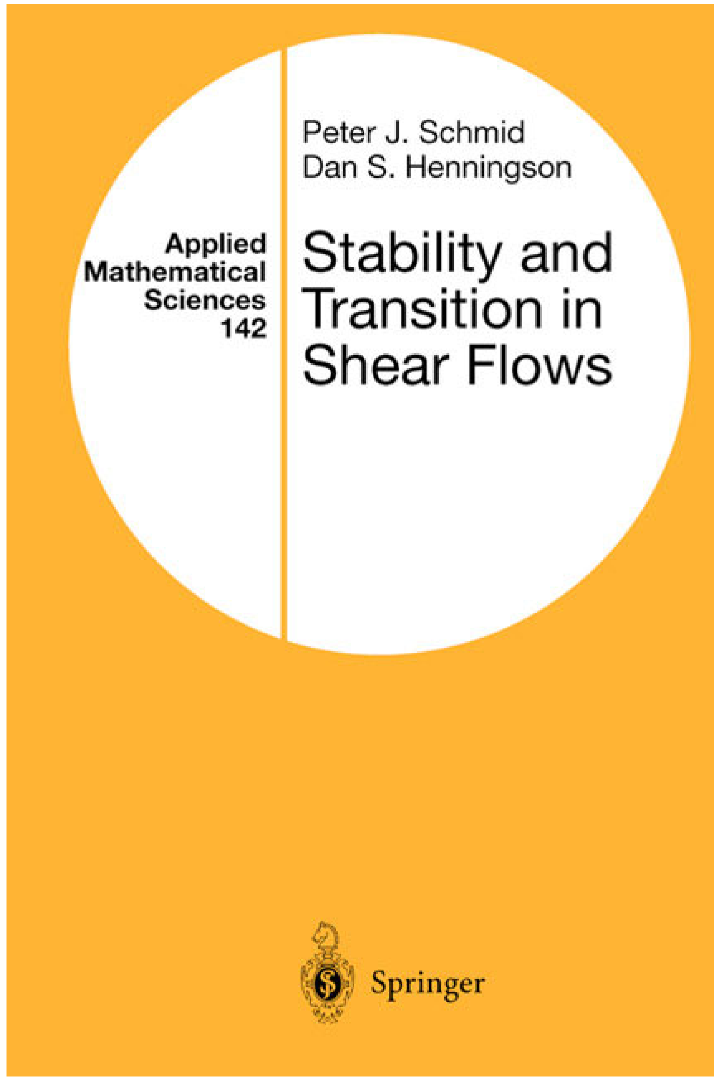
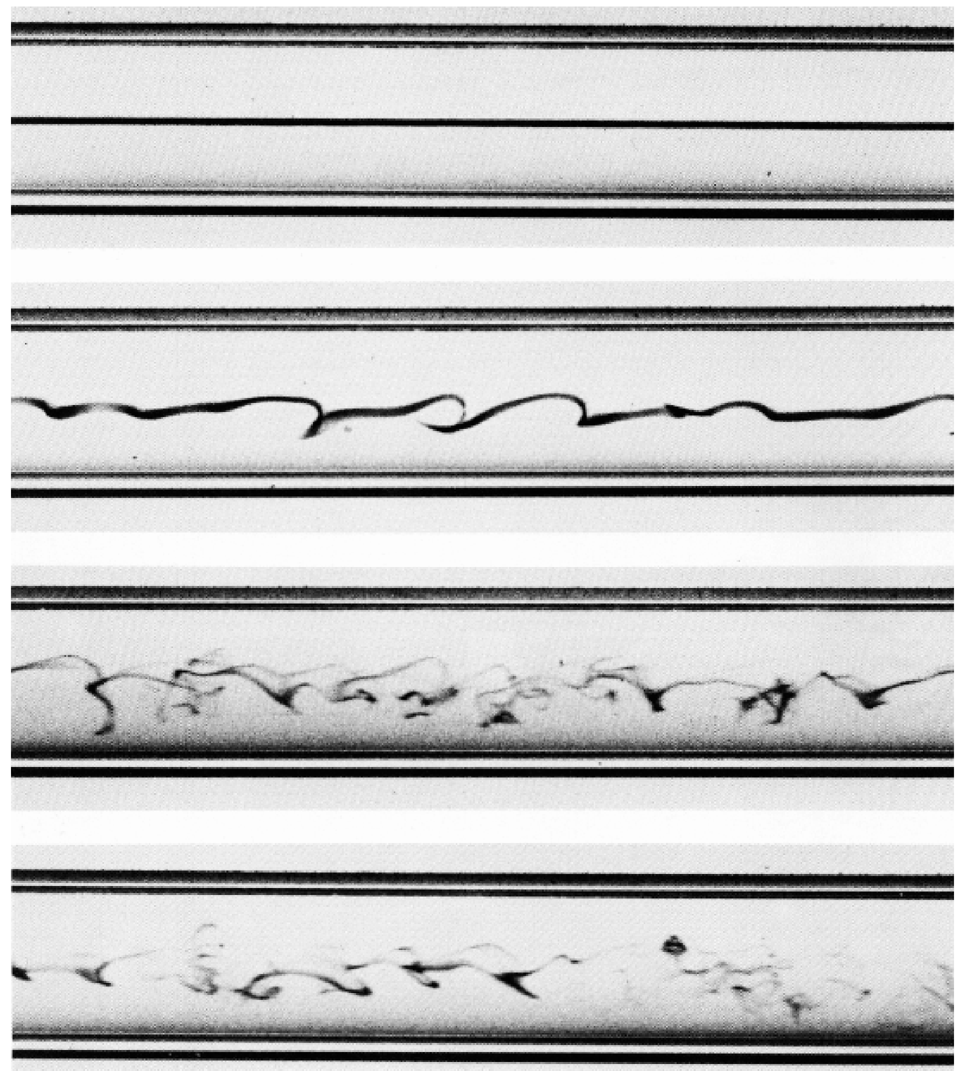
Course on Stability and Transition



Reynolds pipe flow experiment

- *Original 1883 apparatus*
- *Dye into center of pipe*
- *Critical Re=13.000*
- *Lower today due to traffic*



Reynolds-Orr equation

$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{\mathrm{Re}} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j} \left[-\frac{1}{2} u_i u_i U_j - \frac{1}{2} u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{\mathrm{Re}} u_i \frac{\partial u_i}{\partial x_j} \right] \\
&\Rightarrow \\
\frac{dE_V}{dt} &= -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{\mathrm{Re}} \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV
\end{aligned}
$$

Theorem: Linear mechanisms required for energy growth

Proof: $\frac{1}{E_V} \frac{dE_V}{dt}$ independent of disturbance amplitude

Parallel shear flows: $U_i = U(y)\delta_{1i}$

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}
$$

divergence of the momentum equations gives $\nabla^2 p = -2U'\frac{\partial v}{\partial x}$

elliminate pressure in v-equation $\Rightarrow$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4\right] v = 0$$

Parallel shear flows, cont

normal vorticity describes horizontal flow

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta = -U'\frac{\partial v}{\partial z}$$

with the boundary conditions

$v = v' = \eta = 0$ at a solid wall and in the far field

Orr-Sommerfeld and Squire equations

Assume wavelike solutions: $v(x,y,z,t) = \tilde{v}(y)\ e^{i(\alpha x + \beta z - \omega t)} \quad \Rightarrow$

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - k^2)^2\right]\tilde{v} \;=\; 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(D^2 - k^2)\right]\tilde{\eta} \;=\; -i\beta U'\tilde{v}$$

Orr-Sommerfeld modes: $\left\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\right\}_{n=1}^{N}$

Squire modes: $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^{M}$

Interpretation of modal results

$$\begin{aligned}
\omega &= \alpha c \\
v &= \mathrm{Real}\{|\tilde{v}(y)|\ e^{i\phi(y)}\ e^{i[\alpha x+\beta z-\alpha(c_r+ic_i)t]}\} \\
&= |\tilde{v}(y)|\ e^{\alpha c_i t}\ \cos[\alpha(x-c_r t)+\beta z+\phi(y)]
\end{aligned}$$

ω angular frequency

c_r phase speed

c_i temporal growthrate

α streamwise wavenumber

β spanwise wavenumber

Squire's transformation

3D and 2D Orr-Sommerfeld equation with $\omega = \alpha c$

$$(U-c)(D^2-k^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha \mathrm{Re}}(D^2-k^2)^2\tilde{v} = 0$$

$$(U-c)(D^2-\alpha_{2D}^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha_{2D} \mathrm{Re}_{2D}}(D^2-\alpha_{2D}^2)^2\tilde{v} = 0$$

$$\begin{aligned} \alpha_{2D} &= k = \sqrt{\alpha^2+\beta^2} \\ \alpha_{2D}\mathrm{Re}_{2D} &= \alpha \mathrm{Re} \\ &\Rightarrow \\ \mathrm{Re}_{2D} &= \mathrm{Re}\frac{\alpha}{k} < \mathrm{Re} \end{aligned}$$

Squire’s theorem

Each 3D Orr-Sommerfeld mode corresponds a 2D Orr-Sommerfeld mode at a *lower* Re, *i.e.*

$$\mathrm{Re}_{2D} = \mathrm{Re}\frac{\alpha}{k} < \mathrm{Re}$$

$$\Rightarrow$$

$$\mathrm{Re}_c \equiv \min_{\alpha,\beta} \mathrm{Re}_L(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_L(\alpha,0)$$

since growth rate increases with Reynolds number.

Inviscid disturbances

$$\text{Re} \to \infty \Rightarrow \text{Rayleigh equation}$$

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U''\right]\tilde{v} = 0$$

$$\omega = \alpha c \Rightarrow$$

$$\left(D^2 - k^2 - \frac{U''}{U - c}\right)\tilde{v} = 0$$

Rayleigh's inflection point criterion

Theorem: A *necessary* condition for *invicid* instability is an *inflection* point in $U(y)$

$$-\int_{-1}^{1} \tilde{v}^* \left(D^2\tilde{v} - k^2\tilde{v} - \frac{U''}{U-c}\tilde{v} \right) dy \;=$$

$$\int_{-1}^{1} |D\tilde{v}|^2 + k^2|\tilde{v}|^2 \; dy + \int_{-1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 \; dy \;=\; 0$$

$$\mathrm{Im}\left\{ \int_{1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 \; dy \right\} = \int_{-1}^{1} \frac{U'' c_i |\tilde{v}|^2}{|\; U - c \;|^2} \; dy = 0$$

Inviscid algebraic instability

$$\begin{aligned}\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{\eta} &= -i\beta U'\hat{v} \qquad \text{where} \qquad -i\omega \rightarrow \frac{\partial}{\partial t}\\ \hat{\eta}(t=0) &= \hat{\eta}_0\end{aligned}$$

$$\Rightarrow$$

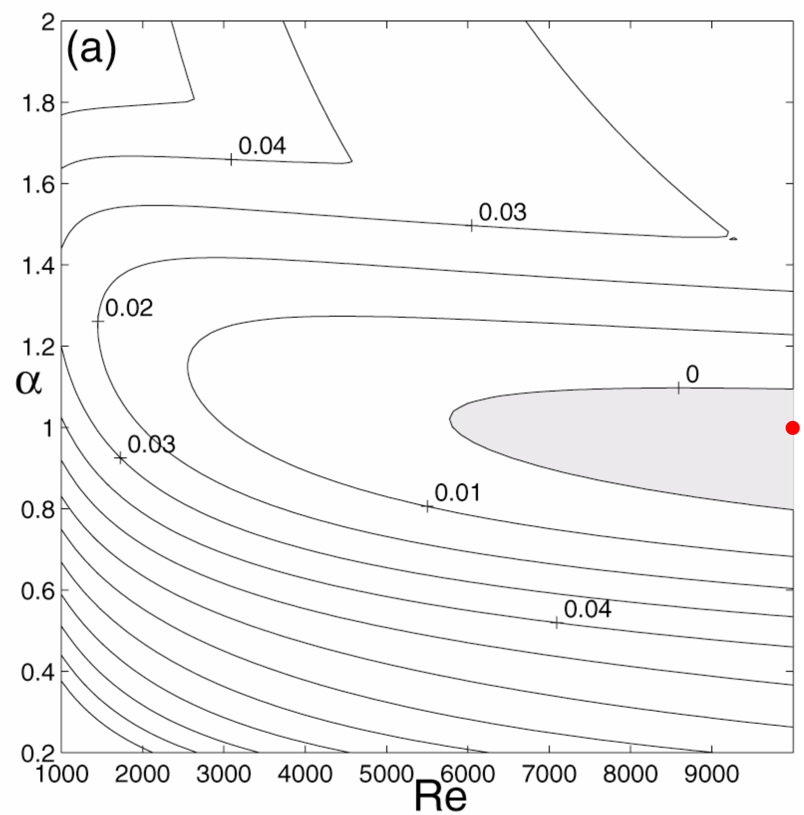
$$\hat{\eta} = \hat{\eta}_0 e^{-i\alpha U t} - i\beta U' e^{-i\alpha U t} \int_0^t \hat{v}(y,t')\ e^{i\alpha U t'} dt'$$

