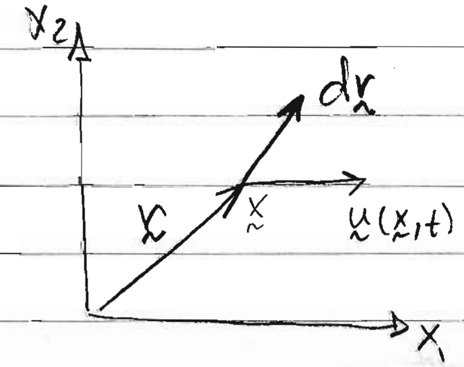
The relative motion near a point.

- Will give mathematical description of how small fluid elements deform in a given flow field.
- Physical understanding of mathematical expressions describing these deformations.
- Rates of deformation of fluid element directly linked to internal viscous forces in the fluid.



consider material line element

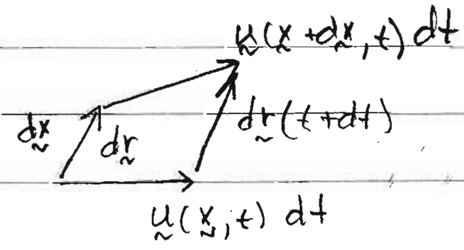
$$d\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{X}^0 + d\underset{\sim}{X}^0, \hat{t}) - \underset{\sim}{r}(\underset{\sim}{X}^0, \hat{t})$$

Translation with velocity

$$\frac{\partial \underset{\sim}{r}(\underset{\sim}{X}^0, \hat{t})}{\partial \hat{t}} = \underset{\sim}{u}(\underset{\sim}{x}, t); \quad \underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{X}^0, \hat{t}), \quad t = \hat{t}$$

The instantaneous rate of change of length and orientation of a material line element is given by the relative motion.

$$\frac{\partial}{\partial \hat{t}} \underset{\sim}{r}(\underset{\sim}{X}^0 + d\underset{\sim}{X}^0, \hat{t}) = \underset{\sim}{u}(\underset{\sim}{x} + d\underset{\sim}{x}, t) \quad ; \quad d\underset{\sim}{x} = d\underset{\sim}{r}(\underset{\sim}{X}^0, d\underset{\sim}{X}^0, \hat{t}), \quad t = \hat{t}$$



$$\frac{\partial\, d\underset{\sim}{r}}{\partial \hat{t}} = \frac{D\, d\underset{\sim}{r}}{Dt} = \underset{\sim}{u}(\underset{\sim}{x} + d\underset{\sim}{x}, t) - \underset{\sim}{u}(\underset{\sim}{x}, t)$$
$$= d\underset{\sim}{u}$$

Taylor expansion around $\underset{\sim}{x}$ $\Rightarrow$

$$du_i = u_i(\underset{\sim}{x} + d\underset{\sim}{x}, t) - u_i(\underset{\sim}{x}, t) =$$

Relative velocity

$$= u_i(\underset{\sim}{x}, t) + \frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_1} dx_1 + \frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_2} dx_2 + \frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_3} dx_3 +$$

$$- u_i(\underset{\sim}{x}, t) = \sum_{j=1}^{3} \frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_j} dx_j = \frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_j} \underbrace{dx_j}$$

du_i are components of relative velocity at $\underset{\sim}{x}$, t

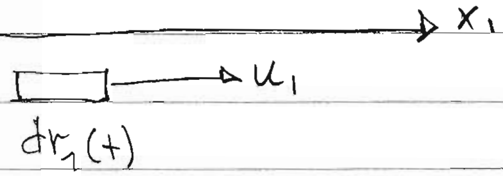
dx_j are components of considered line element

$\frac{\partial u_i(\underset{\sim}{x}, t)}{\partial x_j}$ the velocity gradient tensor (indep. of length & orient. of line element

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

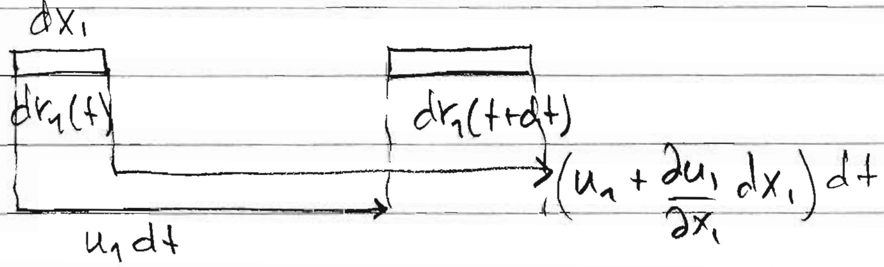
Evaluation of the effect on the relative motion from individual components of $\frac{\partial u_i}{\partial x_j}$.

