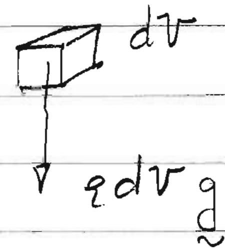
Forces on a fluid element

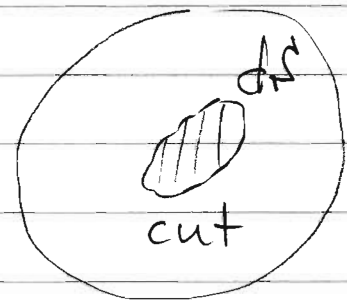
- External body forces
  (long range interaction)

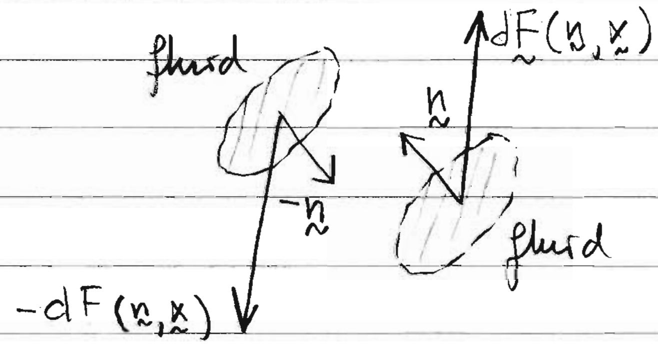


proportional to mass of fluid element $\quad \rho\, dV\, \mathbf{g}$

↑ density

- Surface forces
  (short range interaction)





contact forces cancel

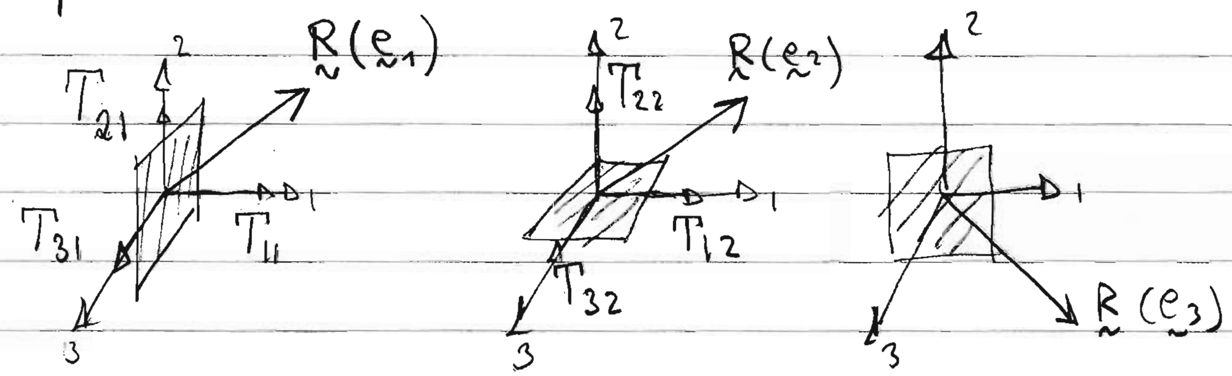
proportional to area of fluid element

$$d\mathbf{F} = \underbrace{\mathbf{R}(\mathbf{n},\mathbf{x})}_{\text{stress force (force/unit area)}}\, dS$$

Stress tensor

The stress state is uniquely determined by a tensor field $T_{ij}(\underset{\sim}{x}, t)$

The components are given by the surface forces on three orthogonal planes



$\underset{\sim}{R}(\underset{\sim}{e}_1) = (T_{11}, T_{12}, T_{13})$ $\underset{\sim}{R}(\underset{\sim}{e}_2) = (T_{12}, T_{22}, T_{32})$

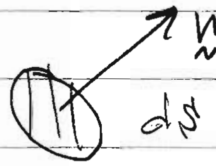
T_{ij} is i-component of stress on surface element with unit normal in j-direction