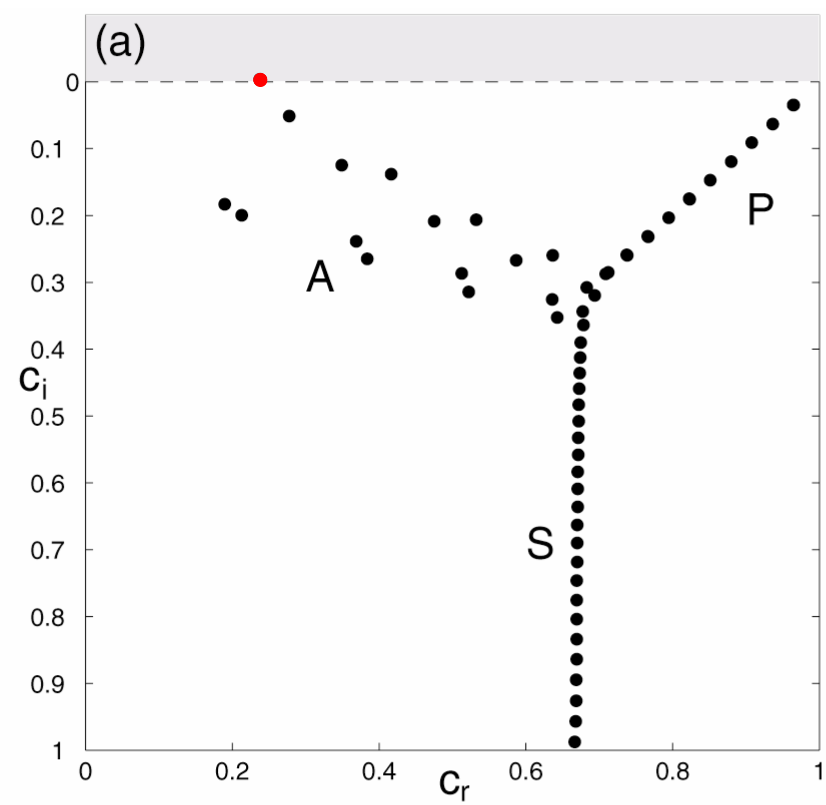
for $\alpha = 0 \quad \Rightarrow \quad \hat{v} = const \quad \Rightarrow$

$$\hat{\eta} = \hat{\eta}_0 - i\beta U' \hat{v}_0 t$$

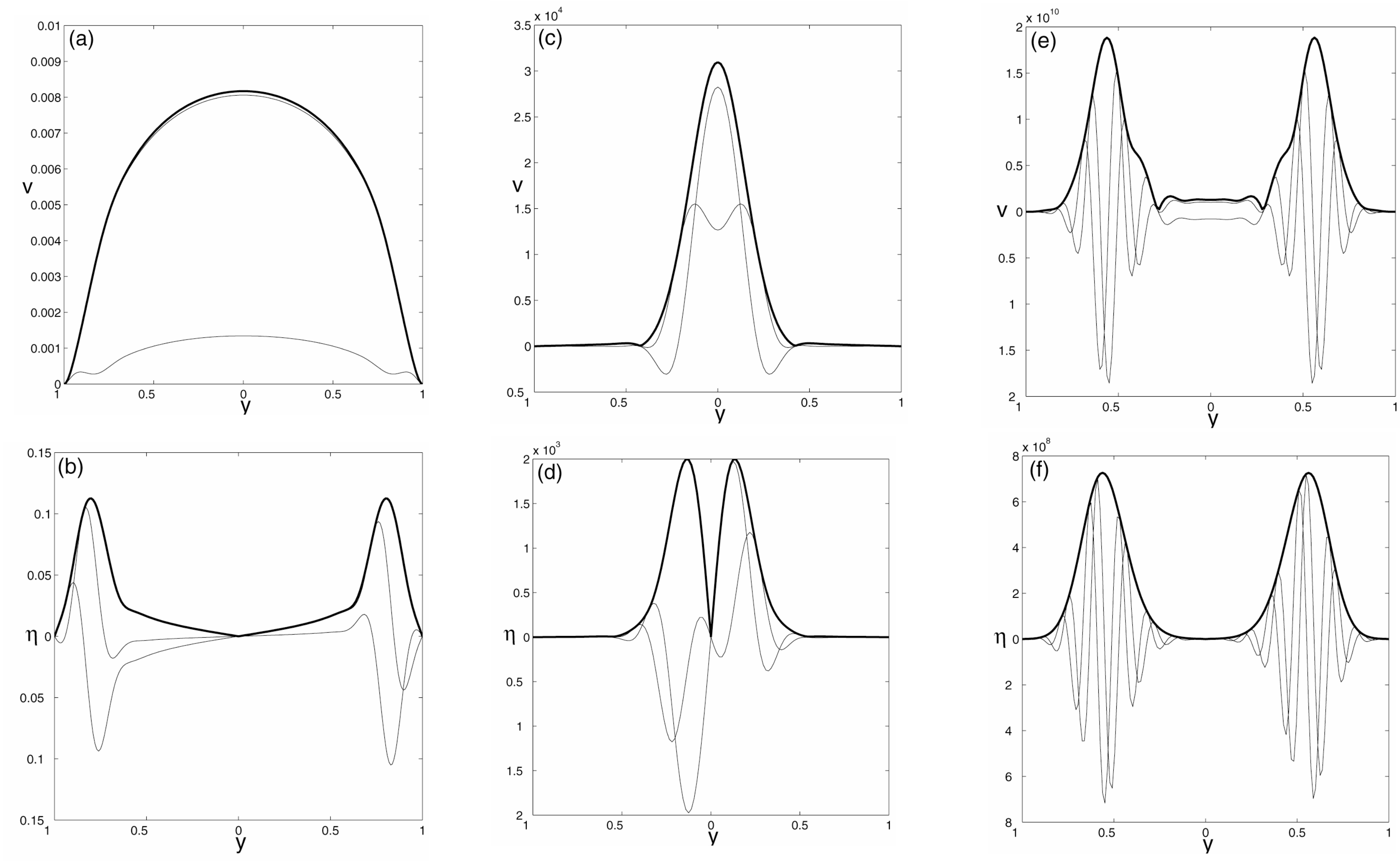
Plane Poiseuille flow

Neutral curve and spectrum

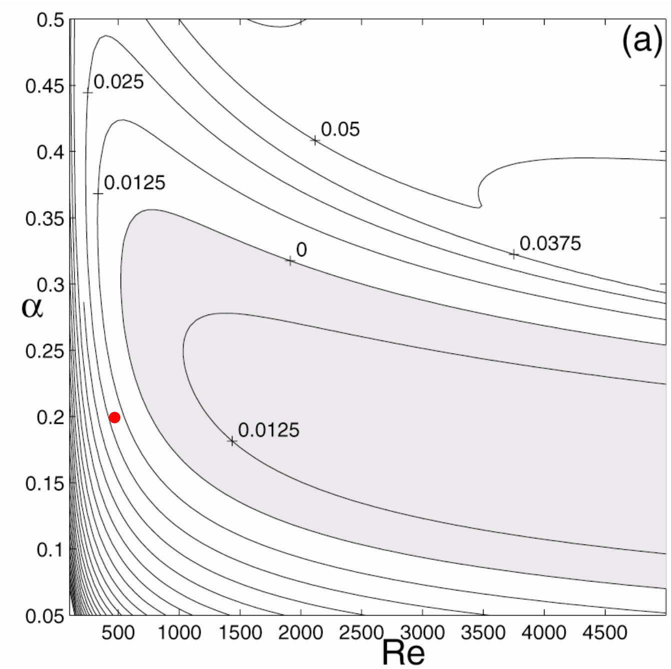


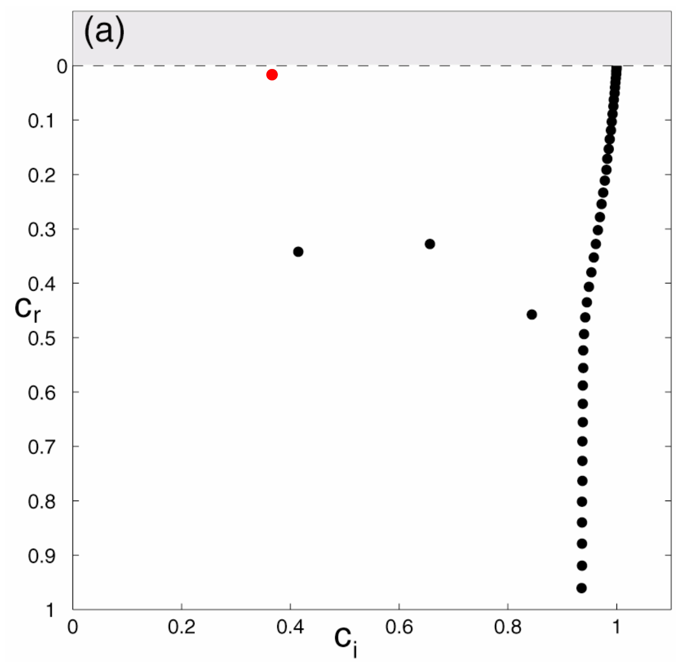


A, P, S- Eigenfunctions for PPF

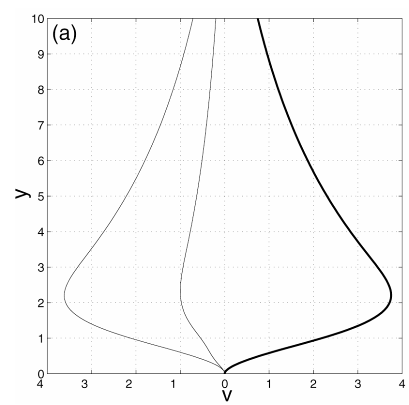


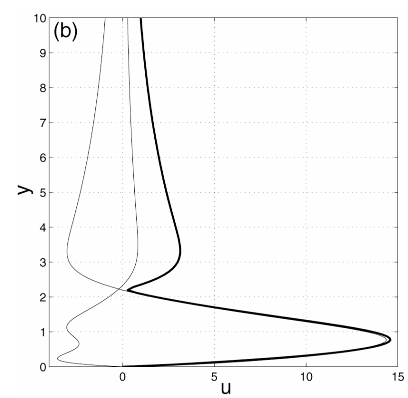
Blasius boundary layer





- Re = 500
- $\alpha = 0.2$
- TS-mode





Continuous spectrum

$$\left(D^2-k^2\right)^2\hat{v} = i\alpha\mathrm{Re}\left[(U_\infty - c)\left(D^2-k^2\right)\right]\hat{v}$$

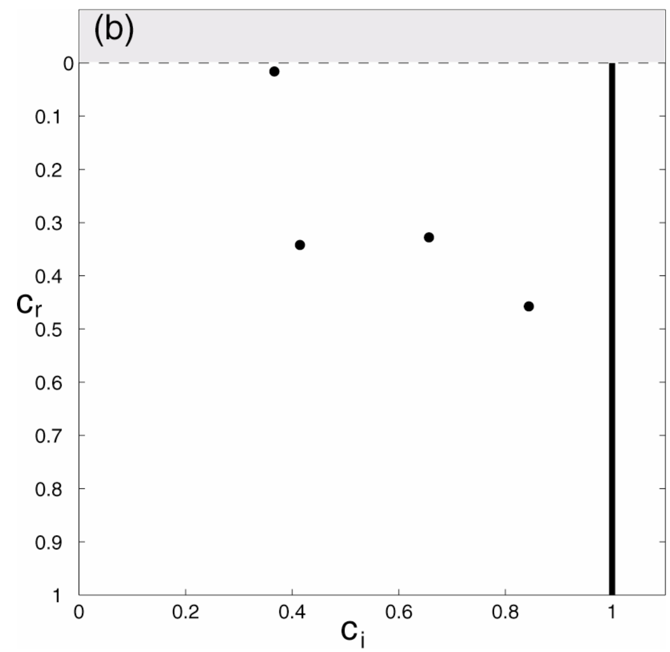
$$\hat{v}_n = \exp(\lambda_n y)$$

$$\lambda_{1,2} = \pm\sqrt{i\alpha\mathrm{Re}(U_\infty - c) + k^2}, \quad \lambda_{3,4} = \pm k$$

$\hat{v}, D\hat{v}$ bounded as $y \to \infty \quad \Rightarrow$

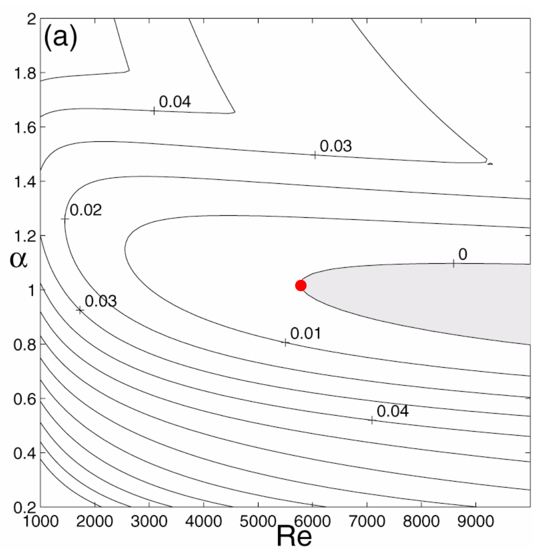
$\alpha\mathrm{Re}c_i + k^2 < 0 \quad \alpha\mathrm{Re}(U_\infty - c_r) = 0 \quad \Rightarrow$

