abel's identity

Abel's identity[1] states that for a system of two first order linear, homogenous differential equations

$$\begin{cases} \dot{x} = p(t)\ y + q(t)\ x \\ \dot{y} = r(t)\ y + s(t)\ x \end{cases}, \tag{7.9}$$

which has the general solution

$$\begin{cases} x = \alpha\, x_1 + \beta\, x_2 \\ y = \alpha\, y_1 + \beta\, y_2 \end{cases}, \tag{7.10}$$

the Wronskian of this system ($\mathrm{W} = x_1\, y_2 - x_2\, y_1$) fulfils

$$\mathrm{W}(t) = \mathrm{W}(t_0)\ \exp\left(\int_{t_0}^{t} \left[q(t') + r(t')\right]\, dt'\right) \tag{7.11}$$

An extension of Abel's identity to nonlinear differential equations[15] and a method to use this to construct solutions to nonlinear autonomous systems[16] was presented in two articles by Jones et al. We will now use this method for our problem. In an attempt to avoid confusion, we will use their notation while describing the solution process and return to our variables later. To be able to use this method the nonlinear system should have the form

$$\begin{cases} \dot{x} = F(x, y) \\ \dot{y} = G(x, y) \end{cases}, \tag{7.12}$$

where

$$F_x + G_y = \mu\,. \tag{7.13}$$

(μ is a constant). To get (7.5) on this form we first formally rewrite it as a system of two differential equations.

$$\begin{cases} \frac{\partial}{\partial x^*}\widetilde{V} = \frac{\partial \widetilde{V}}{\partial x^*} \\ \frac{\partial}{\partial x^*}\frac{\partial \widetilde{V}}{\partial x^*} = \frac{a}{b}\frac{\partial \widetilde{V}}{\partial x^*} - \frac{1}{2\,b}\widetilde{V}^2 + \frac{C(T)}{b} \end{cases}. \tag{7.14}$$

Thereafter we introduce the new notation

$$\begin{cases} x = \widetilde{V} \\ y = \frac{\partial \widetilde{V}}{\partial x^*} \\ t = x^* \end{cases} \tag{7.15}$$

and thus get the system on the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{a}{b}\, y - \frac{1}{2\,b} x^2 - c \end{cases}. \tag{7.16}$$

Theory

We will here quote the relevant theory in chapter 4 of [16], where the construction of integrals (of motion) for systems of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu\, y + g\,(x) \end{cases} \tag{7.17}$$

is described. A first integral on the form

$$h\,(x, y) = c_1 \tag{7.18}$$

can be found by integrating the differential equation

$$y\, y' = \mu\, y + g\,(x)\ . \tag{7.19}$$

If (7.18) is an integral of this equation, $h\,(x, y)$ satisfies the partial differential equation

$$y\, h_x + (\mu\, y + g\,(x))\ h_y = 0\,. \tag{7.20}$$

To simplify the construction of a solution to this equation, a new dependent variable $z = z\,(x, y)$ is introduced. The relation between z and h is defined to be

$$h\,(x, y) = A\,(x) \int_{y_0}^{y} u\, z\,(x, u)\ du + B\,(x)\ , \tag{7.21}$$

where the functions $A\,(x)$ and $B\,(x)$ are to be determined. Here y_0 can be chosen arbitrary. The function $B\,(x)$ fulfils

$$B\,(x) = -\int_{x_0}^{x} A\,(s)\ z\,(s, y_0)\ (\mu\, y_0 + g\,(s))\ ds\,. \tag{7.22}$$

Here the choice of x_0 is free. Substituting (7.21) into (7.20), the equation

$$y\, z_x + (\mu\, y + g\,(x))\ z_y + \left(\mu + \frac{A'}{A}\, y\right)\ z = 0 \tag{7.23}$$

can be derived. Also the factor $\frac{A'(x)}{A(x)}$ can be chosen freely. In the special cases that $z = w^n$ where n is a constant, (7.23) can be written

$$n\, y\, w_x + n\ (\mu\, y + g\,(x))\ w_y + \left(\mu + \frac{A'}{A}\, y\right)\ w = 0\,. \tag{7.24}$$