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

shear stresses

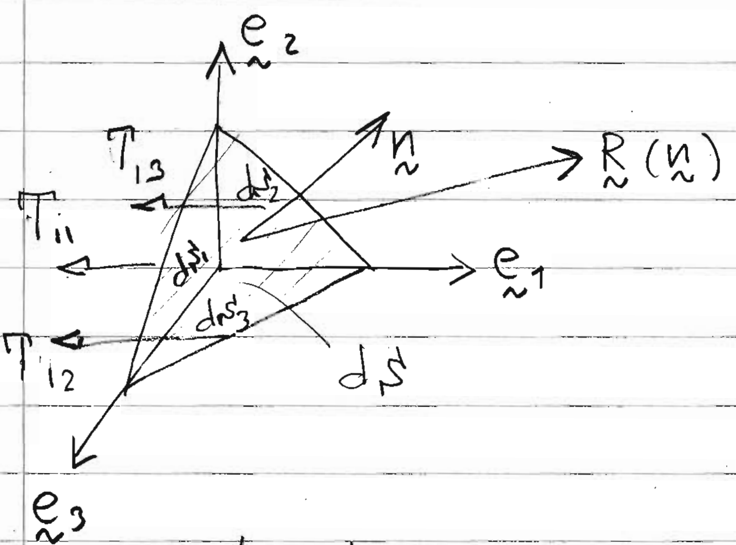
normal stresses

The stress tensor may be used to obtain the stress on a surface with any direction



let $d\underset{\sim}{S} = \underset{\sim}{n}\, dS$

or $dS_i = n_i\, dS$



Surface forces must balance each other to lowest order on small fluid element.

at dS: $dF_i = R_i\, dS$

at orthogonal surfaces:

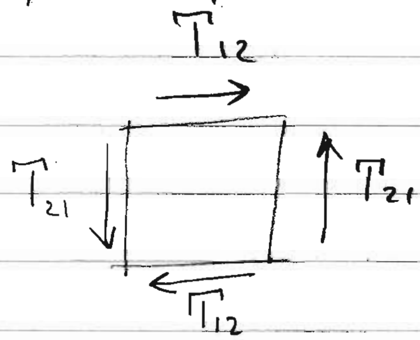
$$-T_{i1}dS_1 - T_{i2}dS_2 - T_{i3}dS_3 = -T_{ij}\,dS_j = -T_{ij}\,n_j\,dS$$

$\Rightarrow$ Net force $R_i\, dS - T_{ij} n_j\, dS = 0$

$\Rightarrow$ $\boxed{R_i = T_{ij}\, n_j}$ $\boxed{\underset{\sim}{R} = \underset{\sim}{T} \cdot \underset{\sim}{n}}$

Moment balance around fluid particle shows that

$$T_{ij} = T_{ji}$$

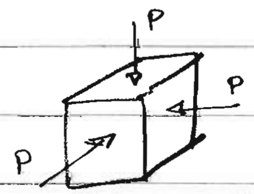


$\Rightarrow$ T_{ij} has only 6 independent components.

Pressure and viscous stresses

For fluid element at rest only isotropic normal stresses are present.
(Otherwise, the fluid element would deform continuously.)

At rest : $T_{ij} = -p\,\delta_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$



At deformation : $T_{ij} = -p\,\delta_{ij} + \tau_{ij}$

p is hydrostatic pressure (directed inward)

τ_{ij} is viscous stress tensor, depends on the fluids rate of deformation.
(Note that τ_{ij} may also have normal component

Newtonian fluid

Viscous stresses proportional to deformation rate of fluid element.

$$\tau_{ij} = 2\mu(T)\bar{e}_{ij} + \mu_B(T)\bar{\bar{e}}_{ij}$$

isotropic strain rate tensor

deviatoric strain rate tensor

τ_{ij} does not depend on rotation tensor ξ_{ij}

$\mu(T)$ dynamic viscosity [kg/ms]

$\mu_B(T)$ bulk viscosity [kg/ms]

(in air $\mu_B \sim 0.6\,\mu$, in CO_2 $\mu_B \sim 1000\,\mu$)

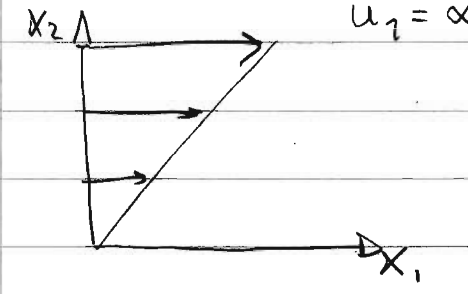
$\nu = \mu/\varrho$ kinematic viscosity [m^2/s]