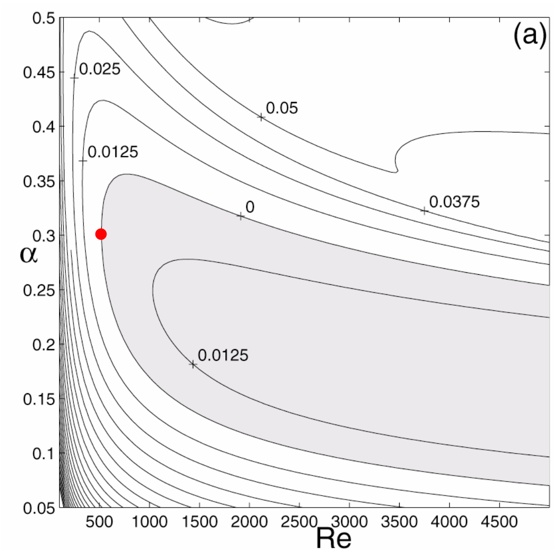
$$c = U_\infty - i(1+\xi^2)\frac{k^2}{\alpha\mathrm{Re}}$$



Critical Reynolds numbers

Flow	α_{crit}	Re_{crit}	$c_r\|_{crit}$
Plane Poiseuille flow	1.02	5772	0.264
Blasius boundary layer flow	0.303	519.4	0.397





General formulation of viscous IVP

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix} -D^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}}\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{L}_{OS} & 0 \\ -i\beta U' & \mathcal{L}_{SQ} \end{pmatrix}}_{\mathbf{L}}\underbrace{\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix}}_{\hat{\mathbf{q}}}$$

$$\mathcal{L}_{OS} = -i\alpha U(k^2-D^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(k^2-D^2)^2$$

$$\mathcal{L}_{SQ} = -i\alpha U - \frac{1}{\mathrm{Re}}(k^2-D^2).$$

$$\frac{\partial}{\partial t}\mathbf{M}\hat{\mathbf{q}} = \mathbf{L}\hat{\mathbf{q}}$$

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{L}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}}$$

Disturbance measure

$$E_V = \int_\alpha \int_\beta E \; d\alpha \; d\beta$$

$$\begin{aligned} E &= \frac{1}{2} \int_{-1}^{1} \left(|\hat{u}|^2 + |\hat{v}|^2 + |\hat{u}|^2 \right) \; dy \\ &= \frac{1}{2k^2} \int_{-1}^{1} \left(|D\hat{v}|^2 + k^2 |\hat{v}|^2 + |\hat{\eta}|^2 \right) \; dy \\ &= \frac{1}{2k^2} \int_{-1}^{1} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix}^H \begin{pmatrix} -D^2 - k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} \; dy \end{aligned}$$

$$2k^2 E = \int_{-1}^{1} \hat{\mathbf{q}}^H \mathbf{M} \hat{\mathbf{q}} \; dy = (\hat{\mathbf{q}}, \hat{\mathbf{q}}) = \|\hat{\mathbf{q}}\|^2$$

Adjoint OS-SQ system

$$
\begin{aligned}
(\tilde{\mathbf{q}}^+, \mathbf{L}_1 \tilde{\mathbf{q}}) &= \int_{-1}^{1} \tilde{\mathbf{q}}^{+H} \mathbf{M} \mathbf{L}_1 \tilde{\mathbf{q}} \, dy \\
&= \int_{-1}^{1} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix}^H \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ i\beta U' & \mathcal{L}_{SQ} \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} dy \\
&= \int_{-1}^{1} [\tilde{\xi}^* \mathcal{L}_{OS} \tilde{v} + i\beta U' \tilde{\zeta}^* \tilde{v} + \tilde{\zeta}^* \mathcal{L}_{SQ} \tilde{\eta}] dy = \{\text{integration by parts}\} \\
&= \int_{-1}^{1} [(\mathcal{L}_{OS}^+ \tilde{\xi})^* \tilde{v} - (i\beta U' \tilde{\zeta})^* \tilde{v} + (\mathcal{L}_{SQ}^+ \tilde{\zeta})^* \tilde{\eta}] dy \\
&= \int_{-1}^{1} \left[\begin{pmatrix} \mathcal{L}_{OS}^+ & -i\beta U' \\ 0 & \mathcal{L}_{SQ}^+ \end{pmatrix} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix} \right]^H \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} dy \\
&= \int_{-1}^{1} [\mathbf{M} \mathbf{L}_1^+ \tilde{\mathbf{q}}^+]^H \tilde{\mathbf{q}} \, dy \\
&= (\mathbf{L}_1^+ \tilde{\mathbf{q}}^+, \tilde{\mathbf{q}})
\end{aligned}
$$

Biorthogonality

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{L}_1 \hat{\mathbf{q}}, \quad \hat{\mathbf{q}} = \tilde{\mathbf{q}}\; e^{\lambda t} \quad \Rightarrow$$

$$0 = \left(\tilde{\mathbf{q}}^{+}, (\mathbf{L}_1 - \lambda \mathbf{I})\tilde{\mathbf{q}}\right) = \left((\mathbf{L}_1^{+} - \lambda^{*}\mathbf{I})\tilde{\mathbf{q}}^{+}, \tilde{\mathbf{q}}\right)$$

$$\begin{aligned} 0 &= \left(\tilde{\mathbf{q}}_n^{+}, \mathbf{L}_1 \tilde{\mathbf{q}}_m\right) - \left(\mathbf{L}_1^{+}\tilde{\mathbf{q}}_n^{+}, \tilde{\mathbf{q}}_m\right) \\ &= \left(\tilde{\mathbf{q}}_n^{+}, \lambda_m \tilde{\mathbf{q}}_m\right) - \left(\lambda_n^{*}\tilde{\mathbf{q}}_n^{+}, \tilde{\mathbf{q}}_m\right) \\ &= (\lambda_m - \lambda_n)\underbrace{\left(\tilde{\mathbf{q}}_n^{+}, \tilde{\mathbf{q}}_m\right)}_{\delta_{mn}} \end{aligned}$$

Component form of adjoint

$$\lambda^* \underbrace{\begin{pmatrix} -D^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{L}_{OS}^+ & i\beta U' \\ 0 & \mathcal{L}_{SQ}^+ \end{pmatrix}}_{\mathbf{L}^+} \underbrace{\begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix}}_{\hat{\mathbf{q}}^+}$$

$$\begin{aligned} \mathcal{L}_{OS}^+ &= i\alpha U(k^2-D^2) - i\alpha 2U'D - \frac{1}{\mathrm{Re}}(k^2-D^2)^2 \\ \mathcal{L}_{SQ}^+ &= i\alpha U - \frac{1}{\mathrm{Re}}(k^2-D^2). \end{aligned}$$

Adjoint Orr-Sommerfeld modes: $\left\{\tilde{\xi}_n, \tilde{\zeta}=0, \omega_n\right\}_{n=1}^N$

Adjoint Squire modes: $\left\{\tilde{\xi}_m^p, \tilde{\zeta}_m, \omega_m\right\}_{m=1}^M$

Solution of IVP using eigenfunction expansions

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{L}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}} \qquad \hat{\mathbf{q}}(t=0) = \hat{\mathbf{q}}_0 \\ \\ \|\hat{\mathbf{q}}\|^2 = (\hat{\mathbf{q}}, \hat{\mathbf{q}}) = \displaystyle\int_{-1}^{1} \hat{\mathbf{q}}^H \mathbf{M}\hat{\mathbf{q}}\; dy \end{cases}$$

$$\hat{\mathbf{q}} = \sum_{n=1}^{\infty} \kappa_n^0 \tilde{\mathbf{q}}_n e^{\lambda_n t} \quad \Rightarrow$$

$$(\tilde{\mathbf{q}}_m^+, \hat{\mathbf{q}}_0) = \left(\tilde{\mathbf{q}}_m^+, \sum_{n=1}^{\infty} \kappa_n^0 \tilde{\mathbf{q}}_n\right) = \sum_{n=1}^{\infty} \kappa_n^0 (\tilde{\mathbf{q}}_m^+, \tilde{\mathbf{q}}_n) = \kappa_m^0$$

Discrete formulation

Project solution on $S^N = \text{span}\{\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \ldots, \tilde{\mathbf{q}}_N\}$

$$\hat{\mathbf{q}} = \sum_{n=1}^{N} \kappa_n^0 \tilde{\mathbf{q}}_n e^{\lambda_n t} = \sum_{n=1}^{N} \kappa_n(t) \tilde{\mathbf{q}}_n \qquad \hat{\mathbf{q}} \in S^N$$

$$\kappa = \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_N \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{pmatrix} \begin{pmatrix} \kappa_1^0 \\ \vdots \\ \kappa_N^0 \end{pmatrix} = e^{\Lambda t} \kappa^0$$

Discrete formulation, cont.

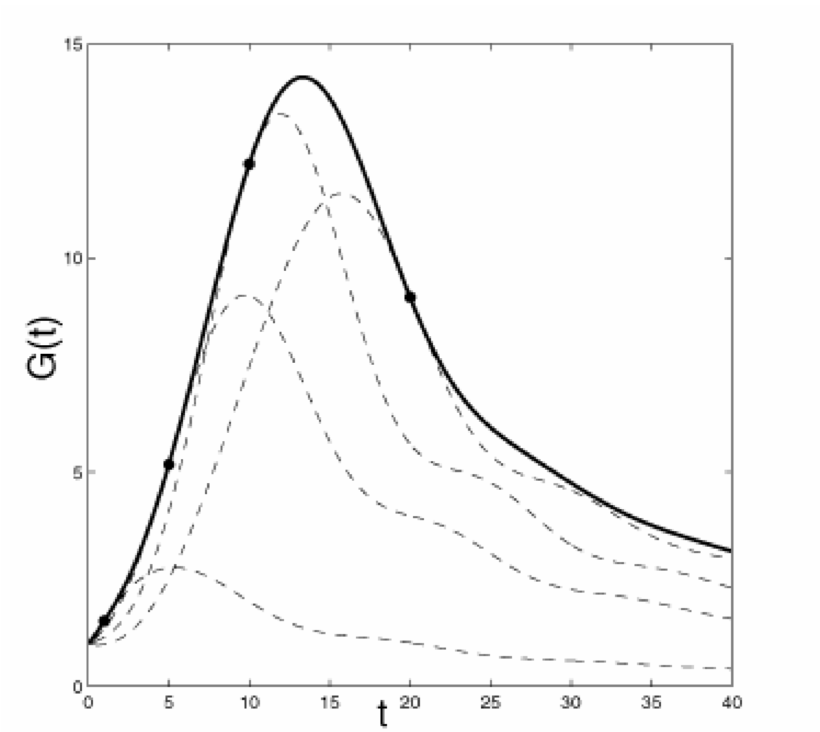
$$
\begin{aligned}
\|\hat{\mathrm{q}}\|^2 &= (\hat{\mathrm{q}},\hat{\mathrm{q}}) = \sum_{m=1}^{N}\sum_{n=1}^{N} \kappa_m \kappa_n^* (\tilde{\mathrm{q}}_n, \tilde{\mathrm{q}}_m) \qquad \hat{\mathrm{q}} \in S^N \\
&= \begin{pmatrix} \kappa_1^* & \dots & \kappa_N^* \end{pmatrix} \begin{pmatrix} (\tilde{\mathrm{q}}_1,\tilde{\mathrm{q}}_1) & (\tilde{\mathrm{q}}_1,\tilde{\mathrm{q}}_2) & \\ & \ddots & \\ & & (\tilde{\mathrm{q}}_N,\tilde{\mathrm{q}}_N) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_N \end{pmatrix} \\
&= \kappa^H A \kappa \\
&= \kappa^H F^H F \kappa \qquad F^H F = A \quad \text{Hermitian} \\
&= \|F\kappa\|_2^2 \qquad \text{2-norm, sum of squares} \\
&= \|\kappa\|_E^2 \qquad \text{energy norm}
\end{aligned}
$$

Maximum amplification

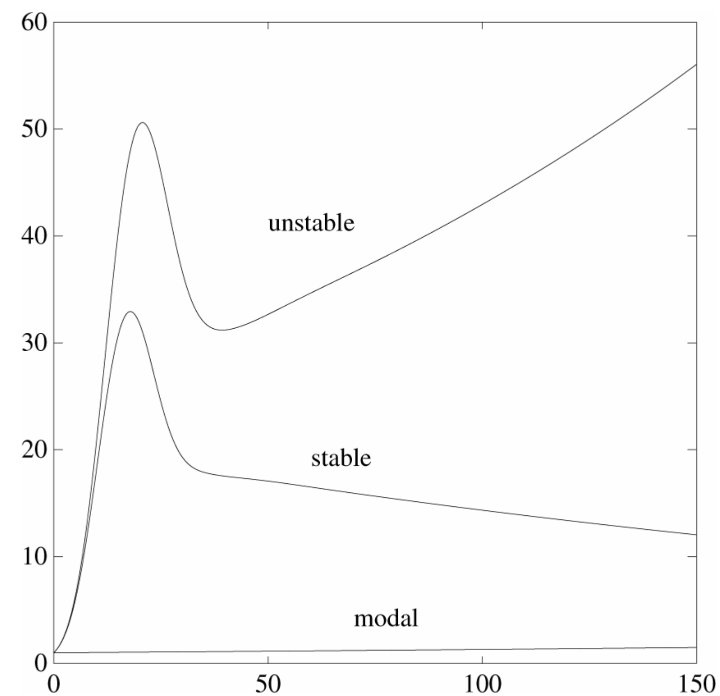
$$
\begin{aligned}
G(t) &= \max_{\hat{\mathbf{q}}_0 \neq 0} \frac{\|\hat{\mathbf{q}}(t)\|^2}{\|\hat{\mathbf{q}}_0\|^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|\kappa\|_E^2}{\|\kappa_0\|_E^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|e^{\Lambda t}\kappa_0\|_E^2}{\|\kappa_0\|_E^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|F e^{\Lambda t} F^{-1} F\kappa_0\|_2^2}{\|F\kappa_0\|_2^2} \\
&= \|\underbrace{F e^{\Lambda t} F^{-1}}_{B}\|_2^2 \\
&\leq \|F\|_2^2 \, \|F^{-1}\|_2^2 \, \|e^{\Lambda t}\|_2^2 \; = \; \mathrm{cond}(F)^2 \; e^{2\Re\{\lambda_{max}\}t}
\end{aligned}
$$

$$\|B\|_2^2 = \lambda_{max}(B^H B) = \sigma_1^2(B) \quad \text{for} \quad F\kappa_0 = v_1$$