* Line element along x_1-axis



$$\frac{D}{Dt} dr_1 = \frac{\partial u_1}{\partial x_1} dr_1 + 0 + 0$$

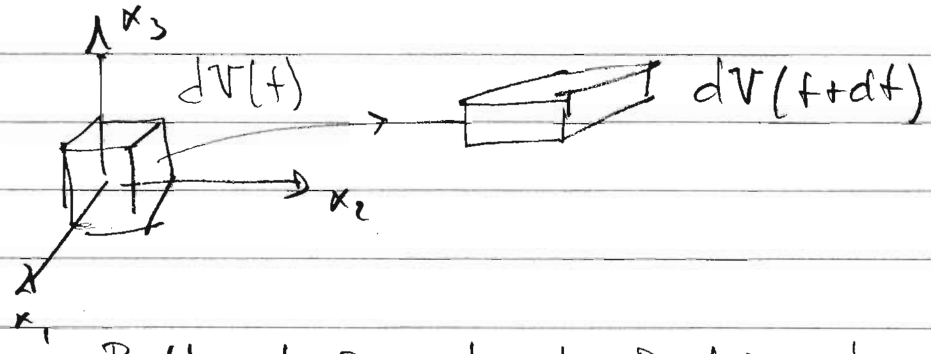
Normal (linear) strain rate $\frac{1}{dr_1} \frac{D \, dr_1}{Dt} = \frac{\partial u_1}{\partial x_1}$



$$u_1 dt + dr_1(t+dt) = dr_1(t) + \left(u_1 + \frac{\partial u_1}{\partial x_1} dx_1\right) dt$$

$$\underbrace{\frac{dr_1(t+dt) - dr_1(t)}{dt}}_{\frac{D \, dr_1}{Dt}} = \frac{\partial u_1}{\partial x_1} dx_1$$

* Material volume $dV = dr_1 dr_2 dr_3$



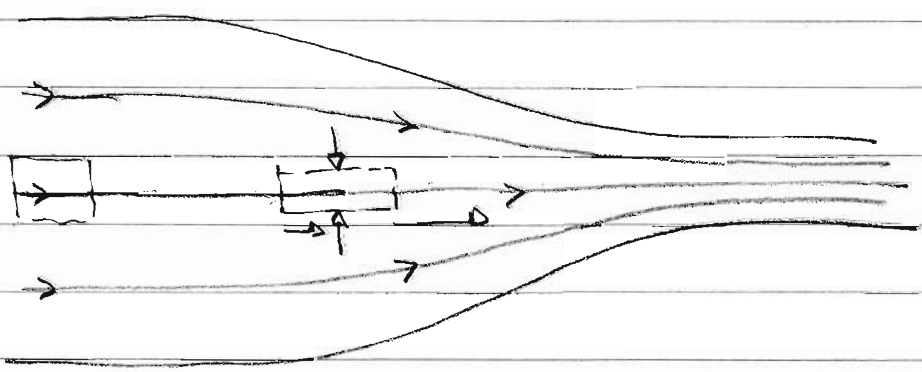
Bulk strain rate $\dfrac{1}{dV}\dfrac{D\,dV}{Dt} = \dfrac{1}{dr_1 dr_2 dr_3}\dfrac{D}{Dt}(dr_1 dr_2 dr_3)$

$$= \frac{1}{dr_1}\frac{D\,dr_1}{Dt} + \frac{1}{dr_2}\frac{D\,dr_2}{Dt} + \frac{1}{dr_3}\frac{D\,dr_3}{Dt} =$$

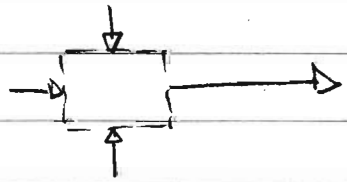
$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \underset{\sim}{u}$$

