Study questions
2010-03-12

1. Describe the Euler and the Lagrange coordinate systems and derive the expression for rate of change of a given quantity F in the Euler coordinate system.

2. State the definition of the material derivative, $\frac{D}{Dt}$, and explain the meaning of its different components.

3. State the compressible continuity, momentum and energy equations in nondimensional form and give the definition of the flow parameters.

4. What is the definition of a Newtonian fluid? Give the expression for viscous stress tensor for such fluid (define the coefficients involved). What is the Stokes hypothesis in this context?

5. Consider the incompressible Navier-Stokes equation in 2D:

   a) derive the boundary-layer equation in x-direction,

   b) which condition for the pressure field is found from the boundary-layer approximations applied to the y-momentum equation?

6. Describe when a system of partical differential equations together with initial and boundary conditions is well posed. Give one example of an equation with initial and boundary conditions that is well posed, and one example that is not well posed.

7. Show that the heat equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

   is parabolic.

8. Describe in words how the solution of elliptic, parabolic and hyperbolic equations behave, *i.e.* what physical processes are described by such equations. Discuss which of the types allow wave-like solutions.

9. Derive the difference formula and the corresponding leading error term for the second derivative using a central three-point scheme. What is the order of that scheme?

10. Consider the integration of an ordinary differential equation $u' = \lambda u$.

    a) (2p) Write down the explicit and implicit first-order Euler discretisation. Discuss briefly the main difference between the two methods, and comment on advantages and disadvantages.

b) (4p) Derive the region of absolute stability for the **explicit** Euler scheme. Sketch the solution in the complex plane $z = \lambda\Delta t$. Is the scheme absolutely stable?

c) (1p) Set $\lambda = -1$. Which integration scheme(s) would you use and what is the maximum possible time step? Motivate your answer.

11. Derive the region of absolute stability for the implicit Euler scheme using the test equation $u' = \lambda u$. Sketch the solution in the complex plane $z = \lambda\Delta t$. Is the scheme absolutely stable?

12. You want to solve the ordinary differential equation $u' = \lambda u$ with $\lambda = 3\sqrt{-1}$. Which integration scheme(s) would you use? Motivate your answer.

13. Consider the advection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \ .$$

Discretise this equation in space using central schemes and in time using the explicit first-order Euler scheme.

a) Write down the discretised equation. Use as abbrevations $\sigma = a\Delta t/\Delta x$ and $\beta = \nu\Delta t/\Delta x^2$.

b) Perform a von-Neumann stability analysis using the Fourier modes $\hat{u}_\xi e^{\mathrm{i}\xi x}$. In particular, compute the amplification factor $\hat{G}(\xi\Delta x)$. A condition for stability depending on β alone can then be derived by setting $\xi\Delta x = \pi$. Another condition relating σ and β can be found close to $\xi\Delta x = 0$ using the expansions $\sin\xi\Delta x \approx \xi\Delta x$ and $\cos\xi\Delta x - 1 \approx -\frac{1}{2}(\xi\Delta x)^2$. Neglect terms of order $(\xi\Delta x)^4$ and above. What are then the conditions for stability in terms of σ and β?

c) Can you derive an explicit condition for the maximum time step? Does it depend on the spatial grid spacing Δx?

14. On the example of the flow around an airplane wing, discuss – as a function of the angle of attack – the regions in which viscosity is important and regions which can be treated inviscidly. Which equations would you use in the different regions of the flow?

15. Write down the compressible Euler equations in conservative form. Briefly discuss the physical meaning of the individual terms, and the physical concept that leads to the formulation of the equations. Write down the system in such a way that it is completely closed, *i.e.* the same number of equations as the number of unknowns. Of what type are the unsteady Euler equations (no derivation needed)? What are the conservative variables? What are the primitive variables?

16. Define and use the Rankine-Hugoniot jump condition to compute the shock speed for the following problem

$$\begin{aligned} u_t + uu_x &= 0 \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

How would the solution look like if the initial condition is reversed, *i.e.*

$$u(x,0) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

17. Define the entropy condition for a scalar conservation law.

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

with a convex flux function $f(u)$. The shock is moving with speed s and the state to the left is given by u_L and the state to the right by u_R.