2D PPF: envelope and selected IC

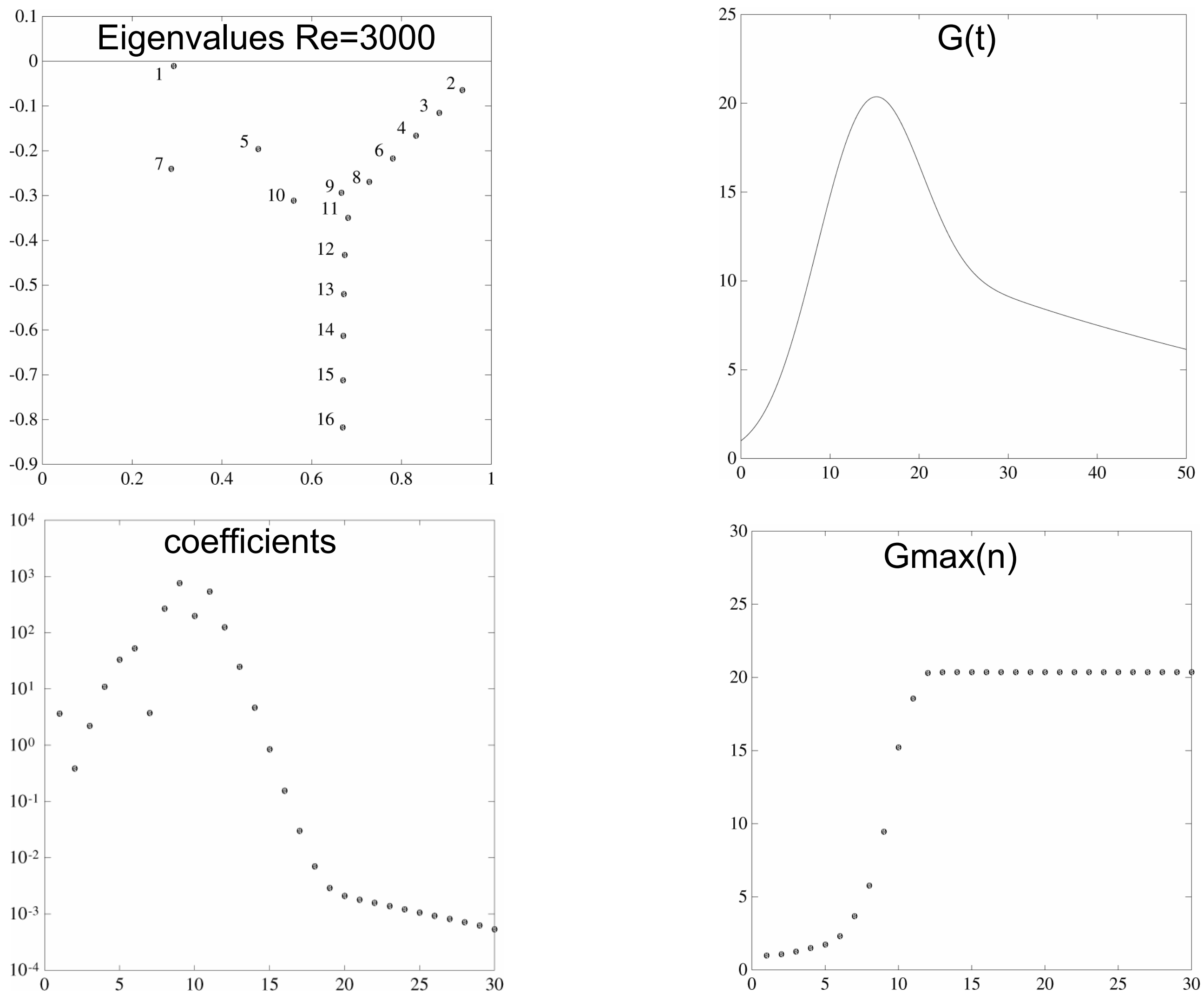


Re=1000



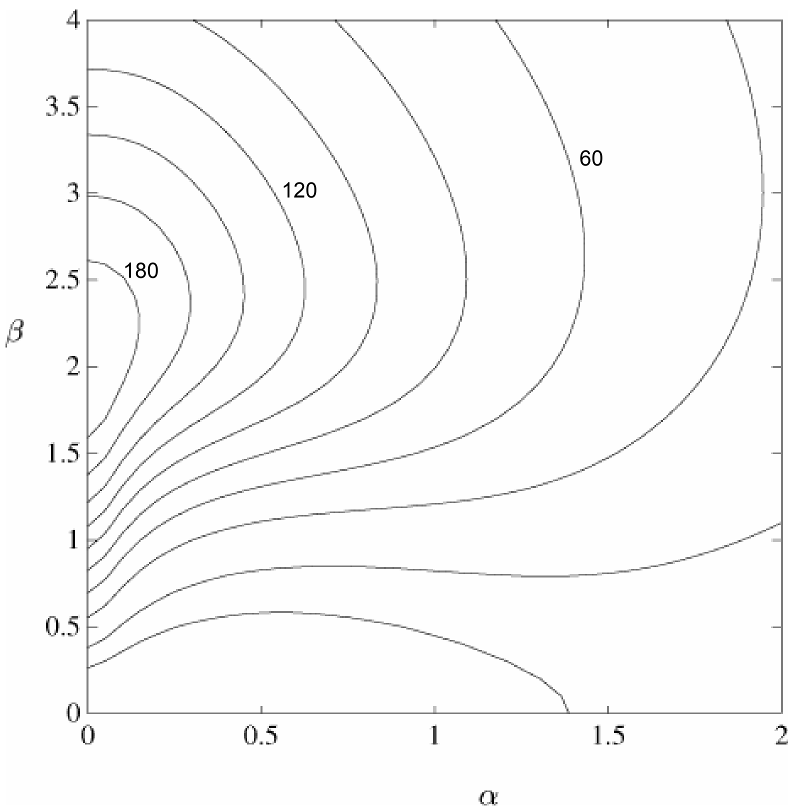
Re=5000, 8000

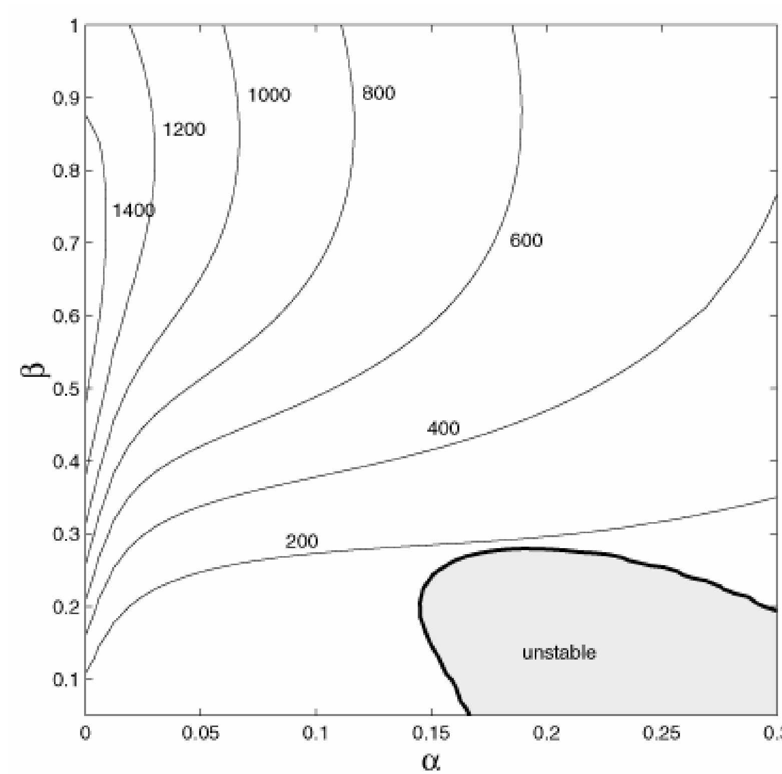
2D PPF: dependence on N



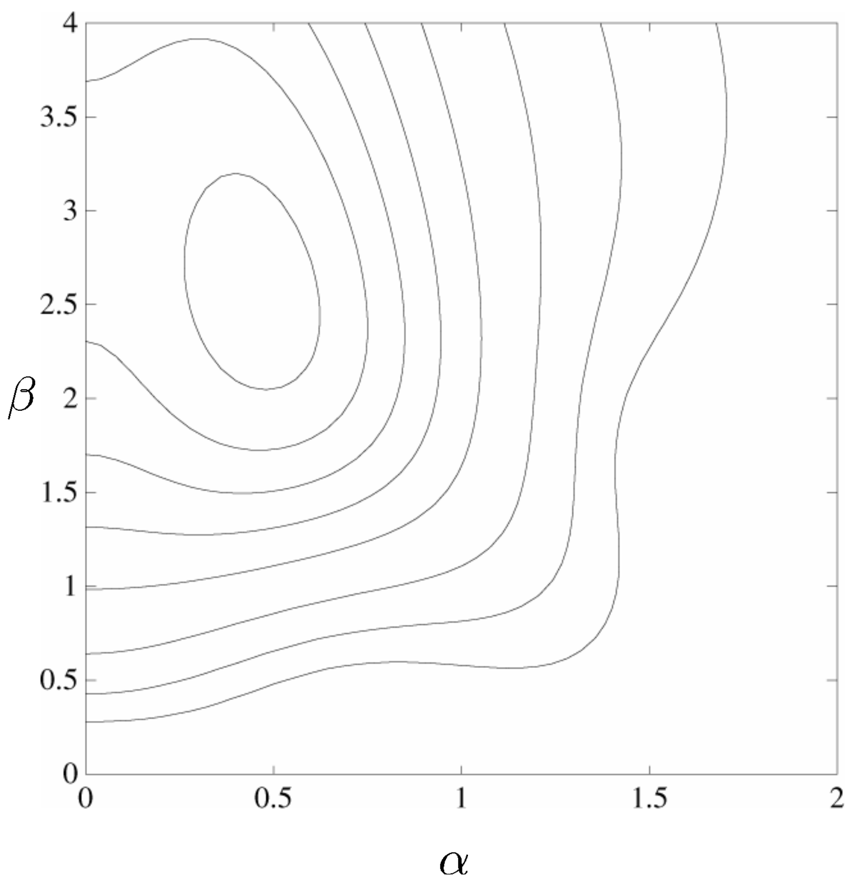
3D PPF and Blasius flow, Re=1000

$$G_{max} = \max_{t \geq 0} G(t)$$

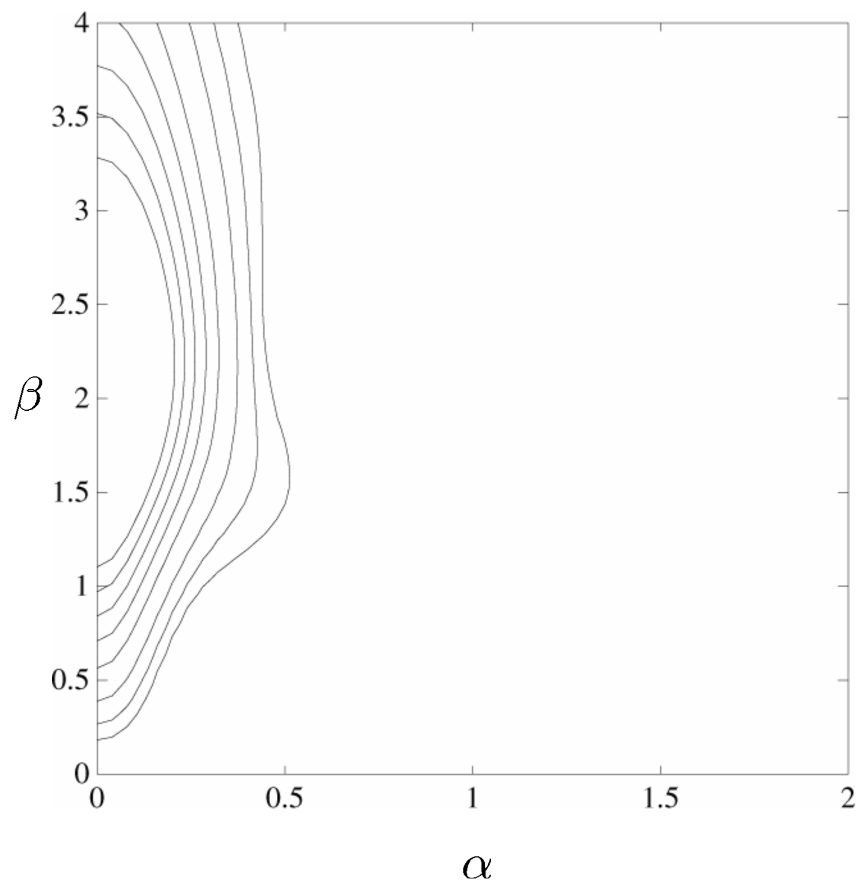




3D PPF: G(t), Re=1000

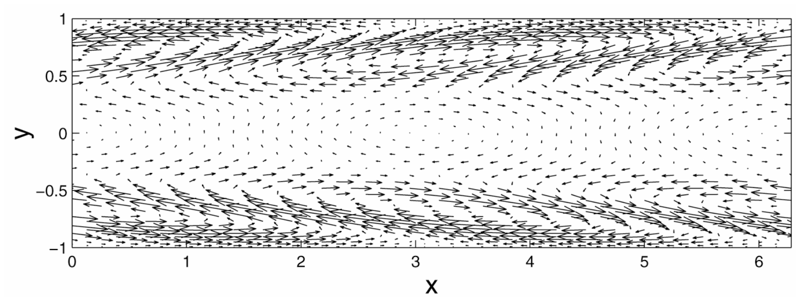
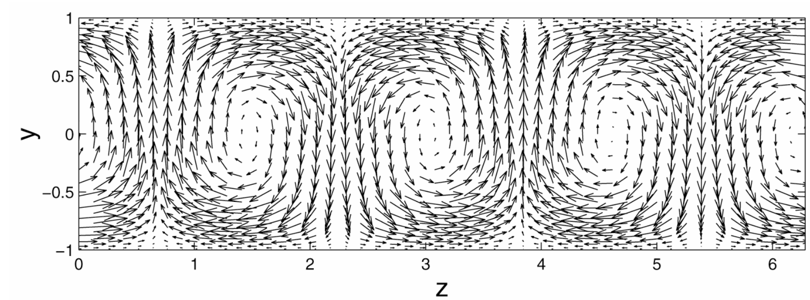
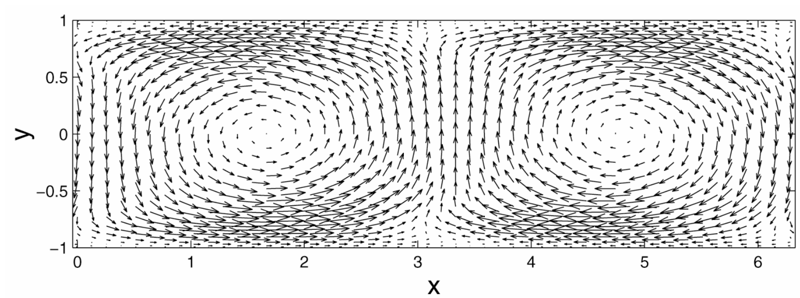
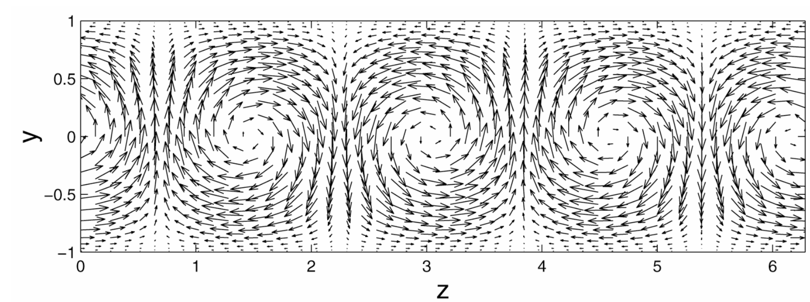


$t = 25$



$t = 75$

Optimal disturbances PPF, Re=1000

	2D disturbance	3D disturbance
$t = 0$		
$t = t_{max}$		
	$\alpha = 1, \beta = 0$	$\alpha = 0, \beta = 2$

The forced problem and the resolvent

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{\mathbf{q}} &= \mathbf{L}_1\hat{\mathbf{q}} + \hat{\mathbf{q}}_f e^{i\omega t} \qquad \Rightarrow \\
\hat{\mathbf{q}} &= e^{\mathbf{L}_1 t}\hat{\mathbf{q}}_0 + (\mathbf{L}_1 - i\omega\mathbf{I})^{-1}\hat{\mathbf{q}}_f e^{i\omega t}
\end{aligned}
$$

$$
\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}} \qquad \tilde{\mathbf{q}} = \int_0^\infty e^{-st}\hat{\mathbf{q}}(t)dt
$$

$$
\begin{aligned}
s\tilde{\mathbf{q}} - \mathbf{L}_1\tilde{\mathbf{q}} &= \hat{\mathbf{q}}_0 \\
\tilde{\mathbf{q}} &= (s\mathbf{I} - \mathbf{L}_1)^{-1}\hat{\mathbf{q}}_0
\end{aligned}