Besides (7.18), a second independent integral of (7.17) can be found on the form

$$\frac{y}{h_y} = c_2 \exp\,(\mu\, t)_{.} \tag{7.25}$$

Keeping (7.21) in mind this means that if $z(x, y)$ is found, this integral can be written

$$A(x)\ z^{-1} = c_2 \exp(\mu\, t)_. \tag{7.26}$$

In [16], also a solution to the (carefully chosen) example

$$g(x) = -\frac{2}{9}\mu^2\, x + \varepsilon\, x^3 \tag{7.27}$$

is stated. For $n = -\frac{3}{4}$ and $A(x) \equiv 1$ a solution of (7.24) (with (7.27)) is

$$w = y^2 - \frac{2}{3}\mu\, x\, y + \frac{1}{9}\mu^2\, x^2 - \frac{1}{2}\varepsilon\, x^4\,. \tag{7.28}$$

This means that

$$z(x, y) = \frac{1}{\left(y^2 - \frac{2}{3}\mu\, x\, y + \frac{1}{9}\mu^2\, x^2 - \frac{1}{2}\varepsilon\, x^4\right)^{\frac{3}{4}}} \tag{7.29}$$

satisfies (7.23) with (7.27) if $A(x) \equiv 1$ is chosen.

Application

Comparing our system (7.16) with the general formulation (7.17), we see that for our case

$$g(x) = -\,c - \frac{1}{2\,b}\, x^2\,. \tag{7.30}$$

Since (7.30) resembles (7.27) but one order lower (in x) in both terms, it seems reasonable to investigate a solution on a form similar to (7.28). We therefore make the ansatz

$$w = y^2 + c_{01}\, y + c_{11}\, x\, y + c_0 + c_{10}\, x + c_{20}\, x^2 + c_{30}\, x^3\,. \tag{7.31}$$

Substituting this into (7.24) and collecting the terms in powers of x and y, we see that a first requirement ("the coefficient of y^3"=0) is

$$\frac{A'}{A} = 0 \tag{7.32}$$

so also in this case $A(x) = 1$ is a convenient choice. We use this in the expression and repeats this process for all the unknowns (n and c included). It turns out that $n = \frac{5}{6}$ is the value which gives a solution here. We end up with a solution, to (7.23) with (7.30), which has the somewhat complicated form

$$z(x, y) = \frac{1}{\left(y^2 + \frac{24}{125}\frac{a^3}{b^2}y - \frac{4}{5}\frac{a}{b}xy + \frac{72}{15625}\frac{a^6}{b^4} - \frac{12}{625}\frac{a^4}{b^3}x - \frac{2}{25}\frac{a^2}{b^2}x^2 + \frac{1}{3b}x^3\right)^{\frac{5}{6}}}. \tag{7.33}$$

This can be rewritten as

$$z(x, y) = \frac{1}{\left(y^2 - \frac{4}{5}\frac{a}{b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)y + \frac{1}{3b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)^2\left(x + \frac{6}{25}\frac{a^2}{b}\right)\right)^{\frac{5}{6}}}. \tag{7.34}$$

First integral

The first integral, $h(x,y) = c_1$, can now be found by using the solution (7.34) in (7.21) and (7.22). This gives us the expression

$$h(x,y) = \int_{y_0}^{y} \frac{u}{\left(u^2 - \frac{4}{5}\frac{a}{b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)u + \frac{1}{3b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)^2\left(x + \frac{6}{25}\frac{a^2}{b}\right)\right)^{\frac{5}{6}}}\,du + B(x) \tag{7.35}$$

where

$$B(x) = -\int_{x_0}^{x} \frac{\left(\frac{a}{b}\,y_0 - \frac{1}{2b}\left(s^2 - \frac{36}{625}\frac{a^4}{b^2}\right)\right)}{\left(y_0^2 - \frac{4}{5}\frac{a}{b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)y_0 + \frac{1}{3b}\left(s - \frac{6}{25}\frac{a^2}{b}\right)^2\left(s + \frac{6}{25}\frac{a^2}{b}\right)\right)^{\frac{5}{6}}}\,ds\,. \tag{7.36}$$