Why do we need an entropy condition ?

18. Investigate the one-sided difference scheme

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

for the advection equation

$$u_t + au_x = 0$$

Consider the cases $a > 0$ and $a < 0$.

a) Prove that the scheme is consistent and find the order of accuracy. Assume $\Delta t/\Delta x$ constant.

b) Determine the stability requirement for $a > 0$ and show that it is unstable for $a < 0$.

19. Apply the Lax-Friedrichs scheme to the advection equation

$$u_t + au_x = 0$$

that is,

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

a) Write down the modified differential equation.

b) What type of equations is this?

c) What kind of behavior can we expect from the solution?

20. Sketch the effect of diffusive and dispersive errors on the advection of a top-hat (a signal with discontinuity) signal. What terms are known to cause such errors? If you consider the advection of a pure sine wave, what are the effects of diffusive and dispersive errors?

21. The three-point centered scheme applied to

$$u_t + a u_x = 0, \quad a > 0$$

yields the approximation

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{2\Delta x}(u_{j+1} - u_{j-1})$$

Show that this approximation is not stable even though the CFL condition is fulfilled.

22. What does Lax(–Richtmyer) equivalence theorem state?

23. What is the condition on the $n \times n$ real matrix $A(\mathbf{u})$ for the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

to be hyperbolic?

24. The barotropic gas dynamic equations

$$\begin{aligned} \rho_t + \rho u_x &= 0 \\ u_t + u u_x + \frac{1}{\rho} p_x &= 0 \end{aligned} \qquad (1)$$

where

$$p = p(\rho) = C\rho^{\gamma}$$

and C a constant, can be linearized by considering small perturbations (ρ', u') around a motionless gas.

a) Let $\rho = \rho_0 + \rho'$ and $u = u_0 + u'$ where $u_0 = 0$. Linearize the system (1) and show that this yields the following linear system (the primes have been dropped)

$$\begin{aligned} \rho_t + \rho_0 u_x &= 0 \\ u_t + \frac{a^2}{\rho_0}\rho_x &= 0 \end{aligned} \qquad (2)$$

where a is the speed of sound. a and ρ_0 are constants.

b) Is the system given by (2) a hyperbolic system? Motivate your answer.

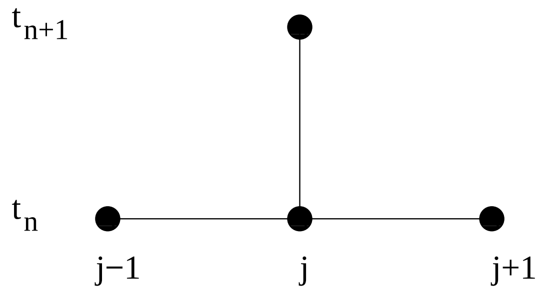
c) Determine the characteristic variables in terms of ρ and u.

d) Determine the partial differential equations that are fulfilled by the characteristic variables, *i.e.* the characteristic formulation.

e) Let $-\infty < x < \infty$ (no boundaries) and the initial conditions at $t = 0$ are

$$\rho(0, x) = \sin(x) \quad u(0, x) = 0 \ .$$

Determine the analytical solution of equation (2) for $t > 0$. Hint: Start from the characteristic formulation.



25. The linearized form of the barotropic gas dynamics equations (1) is given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \qquad (3)$$

where a is the speed of sound. a and ρ_0 are constants.

a) Draw the domain of dependence of the solution to the system (3) in a point P in the x-t plane.

b) The system is solved numerically on a grid given by $x_j = j\Delta x, j = 0, 1, 2...$ and $t_n = n\Delta t, n = 0, 1, 2, ...$ using an explicit three-point scheme, see the figure below.

Draw the domain of dependence of the numerical solution at P (in the same figure as a)) of the three-point scheme in the case when

i) the CFL condition is fulfilled

ii) the CFL condition is NOT fulfilled.

Assume that P is a grid point.