$$

Discrete formulation

$$
\begin{aligned}
\tilde{\mathbf{q}} &= \int_0^\infty e^{-st}\hat{\mathbf{q}}(t)dt \\
&= \int_0^\infty e^{-st} \sum_{n=1}^{N} \kappa_n \tilde{\mathbf{q}}_n e^{\lambda_n t} dt \\
&= \sum_{n=1}^{N} \kappa_n \tilde{\mathbf{q}}_n \int_0^\infty e^{-(s-\lambda_n)t} dt \\
&= \sum_{n=1}^{N} \underbrace{\frac{\kappa_n}{s-\lambda_n}}_{\kappa_n(s)} \tilde{\mathbf{q}}_n
\end{aligned}
$$

$$
\kappa(s) = \begin{pmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_N} \end{pmatrix} \begin{pmatrix} \kappa_1^0 \\ \vdots \\ \kappa_N^0 \end{pmatrix}
$$

Maximum response to forcing

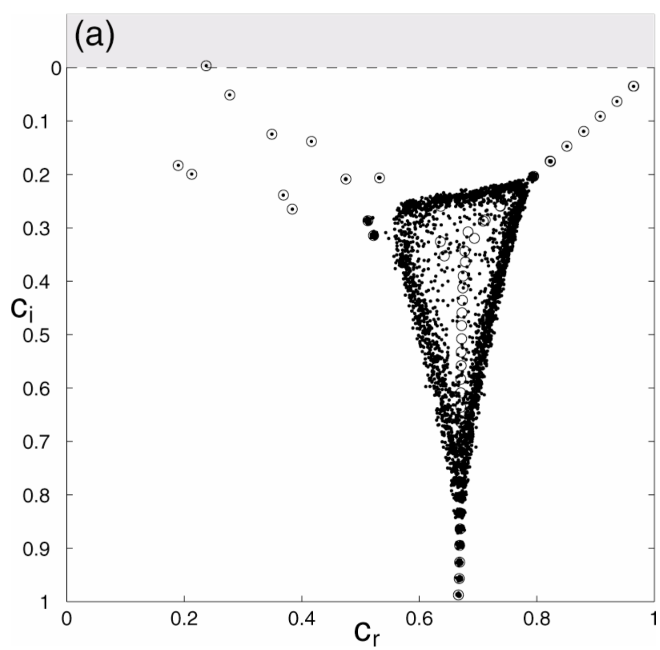
$$
\begin{aligned}
R(s) &= \max_{\hat{\mathbf{q}}_0 \neq 0} \frac{\|(s\mathbf{I} - \mathbf{L}_1)^{-1}\hat{\mathbf{q}}_0\|}{\|\hat{\mathbf{q}}_0\|} \\
&= \max_{\kappa_0 \neq 0} \frac{\|\kappa(s)\|_E}{\|\kappa_0\|_E} \\
&= \|F \;\mathrm{diag}\{\frac{1}{s-\lambda_1}, \dots, \frac{1}{s-\lambda_N}\} F^{-1}\|_2 \\
&\leq \|F\|_2 \; \|F^{-1}\| \; \frac{1}{\mathrm{min.dist}(\lambda - s)}
\end{aligned}
$$

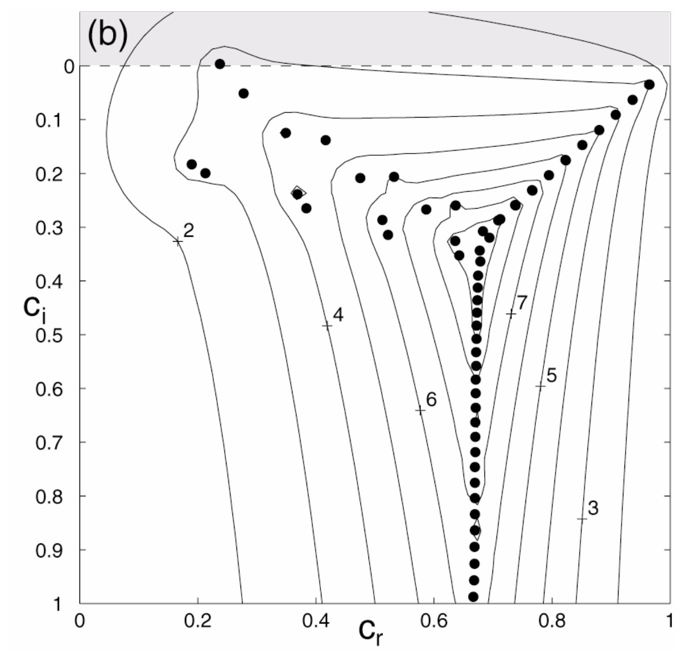
Pseudospectra, resolvents and sensitivity

Definition: for $\epsilon \geq 0$, s is in the ϵ-pseudospectra of $\mathbf{L}$ if any of the following equivalent conditions hold

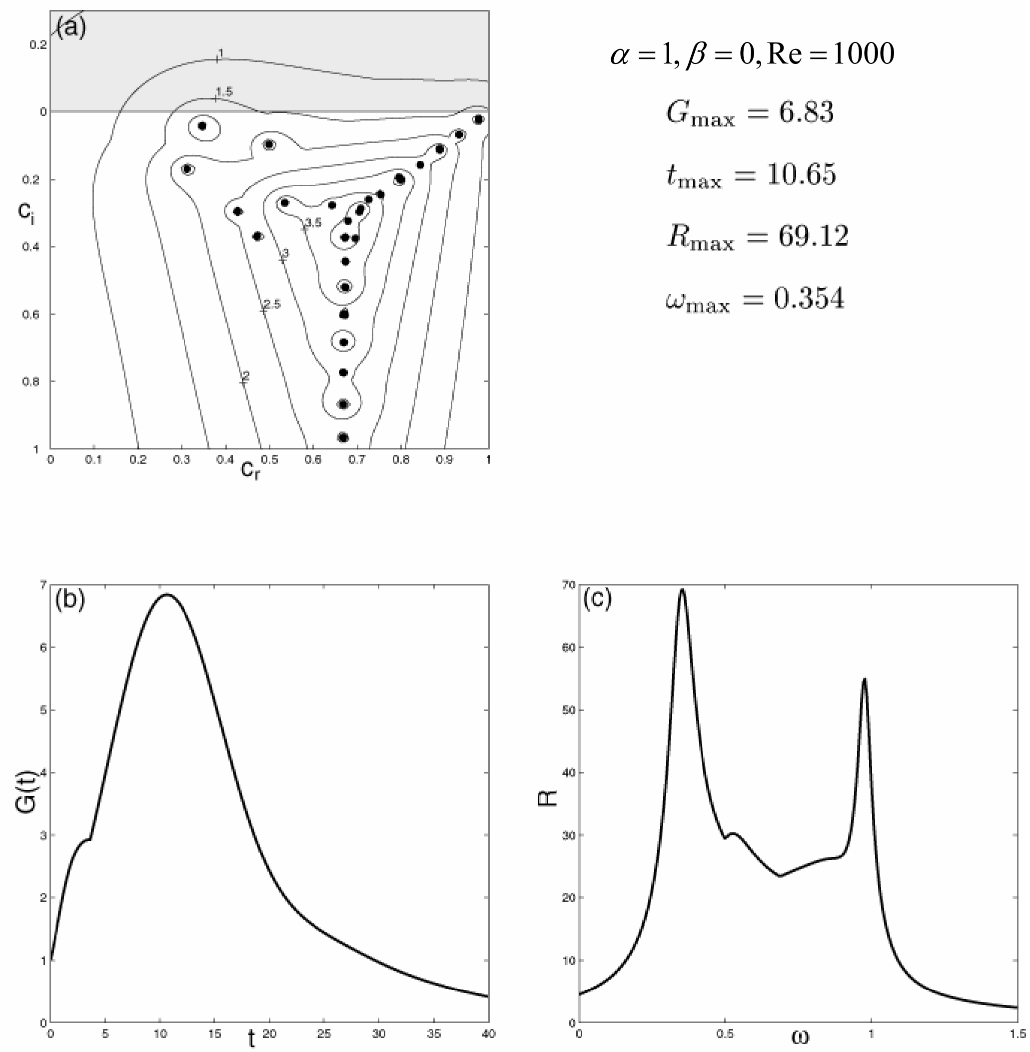
(i) s is an eigenvalue of $\mathbf{L}+\mathbf{E}$, where $\|\mathbf{E}\| \leq \epsilon$