In this case $y_0 = 0$ seems to be a convenient choice as it simplifies (7.36) to the form

$$B(x) = \int_{x_0}^{x} \frac{\frac{1}{2b}\left(s + \frac{6}{25}\frac{a^2}{b}\right)\left(s - \frac{6}{25}\frac{a^2}{b}\right)}{\left(\frac{1}{3b}\left(s - \frac{6}{25}\frac{a^2}{b}\right)^2\left(s + \frac{6}{25}\frac{a^2}{b}\right)\right)^{\frac{5}{6}}}\,ds\,. \tag{7.37}$$

The integration of this integral can be performed in two steps. The first is

$$\int \frac{(s+k)(s-k)}{\left((s+k)(s-k)^2\right)^{\frac{5}{6}}}ds = -2\frac{(s+k)(s-k)}{\left((s+k)^5(s-k)^4\right)^{\frac{1}{6}}} - \frac{2k}{3}\int \frac{ds}{\left((s+k)^5(s-k)^4\right)^{\frac{1}{6}}} \tag{7.38}$$

and the second

$$\int \frac{ds}{\left((s+k)^5(s-k)^4\right)^{\frac{1}{6}}} = 3\left(\frac{2}{k}\right)^{\frac{1}{2}}\left(\frac{s+k}{2k}\right)^{\frac{1}{6}}\,{}_2\mathrm{F}_1\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};\frac{s+k}{2k}\right) \tag{7.39}$$

where F is the hypergeometric function

$${}_2\mathrm{F}_1(p,q;r;\zeta) = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)}\sum_{n=0}^{\infty}\frac{\Gamma(n+p)\Gamma(n+q)}{\Gamma(n+r)}\frac{\zeta^n}{n!}\,. \tag{7.40}$$

Combining (7.38) and (7.39) with (7.37), we get the expression

$$B(x) = -\frac{3^{\frac{5}{6}}}{b^{\frac{1}{6}}}(x+k)^{\frac{1}{6}}(x-k)^{\frac{1}{3}} - (2k)^{\frac{4}{3}}(x+k)^{\frac{1}{6}}\,{}_2\mathrm{F}_1\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};\frac{x+k}{2k}\right) \tag{7.41}$$

where $k = \frac{6}{25}\frac{a^2}{b}$. Here we have chosen x_0 so that $B(x_0) = 0$. Another possibility is to choose $x_0 = 0$ since $B(0)$, according to the expression above, is a constant. This constant can then be considered to be a part of c_1 in (7.18).

In a similar manner the integration of the integral term in (7.35), can be performed as

$$\int \frac{u}{(u^2+\alpha u+\beta)^{\frac{5}{6}}}\,du = $$
$$= 3\left(u^2+\alpha u+\beta\right)^{\frac{1}{6}} - \frac{\alpha}{2}\frac{u+\frac{\alpha}{2}}{\left(-\frac{1}{4}\alpha^2+\beta\right)^{\frac{5}{6}}}\,{}_2\mathrm{F}_1\left(\frac{1}{2},\frac{5}{6};\frac{3}{2};\frac{\left(u+\frac{\alpha}{2}\right)^2}{\frac{\alpha^2}{4}-\beta}\right) \tag{7.42}$$

where

$$\alpha = -\frac{4}{5}\frac{a}{b}\left(x - \frac{6}{25}\frac{a^2}{b}\right) \tag{7.43}$$

and

$$\beta = w(x,0) = \frac{1}{3b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)^2\left(x + \frac{6}{25}\frac{a^2}{b}\right). \tag{7.44}$$