26. Consider the Euler equations in 1D

$$\begin{aligned} \rho_t + \rho u_x + u\rho_x &= 0 \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0 \\ p_t + \rho c^2 u_x + up_x &= 0 \ . \end{aligned}$$

How many boundary conditions must be added at the

inflow boundary when the flow is

a) Supersonic

b) Subsonic

outflow boundary when the flow is

c) Supersonic

d) Subsonic

Motivate your answer!

27. Consider the shock tube problem described by the isentropic Euler equations in one space dimension:

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix}_x = 0 \ . \qquad (4)$$

    At $t = 0$ a membrane is separating a region with a high-pressure gas from a region with gas at a lower pressure.

    a) Describe how the solution is evolving as a function of time once the membrane is removed.

    b) What type of discontinuity is excluded when solving equation (4) instead of the full Euler equations, and why?

28. Give at least one reason for using artificial viscosity when solving a conservation law using the MacCormack scheme. Why does one not use any artificial viscosity in an upwind discretisation?

29. Projection on a divergence-free space

    a) Show that a vector field w_i can be decomposed into

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

    where u is divergence free and parallel to the boundary.

    b) Apply this to the Navier-Stokes equations, show that the pressure term disappears and recover an equation for the pressure from the gradient part.

30. From the differential form of the Navier-Stokes equations obtain the Navier-Stokes equations in integral form used in finite-volume discretizations.

31. Finite volume (FV) discretization

    (a) Derive the finite volume (FV) discretization on arbitrary grids of the continuity equation ($\partial u_i / \partial x_i = 0$),

    (b) derive the FV discretization for the Laplace equation on a Cartesian grid,

    (c) show that both are equivalent to a central difference approximation for Cartesian grids.

32. State the difficulties associated with the finite-volume discretizations of the Navier-Stokes equations on a co-located grid and show the form of the spurious solution which exist.

33. Staggered grid

    (a) Define an appropriate staggered grid that can be used for the discretization of the Navier-Stokes equations,

(b) write down the FV discretization of the Navier-Stokes equations on a staggered cartesian grid,

(c) discuss how to treat noslip and inflow/outflow boundary conditions.

34. Time dependent flows.

(a) Define a simple projection method for the time dependent incompressible Navier-Stokes equations

$$\frac{d}{dt}\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{N}(\mathbf{u}) & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

(b) show in detail the equation for the pressure to be solved at each time step and discuss the boundary conditions for the pressure.

35. Iterative techniques for linear systems.

(a) Define Gauss-Seidel iterations for the Laplace equation, give the convergence rate and derive an approximation for number of iterations required for error reduction of $\mathcal{O}(h^2)$.

(b) Describe the idea behind multigrid methods.

(c) Dedcribe the 2-level multigrid method for the Laplace equation.

36. Coordinate transformation

(a) Define the coordinate transformation from a Cartesian one (x, y, z) to a general one (ξ, η, ζ). State the Jacobian matrix of transformation and describe a practical way of computing it.

(b) Derive the transformation of the 2D Navier-Stokes equations

$$\text{from} \quad \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \quad \text{to} \quad \frac{\partial \mathbf{U}'}{\partial t} + \frac{\partial \mathbf{F}'}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} = 0,$$

and give the vectors $\mathbf{U}', \mathbf{F}'$ and $\mathbf{G}'$ in terms of $\mathbf{U}, \mathbf{F}$ and $\mathbf{G}$.

37. Compact finite-difference scheme

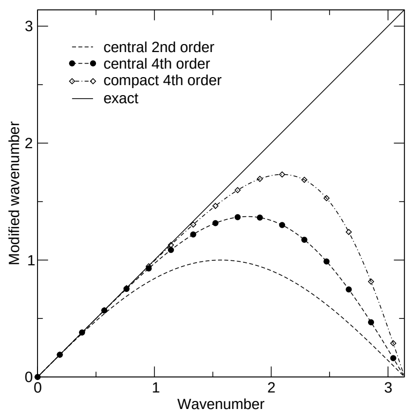
Consider the general approximation of type

$$\beta(f'_{i+2} + f'_{i-2}) + \alpha(f'_{i+1} + f'_{i-1}) + f'_i =$$
$$\frac{c}{6h}(f_{i+3} - f_{i-3}) + \frac{b}{4h}(f_{i+2} - f_{i-2}) + \frac{a}{2h}(f_{i+1} - f_{i-1}),$$

(a) and derive the equations which should be satisfied to get different order of accuracy for discretization of first derivative f'_i.