(ii) $\|(s\mathbf{I}-\mathbf{L})^{-1}\| \geq \dfrac{1}{\epsilon}$





PPF: resolvents, growth and forcing



$$\alpha = 1, \beta = 0, \mathrm{Re} = 1000$$

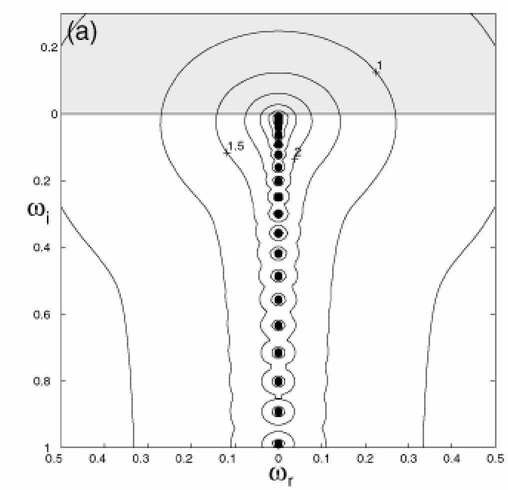
$$G_{\max} = 6.83$$

$$t_{\max} = 10.65$$

$$R_{\max} = 69.12$$

$$\omega_{\max} = 0.354$$

PPF: resolvents, growth and forcing



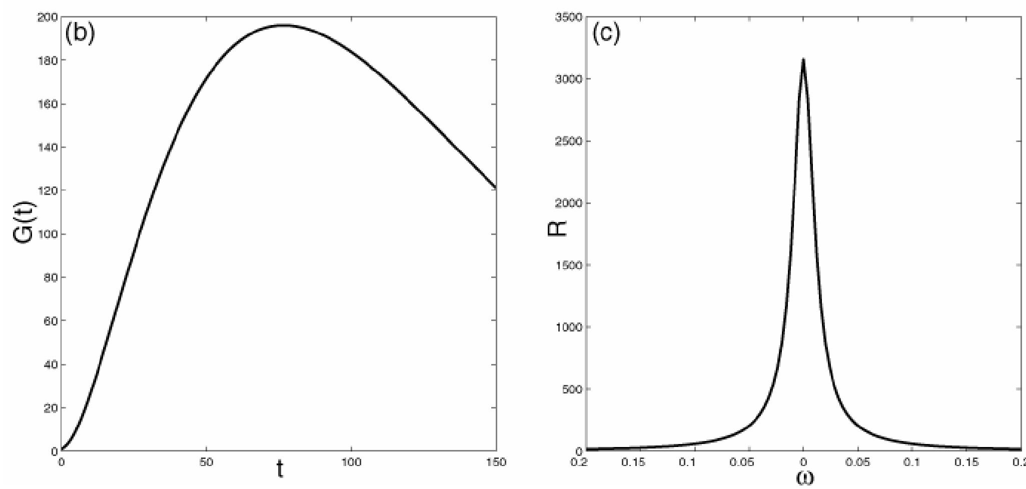
$\alpha = 0, \beta = 2, \mathrm{Re} = 1000$

$G_{\max} = 196.04$

$t_{\max} = 76.5$

$R_{\max} = 3156.32$

$\omega_{\max} = 0.$



Model problem

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}$$

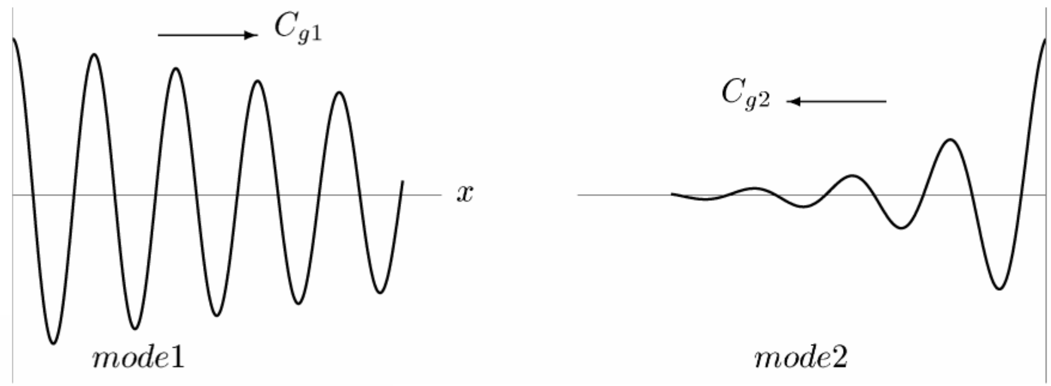
spatial evolution: ω is real valued

$$u = \hat{u}e^{-i\omega t} \quad \Rightarrow \quad -i\omega\hat{u} + U\frac{\partial \hat{u}}{\partial x} = \epsilon\frac{\partial^2 \hat{u}}{\partial x^2}$$

$$\hat{v} = \frac{\partial \hat{u}}{\partial x} \quad \Rightarrow \quad \frac{\partial}{\partial x}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -i\omega/\epsilon & U/\epsilon\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} \qquad \text{IVP in x}$$

$$\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} \sim e^{i\alpha x} \quad \Rightarrow \quad -\det\begin{pmatrix}-i\alpha & 1\\ -\frac{i\omega}{\epsilon} & \frac{U}{\epsilon} - i\alpha\end{pmatrix} = \alpha^2 + \frac{iU}{\epsilon}\alpha - \frac{i\omega}{\epsilon} = 0$$

Burger's eq., cont.



$$
\begin{aligned}
\alpha &= -\frac{iU}{2\epsilon} \pm \sqrt{\frac{i\omega}{\epsilon} - \frac{U^2}{4\epsilon^2}} \\
&= \frac{iU}{2\epsilon} \pm \frac{iU}{2\epsilon}\left[1 - \frac{2i\omega\epsilon}{U^2} + \frac{2\omega^2\epsilon^2}{U^4} + \mathcal{O}(\epsilon^3)\right] \\
&= \begin{cases} \dfrac{\omega}{U} + i\dfrac{\omega^2}{U^3}\epsilon & c_g = \dfrac{\partial\omega}{\partial\alpha} = U \qquad mode1 \\ -\dfrac{\omega}{U} - i\dfrac{U}{\epsilon} & c_g = \dfrac{\partial\omega}{\partial\alpha} = -U \qquad mode2 \end{cases}
\end{aligned}
$$

Spatial OS-SQ system

ω, β given, non-linear eigenvalue problem in α

$$\left[(-i\omega + i\alpha U)(D^2 - \alpha^2 - \beta^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - \alpha^2 - \beta^2)^2\right]\tilde{v} = 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(D^2 - \alpha^2 - \beta^2)\right]\tilde{\eta} = -i\beta U'\tilde{v}$$

$$\begin{pmatrix}\tilde{v}\\ \tilde{\eta}\end{pmatrix} = \begin{pmatrix}\tilde{V}\\ \tilde{E}\end{pmatrix}\exp(-\alpha y) \quad \text{reduces order in } \alpha$$

$$(i\omega - i\alpha U)(D^2 - 2\alpha D - \beta^2)\hat{V} + i\alpha U''\hat{V} + \frac{1}{\mathrm{Re}}(D^2 - 2\alpha D - \beta^2)^2\hat{V} = 0$$

$$(i\omega - i\alpha U)\hat{E} - i\beta U'V + \frac{1}{\mathrm{Re}}(D^2 - 2\alpha D - \beta^2)\hat{E} = 0.$$

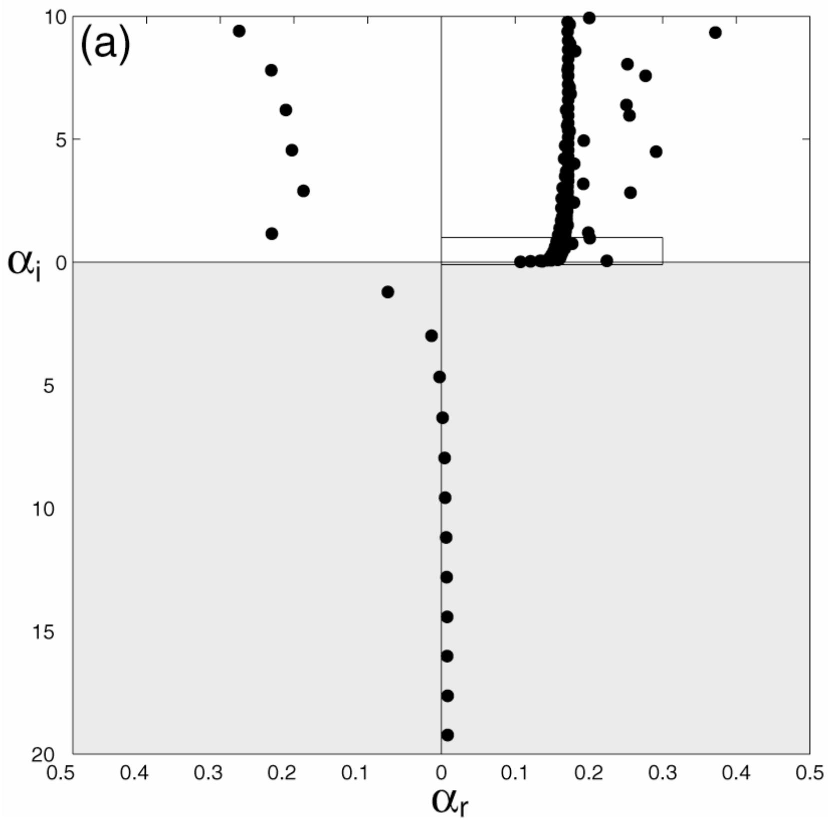
Spatial OS-SQ system, cont.

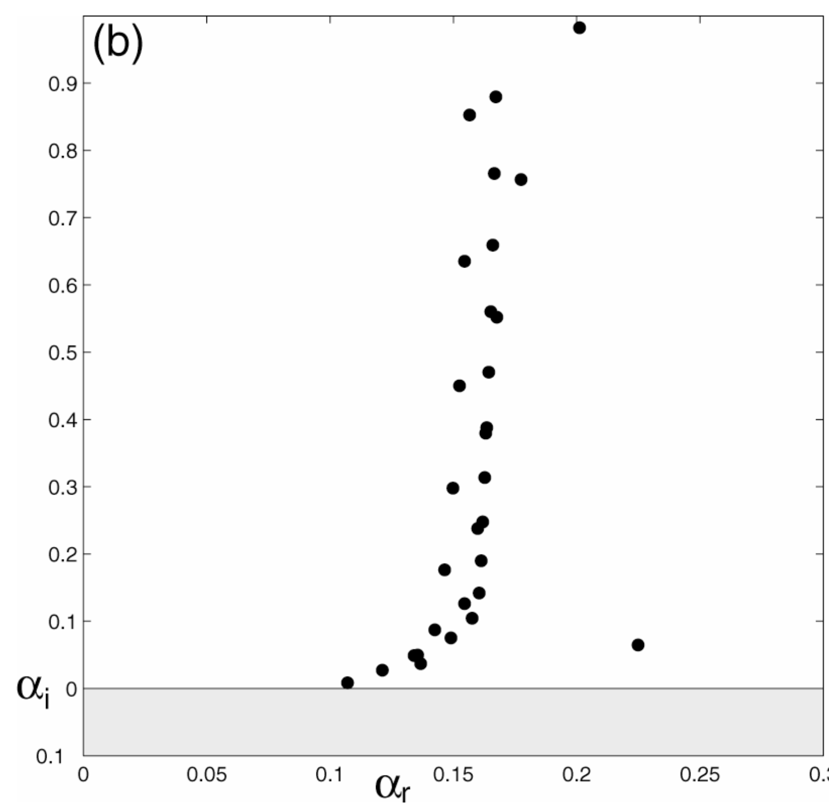
$$\tilde{\mathrm{q}} = (\alpha\hat{V}, \hat{V}, \hat{E})^T$$

$$\begin{pmatrix} -R_1 & -R_0 & 0 \\ I & 0 & 0 \\ 0 & -S & -T_0 \end{pmatrix} \begin{pmatrix} \alpha\hat{V} \\ \hat{V} \\ \hat{E} \end{pmatrix} = \alpha \begin{pmatrix} R_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_1 \end{pmatrix} \begin{pmatrix} \alpha\hat{V} \\ \hat{V} \\ \hat{E} \end{pmatrix}$$

$$\begin{aligned}
R_2 &= \frac{4}{\mathrm{Re}} D^2 + 2iUD \\
R_1 &= -2i\omega D - \frac{4}{\mathrm{Re}} D^3 + \frac{4}{\mathrm{Re}} \beta^2 D - iUD^2 + iU\beta^2 + iU'' \\
R_0 &= i\omega D^2 - i\omega\beta^2 + \frac{1}{\mathrm{Re}} D^4 - \frac{2}{\mathrm{Re}} \beta^2 D^2 + \frac{1}{\mathrm{Re}} \beta^4 \\
T_1 &= \frac{2}{\mathrm{Re}} D + iU \\
T_0 &= -i\omega - \frac{1}{\mathrm{Re}} D^2 + \frac{1}{\mathrm{Re}} \beta^2 \\
S &= i\beta U'
\end{aligned}$$