For incompressible fluid $\dfrac{D\,dV}{Dt} = 0 \Leftrightarrow \nabla \cdot \underset{\sim}{u} = 0$

2D ex

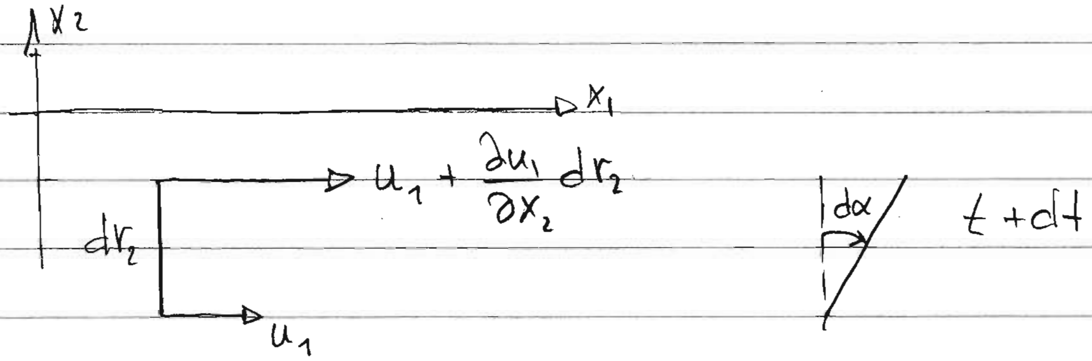


Fix control volume

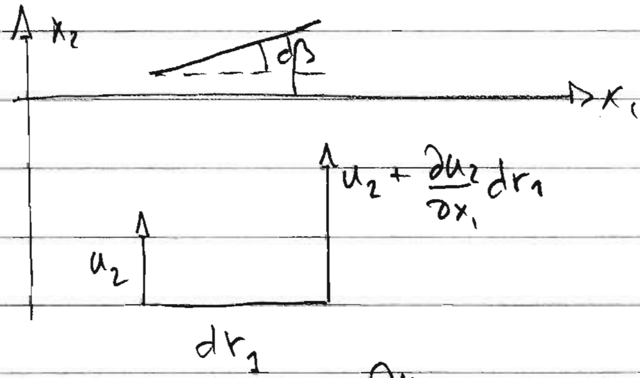


$$\underbrace{\frac{\partial u_1}{\partial x_1}}_{>0} + \underbrace{\frac{\partial u_2}{\partial x_2}}_{<0} = 0$$

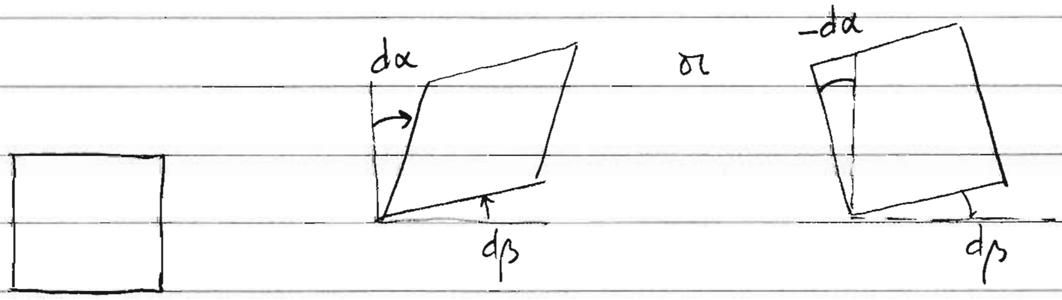
* Line element ⊥ to the flow



$$\underbrace{\tan(d\alpha)}_{\approx d\alpha} = \frac{dt \frac{\partial u_1}{\partial x_2} dr_2}{dr_2} \Rightarrow \frac{d\alpha}{dt} = \frac{\partial u_1}{\partial x_2}$$



$$\tan(d\beta) = \frac{dt \frac{\partial u_2}{\partial x_1} dr_1}{dr_1} \Rightarrow \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1}$$



Definitions: Shear strain rate

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

gives deformation in shape

if $d\beta = -d\alpha \Rightarrow$ no shape change, just rotation

Vorticity

$$-\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \omega_3$$

The average rotation rate $= \frac{1}{2}\,\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$

Velocity gradient tensor

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric part}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{antisymmetric part}}$$