(b) By Fourier analysis of the differencing error of the scheme above derive an expression for the modified wavenumber.

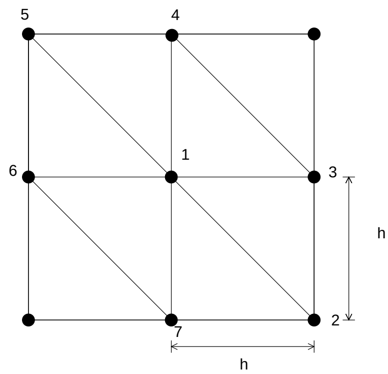
(c) What do the curves in the figure tell us?



38. Unstructured Node-Centered finite volume.

(a) Define the dual grid.

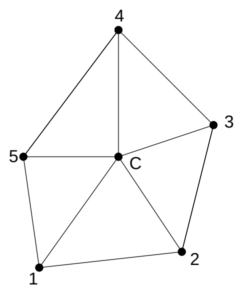
(b) Present a finite-volume approximation of $u_t = u_{xx} + u_{yy}$. Examine the consistency of the scheme and give the order of the accuracy (use the grid given here).



(c) Show that the u_x at node c can be approximated by the following finite-volume approximation and proof that its accuracy is $\mathcal{O}(h)$ (first order),

$$(u_x)_c \approx \frac{1}{V_c} \sum_i \frac{u_c + u_i}{2} \delta y_i.$$

(V_c is the volume of the dual grid)



39. Upwind discretization

(a) Consider equation $u_t + a u_x = 0$, where a is the convective velocity. Give a first-order accurate upwind discretization of his equation which is stable independent of the sign of a.

(b) Define a flux spliting scheme for discretization of one-dimensional

Euler equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} = 0, \quad \mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ E_t \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \rho u \\ \rho u^2 \\ (E_t + p)u \end{pmatrix}.$$

40. You want to solve the Helmholtz equation

$$u''(x) - a^2 u(x) = b \tag{5}$$

on a domain $|x| \leq 1$ using a Chebyshev collocation method.

a) Write down the general series ansatz for the solution $u_N(x)$.

b) Using the derivative matrix $\underline{\underline{D}}$, derive the discretised version of equation (5) as a function of the vector $\underline{u}_N$ of the solution at the grid points x_j.

c) Given are the inhomogeneous boundary conditions $u(-1) = -1$ and $u(1) = 1$. Transform equation (5) into an equation with homogeneous boundary conditions via introducing a function u_B. Use $u_B = x$:

- Explain why this is a good choice for u_B?
- Explain how the discretised system from b) is changing and how you would then incorporate the homogeneous boundary conditions.

d) Now consider the boundary conditions to be $u(-1) = -1$ and $u'(1) = 1$. Directly solving this problem with inhomogeneous boundary conditions, how would you incorporate the boundary conditions then?

41. Consider the advection equation $u_t + au_x = 0$ on a periodic domain with $L = 2\pi$.

a) Derive a Fourier-Galerkin approximation. Go through all steps involved, *i.e.* expansion in trial functions, definition of the residual, multiplication with test functions, simplifications.

b) Derive a Fourier-collocation approximation of the same equation. Use a derivative matrix $\underline{\underline{D}}_F$ to express the spatial derivatives.

c) Can you describe how to compute the Fourier derivative matrix $\underline{\underline{D}}_F$

d) Are there any differences to be expected from a numerical solution of a) and b)?

42. Describe the pseudo-spectral method. What is the difference of the evaluation of a non-linear term $w_j = u_j v_j$ by using a true Galerkin approximation for $\hat{w}_k^{\mathrm{GAL}}$ or the pseudo-spectral evaluation $\hat{w}_k^{\mathrm{PS}}$?

Are there possibilities to avoid these problems?