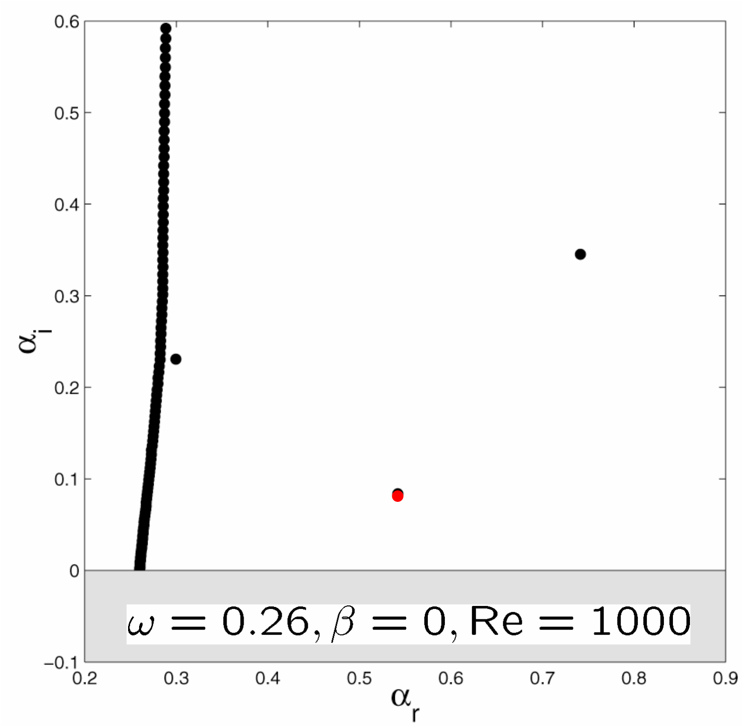
Spatial PPF spectra

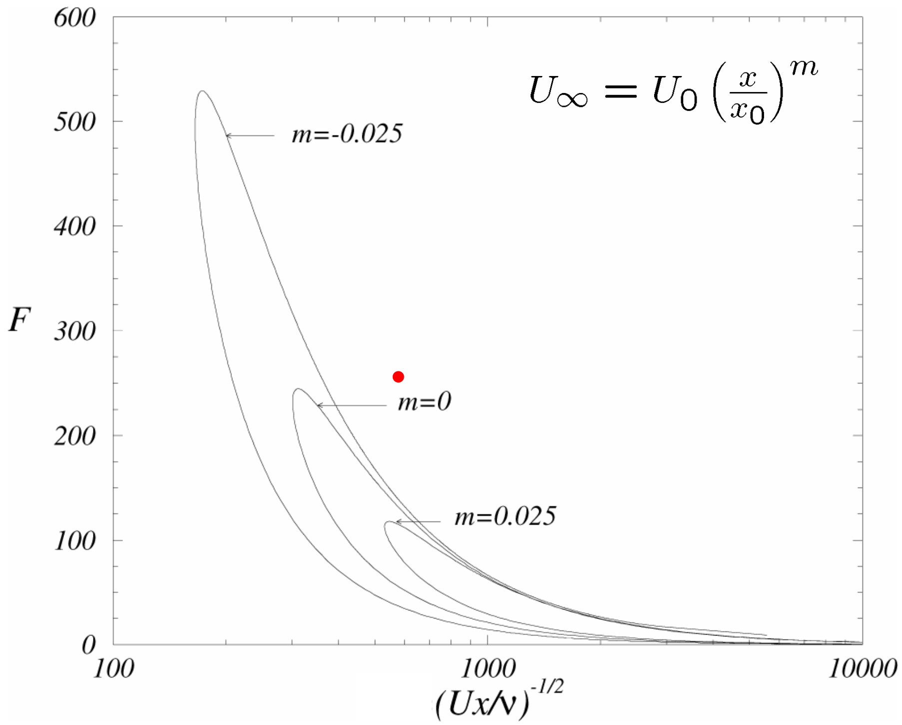




$$\omega = 0.3, \beta = 0, \mathrm{Re} = 2000$$

Boundary layer flow





$$F = 10^6 \omega \nu / U_\infty^2 = 10^6 \omega / \mathrm{Re}$$

$$\mathrm{Re} = 1.72 \sqrt{Ux/\nu}$$

Spatial IVP

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix} = \begin{pmatrix} i\alpha D/k^2 & -i\beta/k^2 \\ 1 & 0 \\ i\beta D/k^2 & i\alpha \end{pmatrix} \begin{pmatrix}\tilde{V}\\ \tilde{E}\end{pmatrix} e^{-\alpha y}$$

$$\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{w}\end{pmatrix} = \sum_{j=1}^{N} \kappa_j(x) \begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix}_j = \sum_{j=1}^{N} \kappa_j^0(x) \begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix}_j e^{i\alpha_j x}$$

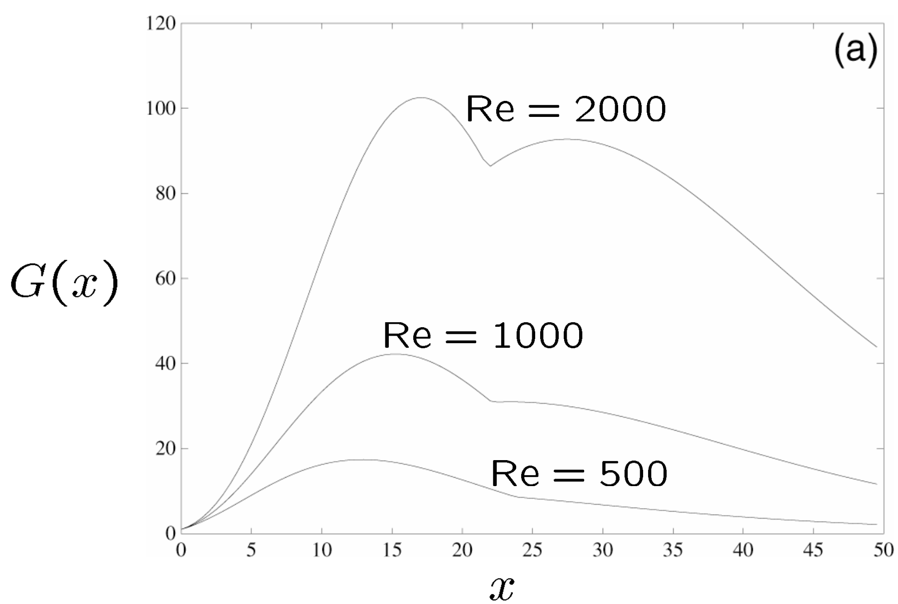
$$\kappa^T = (\kappa_1, \kappa_2, \ldots, \kappa_N)^T$$

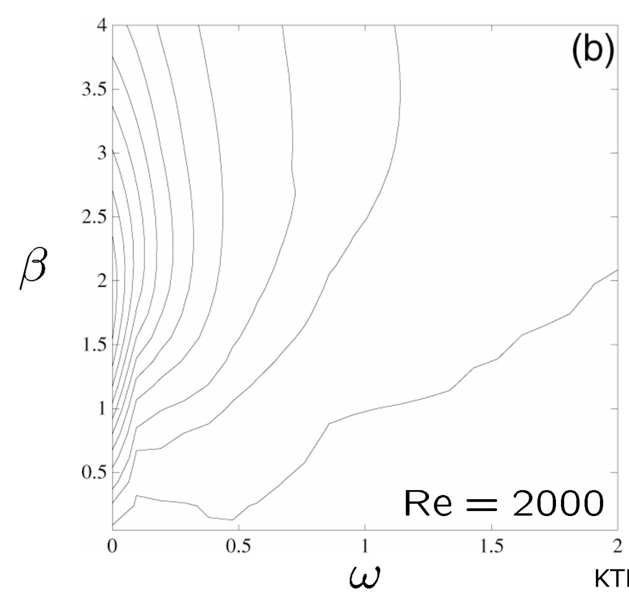
$$\Lambda = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$$

$$\frac{d\kappa}{dx} = i\Lambda\kappa, \quad \kappa(0) = \kappa^0 \quad \Rightarrow \quad \kappa = e^{i\Lambda x}\kappa^0$$

Optimal spatial growth for PPF

$$
\begin{aligned}
E(\kappa) &= \kappa^H A \kappa = \kappa^H F^H F \kappa = \|F\kappa\|^2 \\
A_{ij} &= \frac{1}{2}\int_{-1}^{1} \left(\tilde{u}_i^* \tilde{u}_j + \tilde{v}_i^* \tilde{v}_j + \tilde{w}_i^* \tilde{w}_j\right) dy \\
G(x) &= \sup_{\kappa_0} \frac{E(\kappa)}{E(\kappa_0)} = \|F \exp(i\Lambda x) F^{-1}\|_2^2
\end{aligned}
$$





Non-linear disturbance equations

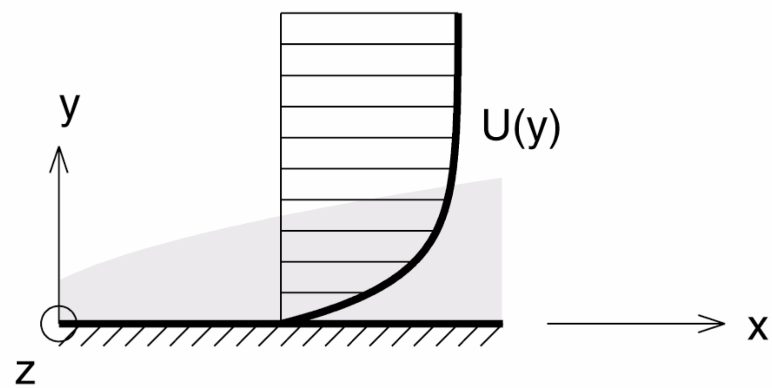
$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i \\
\frac{\partial u_i}{\partial x_i} &= 0
\end{aligned}
$$

$$
\begin{aligned}
u_i(x_i, 0) &= u_i^0(x_i) \\
u_i(x_i, t) &= 0 \quad \text{on solid boundaries}
\end{aligned}
$$

$$
\mathrm{Re} = U_\infty \delta_* / \nu
$$

$$
\begin{aligned}
u_i &= U_i + u_i' \\
p &= P + p' \quad \text{drop primes}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j} \\
\frac{\partial u_i}{\partial x_i} &= 0
\end{aligned}
$$



Stability definitions

$$E_V = \frac{1}{2}\int_V u_i u_i \, dV.$$

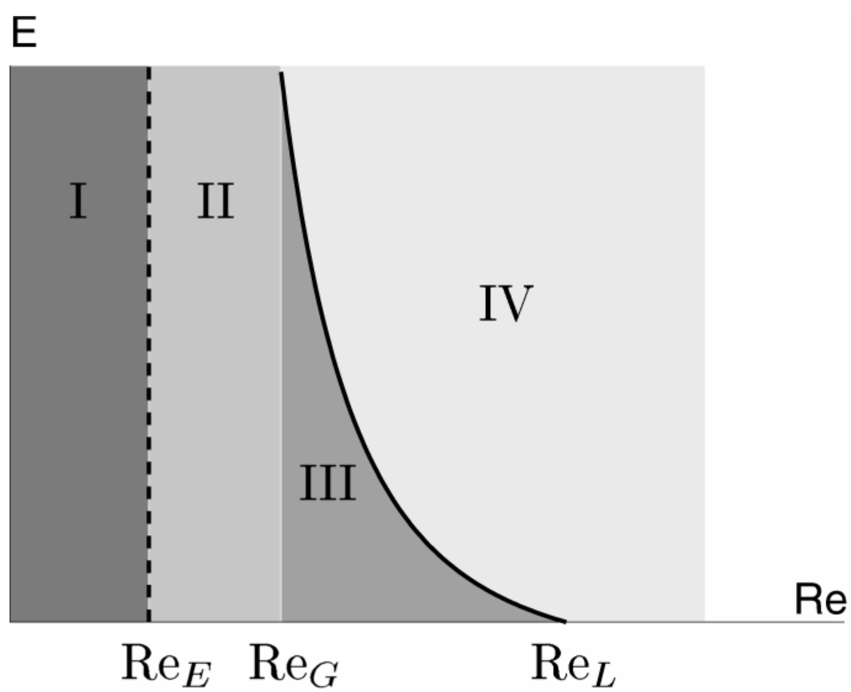
Stable:	$\lim_{t\to\infty} \frac{E_V(t)}{E_V(0)} \to 0$
Conditionally stable:	$\exists \; \delta : E(0) < \delta \;\Rightarrow$ stable
Globally stable:	Conditionally stable with $\delta \to \infty$
Monotonically stable	$\frac{dE}{dt} \leq 0 \;\forall\; t > 0$

Critical Reynolds numbers

Re_E : $\mathrm{Re} < \mathrm{Re}_E$ flow monotonically stable

Re_G : $\mathrm{Re} < \mathrm{Re}_G$ flow globally stable

Re_L : $\mathrm{Re} > \mathrm{Re}_L$ flow linearly unstable ($\delta \to 0$)



Quadratic non-linear interactions

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = -u\frac{\partial u}{\partial x} \equiv -\frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

$$u = \sum_{k=-\infty}^{\infty} a_k(t) e^{ik\alpha x}$$

$$\sum_{k=-\infty}^{\infty} \left[\frac{da_k}{dt} + ik\alpha U a_k + \nu k^2\alpha^2 a_k\right] e^{ik\alpha x}$$
$$= \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} i(m+n)\alpha\,[a_m(t)\ a_n(t)]\ e^{i(m+n)\alpha x}$$
$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} i\alpha k \sum_{m+n=k} [a_m(t)\ a_n(t)]\ e^{ik\alpha x}$$

$$\frac{da_k}{dt} + ik\alpha U a_k + \nu k^2\alpha^2 a_k = \frac{1}{2} ik\alpha \sum_{m+n=k} a_m\ a_n$$

Non-linear v-eta formulation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4 v = \underbrace{-\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)S_2 - \frac{\partial^2 S_1}{\partial x\,\partial y} - \frac{\partial^2 S_3}{\partial y\,\partial z}\right]}_{N_v}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta + U'\frac{\partial v}{\partial z} - \frac{1}{\mathrm{Re}}\nabla^2 \eta = \underbrace{-\left(\frac{\partial S_1}{\partial z} - \frac{\partial S_3}{\partial x}\right)}_{N_\eta}$$

$$S_1 = \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}$$

$$S_2 = \frac{\partial(uv)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(vw)}{\partial z}$$

$$S_3 = \frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y} + \frac{\partial(w^2)}{\partial z}$$