From (7.35), (7.42) and (7.41) we now have the first integral as

$$\begin{aligned} h(x,y) = 3\left(y^2+\alpha y+\beta\right)^{\frac{1}{6}} - \frac{\alpha}{2}\frac{y+\frac{\alpha}{2}}{\left(-\frac{1}{4}\alpha^2+\beta\right)^{\frac{5}{6}}}\,{}_2\mathrm{F}_1\left(\frac{1}{2},\frac{5}{6};\frac{3}{2};\frac{\left(y+\frac{\alpha}{2}\right)^2}{\frac{\alpha^2}{4}-\beta}\right) - 3\beta^{\frac{1}{6}} + \\ + \frac{\frac{1}{4}\alpha^2}{\left(-\frac{1}{4}\alpha^2+\beta\right)^{\frac{5}{6}}}\,{}_2\mathrm{F}_1\left(\frac{1}{2},\frac{5}{6};\frac{3}{2};\frac{\frac{1}{4}\alpha^2}{\frac{\alpha^2}{4}-\beta}\right) - 3\beta^{\frac{5}{6}} + \\ - (2k)^{\frac{4}{3}}(x+k)^{\frac{1}{6}}\,{}_2\mathrm{F}_1\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};\frac{x+k}{2k}\right) = c_1 \end{aligned} \tag{7.45}$$

where we also note that

$$\begin{cases} \frac{1}{4}\alpha^2 = \frac{1}{3b}2k(x-k)^2 \\ -\frac{1}{4}\alpha^2+\beta = \frac{1}{3b}(x-k)^3 \end{cases}. \tag{7.46}$$

Second integral

The construction of the second integral, i e the one on the form $\frac{y}{h_y} = c_2\exp(\mu\, t)$, is considerably more straight forward. It is found directly from our solution $z(x,y)$ (7.34). This since the defined relation (7.21) between h and z makes

$$\frac{y}{h_y(x,y)} = \frac{1}{z(x,y)}. \tag{7.47}$$

Thus we can write the second integral as

$$\left(y^2 - \frac{4}{5}\frac{a}{b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)y + \frac{1}{3b}\left(x - \frac{6}{25}\frac{a^2}{b}\right)^2\left(x + \frac{6}{25}\frac{a^2}{b}\right)\right)^{\frac{5}{6}} = c_2 \exp\left(\frac{a}{b}\,t\right). \tag{7.48}$$

It seems a bit redundant use the notations α and β to get this integral on a more compact form. However the two integrals are both needed to construct the solution for the system. Thus we note that the second integral can be written on the fairly compact form

$$\left(y^2 + \alpha y + \beta\right)^{\frac{5}{6}} = c_2 \exp\left(\frac{a}{b}\,t\right). \tag{7.49}$$

7.2 Construction of solution

With the two independent integrals (7.45) and (7.48), we now have a general solution to our system. However, it is not completely transparent what this solution looks like. We will now try to make this somewhat clearer. Comparing the two equations, it appears to be reasonable to use the second integral to eliminate the variable y from the first integral. First we rewrite (7.49) as

$$y^2 + \alpha y + \beta = c_3 \exp\left(\frac{6}{5}\frac{a}{b}\,t\right) \tag{7.50}$$

and thereafter solve this for y.

$$y = -\frac{1}{2}\alpha \pm \sqrt{\frac{1}{4}\alpha^2 - \beta + c_3 e^{\frac{6}{5}\frac{a}{b}t}} \tag{7.51}$$