$e_{ij} = e_{ji}$ $\qquad$ $\xi_{ij} = -\xi_{ji}$

strain rate tensor $\qquad$ rotation tensor

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$e_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \underset{\sim}{u}$$

$$\xi_{ij} = \begin{bmatrix} 0 & -\frac{1}{2}\omega_3 & \frac{1}{2}\omega_2 \\ \frac{1}{2}\omega_3 & 0 & -\frac{1}{2}\omega_1 \\ -\frac{1}{2}\omega_2 & \frac{1}{2}\omega_1 & 0 \end{bmatrix}$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

Shorter notation

$$\xi_{ij} = -\frac{1}{2} \underline{\underline{\varepsilon_{ijk}}}\, \omega_k$$

permutation tensor

$\varepsilon_{ijk} = 1$ for cyclic permutations of 123

$\varepsilon_{ijk} = -1$ for non-cyclic —— " ——

$\varepsilon_{ijk} = 0$ otherwise

Cross product $(\underset{\sim}{a} \times \underset{\sim}{b})_i = \varepsilon_{ijk}\, a_j b_k$

Rotation of 3D line element

$$\left(\frac{D}{Dt} d\underset{\sim}{r}\right)_{rot.} = \xi_{ij}\, dx_j = -\frac{1}{2} \varepsilon_{ijk}\, \omega_k\, dx_j = -\frac{1}{2} d\underset{\sim}{x} \times \underset{\sim}{\omega}$$

$$= \frac{1}{2} \underset{\sim}{\omega} \times d\underset{\sim}{x}$$

i.e. solid body rotation at angular velocity $\frac{1}{2}\underset{\sim}{\omega}$

(c.f. $\dot{r} = \underset{\sim}{\Omega} \times \underset{\sim}{r}$)

One also finds that

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Tensor notation $\omega_i = \varepsilon_{ijk} \dfrac{\partial u_k}{\partial x_j} =$

$$= \underbrace{\underbrace{\varepsilon_{ijk}}_{as} \underbrace{e_{kj}}_{s}}_{=0} + \underline{\underline{\varepsilon_{ijk} \xi_{kj}}}$$

$$\left\{ \underset{s_{ij}}{\begin{bmatrix} a & c \\ c & b \end{bmatrix}} \underset{a_{ij}}{\begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}} = 0 + cd - cd + 0 = 0 \right\}$$

$$\left[-\frac{1}{2} \varepsilon_{ijk} \omega_k = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{klm} \xi_{ml} = -\frac{1}{2} \varepsilon_{kij} \varepsilon_{klm} \xi_{ml} \right.$$

$$= -\frac{1}{2} \xi_{ml} \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) =$$

$$= -\frac{1}{2} \xi_{ji} + \frac{1}{2} \xi_{ij} = \frac{1}{2} \xi_{ij} + \frac{1}{2} \xi_{ij} = \xi_{ij}$$

Partition of e_{ij}

– isotropic part

$$\bar{\bar{e}}_{ij} = \begin{bmatrix} \frac{1}{3}e_{kk} & 0 & 0 \\ 0 & \frac{1}{3}e_{kk} & 0 \\ 0 & 0 & \frac{1}{3}e_{kk} \end{bmatrix} = \frac{1}{3} e_{kk} \delta_{ij} = \frac{1}{3} \nabla \cdot \underline{u} \, \delta_{ij}$$

gives deformation due to volume expansion

– deviatoric part

$$\bar{e}_{ij} = e_{ij} - \bar{\bar{e}}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right)$$

gives deformation rate without change of volume.

traceless $\bar{e}_{kk} = 0$

Summary of relative motion

$$\frac{D \, dr_i}{Dt} = du_i = \frac{\partial u_i}{\partial x_k} dx_k = \underbrace{\bar{e}_{ik} dx_k}_{(1)} + \underbrace{\bar{\bar{e}}_{ik} dx_k}_{(2)} + \underbrace{\xi_{ik} dx_k}_{(3)}$$