Fourier-transformed equations

$$v = \sum_m \sum_n \hat{v}_{mn}(y,t) e^{i\alpha_m x + i\beta_n z}$$

$$\frac{\partial}{\partial t} \underbrace{\begin{pmatrix} -D_{mn}^2 + k_{mn}^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}_{mn}} \begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix} - \underbrace{\begin{pmatrix} \mathcal{L}_{OS}^{mn} & 0 \\ -i\beta_n U' & \mathcal{L}_{SQ}^{mn} \end{pmatrix}}_{\mathbf{L}_{mn}} \underbrace{\begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix}}_{\hat{\mathbf{q}}_{mn}} = \sum_{k+p=m} \sum_{l+q=n} \underbrace{\begin{pmatrix} \hat{N}_v^{mn} \\ \hat{N}_\eta^{mn} \end{pmatrix}}_{\mathbf{n}_{mn}}$$

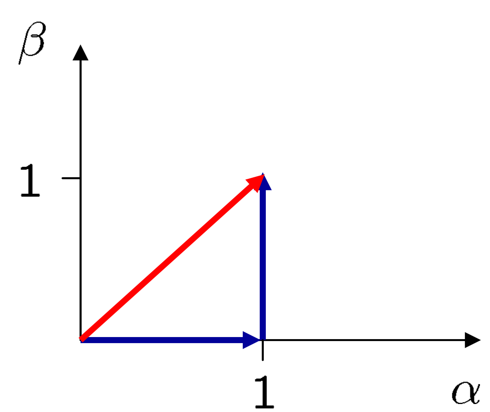
$$\left(\frac{\partial}{\partial t} \mathbf{M}_{mn} - \mathbf{L}_{mn} \right) \hat{\mathbf{q}}_{mn} = \sum_{k+p=m} \sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})$$

Convolution sums and triad interactions

$$\left(\frac{\partial}{\partial t}\mathbf{M}_{mn} - \mathbf{L}_{mn}\right)\hat{\mathbf{q}}_{mn} = \sum_{k+p=m}\sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})$$

non-linear interactions by $\hat{\mathbf{q}}_{kl}$ and $\hat{\mathbf{q}}_{pq}$ contributes to $\hat{\mathbf{q}}_{mn}$ if

$$(\alpha_m, \beta_n) = (\alpha_k, \beta_l) + (\alpha_p, \beta_q)$$



$$(1,1) = (1,0) + (0,1)$$

Example

- interaction possible with $(1,1)$ and $(1,-1)$

- real solution vector $\widehat{v}^*_{mn} = \widehat{v}_{-m,-n}$

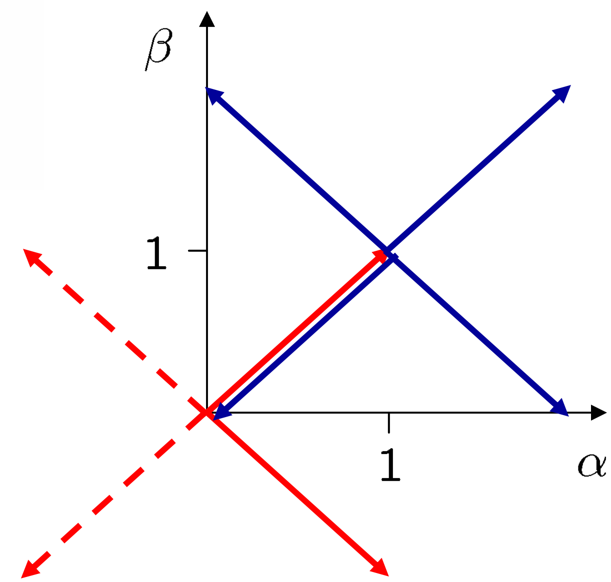
- spanwise symmetry $\widehat{v}_{mn} = \widehat{v}_{m,-n}$

$$(2,2) = (1,1) + (1,1)$$

$$(2,0) = (1,1) + (1,-1)$$

$$(0,2) = (1,1) + (-1,1)$$

$$(0,0) = (1,1) + (-1,-1)$$



Rate of change of energy

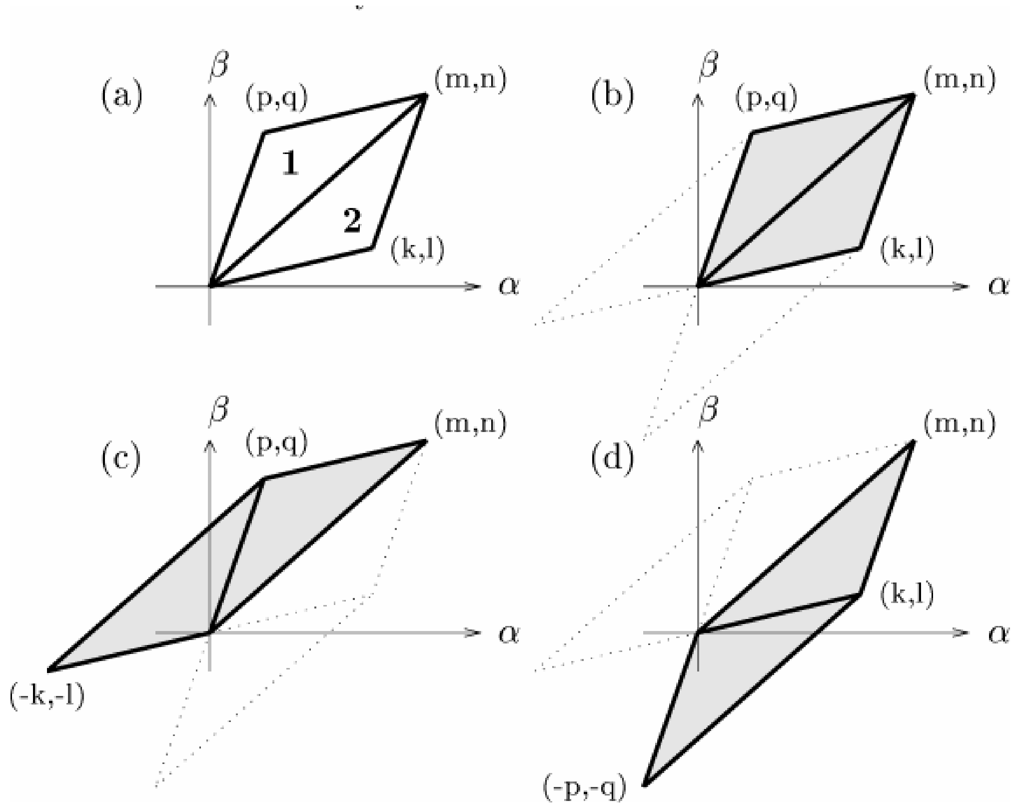
$$E_{mn} = \frac{1}{2k_{mn}^2}\int_y \mathbf{q}_{mn}^* \mathbf{M}_{mn}\mathbf{q}_{mn}dy$$

$$\frac{d}{dt}E_{mn} = \frac{1}{k_{mn}^2}\Re\left\{\int_y \frac{\partial \mathbf{q}_{mn}^*}{\partial t}\mathbf{M}_{mn}\mathbf{q}_{mn}dy\right\}$$

$$= \frac{1}{k_{mn}^2}\Re\left\{-\int_y \mathbf{q}_{mn}^* \mathbf{L}_{mn}\mathbf{q}_{mn}dy\right\}$$

$$+ \frac{1}{k_{mn}^2}\Re\left\{\underbrace{\sum_{k+p=m}\sum_{l+q=n}\int_y \mathbf{q}_{mn}^* \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})dy}_{\dot{E}([m,n],[p,q],[k,l])}\right\}$$

Conservation of energy in triads



$$T([m,n],[p,q],[k,l]) \;\equiv\; \dot{E}([m,n],[p,q],[k,l]) + \dot{E}([m,n],[k,l],[p,q])$$

$$T([m,n],[p,q],[k,l]) + T([p,q],[-k,-l],[m,n]) + T([k,l],[-p,-q],[m,n]) \;=\; 0$$

Form of the solution

- Shape assumption $u^{2D}(x', y) = \tilde{u}_{TS}(y)e^{i\alpha x'} + \tilde{u}^*_{TS}(y)e^{-i\alpha x'}$

- Floquet theory for PDEs with periodic coefficients

$$u(x', y, z, t) = \hat{u}(x', y)\ e^{\gamma x'}\ e^{\sigma t}\ e^{i\beta z}$$

- Temporal instability $\gamma_r = 0, \quad \sigma_r \neq 0$

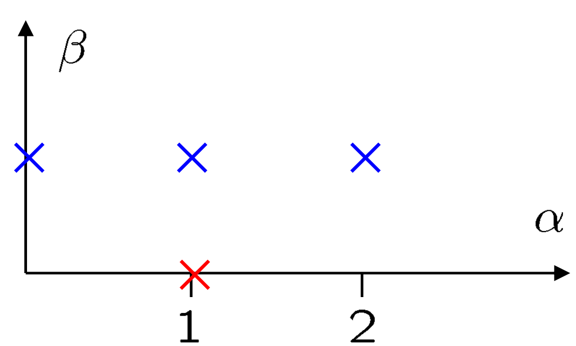
- Expand in Fourier series

$$u(x', y, z, t) = e^{\sigma_r t} e^{i\beta z} \sum_m \tilde{u}_m(y) e^{i(m\alpha + \gamma_i)x'}$$

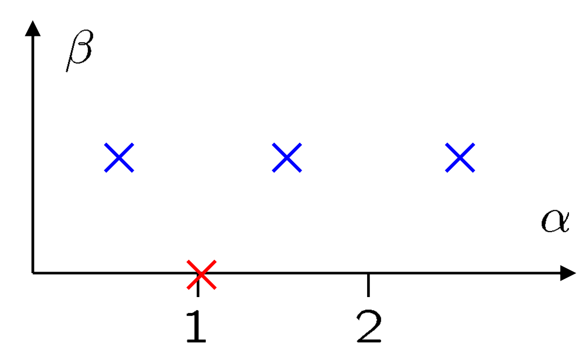
Classification of modes

$$u(x', y, z, t) = e^{\sigma_r t} e^{i\beta z} \sum_m \tilde{u}_m(y) e^{i(m\alpha + \gamma_i)x'}$$

$\gamma_i = 0$ fundamental instability



$\gamma_i = \alpha/2$ subharmonic instability



Secondary instability equations

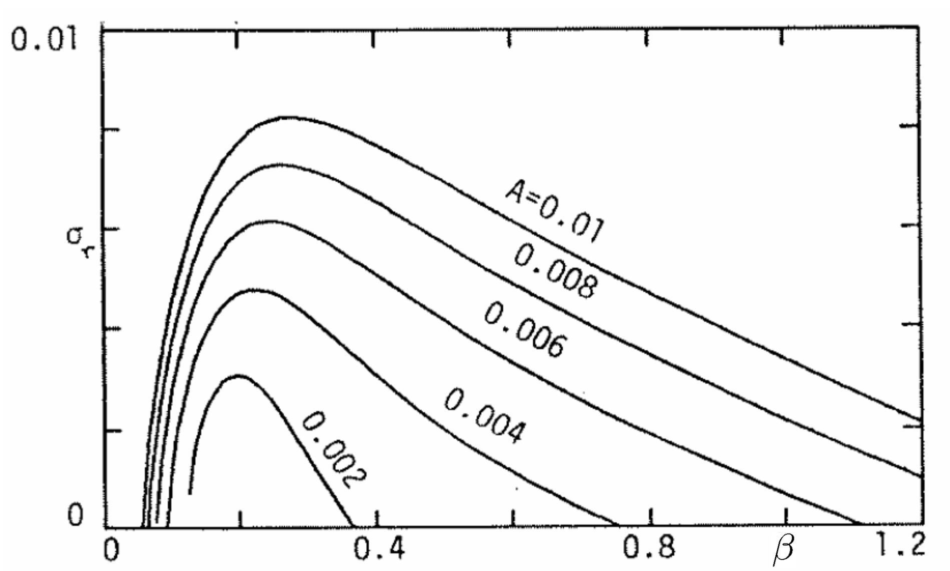
$$
\begin{aligned}
\sigma\tilde{u}_m + i\alpha_m(U-C)\tilde{u}_m + \tilde{v}_m U' + i\alpha_m\tilde{p} - \frac{1}{\mathrm{Re}}(D^2 - k_m^2)\tilde{u}_m &= N_v \\
\sigma\tilde{v}_m + i\alpha_m(U-C)\tilde{v}_m + D\tilde{p} - \frac{1}{\mathrm{Re}}(D^2 - k_m^2)\tilde{v}_m &= N_v \\
\sigma\tilde{w}_m + i\alpha_m(U-C)\tilde{w}_m + i\beta\tilde{p} - \frac{1}{\mathrm{Re}}(D^2 - k_m^2)\tilde{w}_m &= 0 \\
i\alpha_m\tilde{u}_m + D\tilde{v}_m + i\beta\tilde{w}_m &= 0
\end{aligned}
$$

$$
\begin{aligned}
N_u &= -A\left[i\alpha_m\tilde{u}_{m\pm1}\tilde{u}^{TS} + D(\tilde{v}_{m\pm1}\tilde{u}^{TS} + \tilde{u}_{m\pm1}\tilde{v}^{TS}) - i\beta\tilde{w}_{m\pm1}\tilde{u}^{TS}\right] \\
N_v &= -A\left[i\alpha_m(\tilde{u}_{m\pm1}\tilde{v}^{TS} + \tilde{v}_{m\pm1}\tilde{u}^{TS}) + D(v_{m\pm1}\tilde{v}^{TS}) - i\beta\tilde{w}_{m\pm1}\tilde{v}^{TS}\right]
\end{aligned}
$$

Secondary instability of 2D TS waves

Subharmonic secondary instability most unstable:

dependence on amplitude



comparisons with experiments