The left hand side of (7.50) can be found in the first term of (7.45) and in the second term the dependence on y only occurs at the form $y + \frac{1}{2}\alpha$. We now use (7.50) and (7.51) to write (7.45) as

$$\begin{aligned} &h(x,y) = \\ &3\,c_3^{\frac{1}{6}} \exp\left(\frac{1}{5}\frac{a}{b}\,t\right) \mp \frac{\alpha}{2}\frac{\left(\frac{1}{4}\alpha^2 - \beta + c_3 e^{\frac{6}{5}\frac{a}{b}t}\right)^{\frac{1}{2}}}{\left(-\frac{1}{4}\alpha^2 + \beta\right)^{\frac{5}{6}}}\, {}_2\mathrm{F}_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; \frac{\frac{1}{4}\alpha^2 - \beta + c_3 e^{\frac{6}{5}\frac{a}{b}t}}{\frac{1}{4}\alpha^2 - \beta}\right) + \\ &\quad - 3\beta^{\frac{1}{6}} + \frac{\frac{1}{4}\alpha^2}{\left(-\frac{1}{4}\alpha^2 + \beta\right)^{\frac{5}{6}}}\, {}_2\mathrm{F}_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; \frac{\frac{1}{4}\alpha^2}{\frac{1}{4}\alpha^2 - \beta}\right) - 3\beta^{\frac{5}{6}} + \\ &\quad - (2k)^{\frac{4}{3}}(x+k)^{\frac{1}{6}}\, {}_2\mathrm{F}_1\left(\frac{1}{6}, \frac{2}{3}; \frac{7}{6}; \frac{x+k}{2k}\right) = c_1\,. \end{aligned} \tag{7.52}$$

The next step is to solve this for $x(t)$, note that $\alpha = \alpha(x)$ and $\beta = \beta(x)$, and determine the constants c_1 and c_3. We have thus converted our differential equation (7.5) to the algebraic equation (7.52). ($\widetilde{V}(x^*) = x(t)$)

Chapter 8

Summary and conclusions

We have investigated the behaviour of initial N-waves in media for which the wave propagation is nonlinear. The study of N-waves as initial condition was partly motivated by that if neither dispersion nor dissipation occurs, N-waves will be formed after long times regardless of the initial waveform. Korteweg-de Vries-Burgers' equation offers a fairly general description of nonlinear wave propagation for which the effects of dispersion and dissipation is important.

For the special case that the dissipation is zero, i e as described by Korteweg-de Vries' equation, we have presented an analytical solution using the inverse scattering transform. This in the form of a zeroth order iteration solution of Marchenko's equation. In the zeroth order, this method gives accurate solutions only for values of the dispersion coefficient σ which are not too small. The solution shows a quickly decaying shock front, which leaves an oscillating tail behind. This solution agrees well with numerical and experimental investigations by others. The solution contains a continuous and a discrete spectrum. For smaller values of σ the discrete part becomes more dominant and a consists of larger number of solitons. The numerical integration, performed in the zeroth order iteration solution, demands a very large number of terms to describe the shock. This means that higher order iteration solutions become computational expensive.

For small values of the dispersion coefficient we also have given an asymptotic solution, for KdV with an N-wave as the initial waveform, describing the decay of the shock front. In the shock tail region this solution is reduced to an exponential decay. Contrary to the asymptotic analysis of Burgers' equation, for KdV it is not possible to obtain an asymptotic matching between the inner and outer solutions. For the two discontinuities of the N-wave, the matching can in both cases be performed in the positive direction but not in the negative. Probably this is caused by the oscillatory behaviour in those parts.

In most cases, an asymptotic analysis of KdVB leads either to the asymptotic solution of Burgers' equation or to the asymptotic solution of KdV. The exception is if the dispersion coefficient σ and the dissipation coefficient δ are related by $\delta = o(\varepsilon)$ and $\sigma = o\left(\varepsilon^2\right)$ where ε is small. For this special case we have constructed two independent integrals which gives an implicit form of the asymptotic solution. This was done using an extension of Abel's identity.

Chapter 9

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