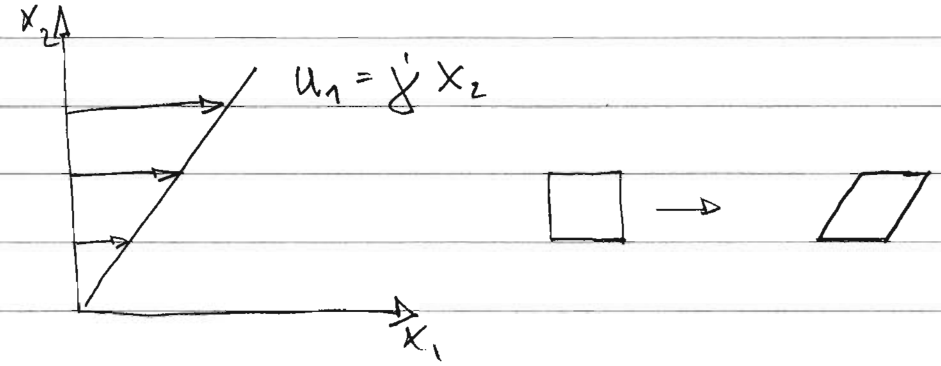
The terms of du_i are results of

(1) pure deformation, without change of volume

(2) isotropic change of volume

(3) local solid body rotation

Ex: Shear flow

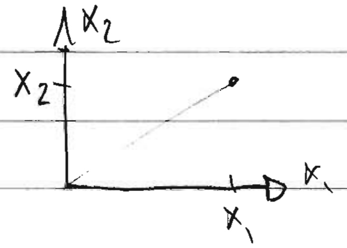


$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_{ij}} + \underbrace{\begin{bmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ -\frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\xi_{ij}}$$

$$e_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 + 0 + 0$$ no vol. def.

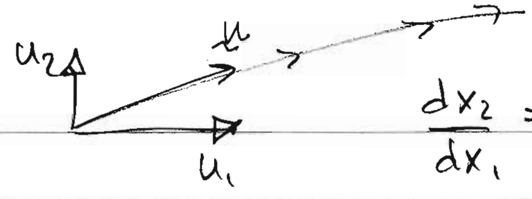
Isotropic strain rate tensor $\bar{e}_{ij} = 0$

Relative velocity field $du_i = \frac{\partial u_i}{\partial x_k} dx_k =$



$$= \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \underbrace{\frac{\gamma}{2}\begin{bmatrix} x_2 \\ x_1 \\ 0 \end{bmatrix}}_{(du_i)_{def}} + \underbrace{\frac{\gamma}{2}\begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}}_{(du_i)_{rot}}$$

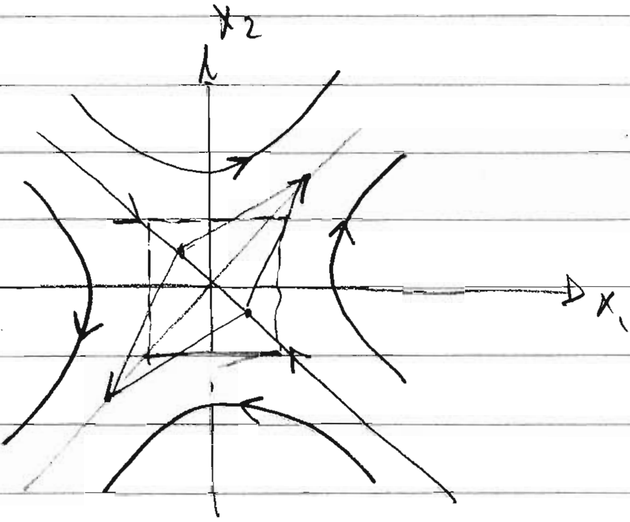
Streamlines



$$\frac{dx_2}{dx_1} = \frac{u_2}{u_1}$$

Deformation $$\frac{dx_2}{dx_1} = \frac{\frac{1}{2}\gamma x_1}{\frac{1}{2}\gamma x_2} = \frac{x_1}{x_2} \Rightarrow x_2\,dx_2 = x_1\,dx_1$$

$$\Rightarrow \frac{x_2^2}{2} = \frac{x_1^2}{2} + C$$



Rotation $$\frac{dx_2}{dx_1} = \frac{-\frac{1}{2}\gamma x_1}{\frac{1}{2}\gamma x_2} = -\frac{x_1}{x_2} \Rightarrow x_2\,dx_2 + x_1\,dx_1 =$$

$$\Rightarrow \frac{x_2^2}{2} + \frac{x_1^2}{2} = C$$