μ_B often disregarded since usually

$$\left|\frac{\partial u_n}{\partial x_n}\right| \ll |\bar{e}_{ij}|$$

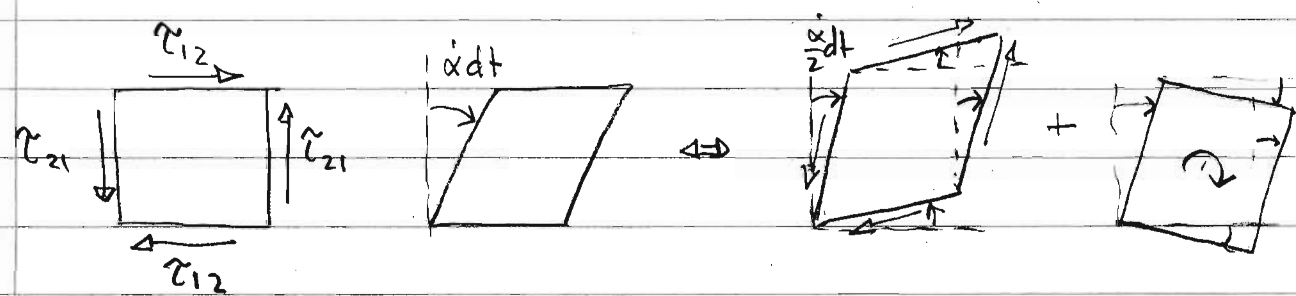
(exception, e.g. strong shock waves)

Ex.



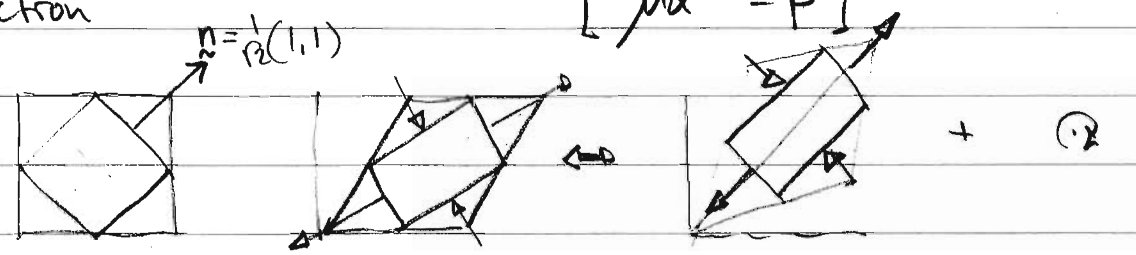
$u_1 = \dot{\alpha}\, x_2 \qquad u_2 = 0 \qquad \dfrac{\partial u_k}{\partial x_k} = 0$

$$\bar{e}_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\dot{\alpha} \\ \frac{1}{2}\dot{\alpha} & 0 \end{bmatrix} ; \quad \bar{\bar{e}}_{ij} = 0 ; \quad \xi_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\dot{\alpha} \\ -\frac{1}{2}\dot{\alpha} & 0 \end{bmatrix}$$



$$T_{ij} = -p\,\delta_{ij} + 2\mu\,\bar{e}_{ij} = \begin{bmatrix} -p & \mu\dot{\alpha} \\ \mu\dot{\alpha} & -p \end{bmatrix}$$

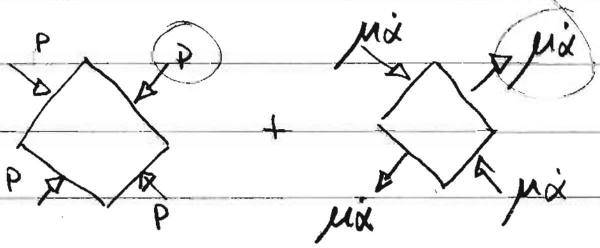
New direction



Here viscous stresses appear normal to surface

$$R_i = T_{ij} n_j = \begin{bmatrix} -p & \mu\dot{\alpha} \\ \mu\dot{\alpha} & -p \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -\frac{p}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\mu\dot{\alpha}}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underset{\sim}{R}(\underset{\sim}{n}) = (-p + \mu\dot{\alpha})\, \underset{\sim}{n}$



Conservation of momentum

- Application of Newton's law of motion on material fluid volume.

- Transformation to fixed open control volume.

- Transformation to differential form, the Navier–Stoke's equations.

{Show OH of control volume & forces}

Newton ⇒

$$\frac{D}{Dt}\int_{\underset{\text{material volume}}{V(t)}} \rho u_i \, dV = \int_V \rho f_i \, dV + \oint\!\!\oint_S T_{ij} n_j \, dS$$

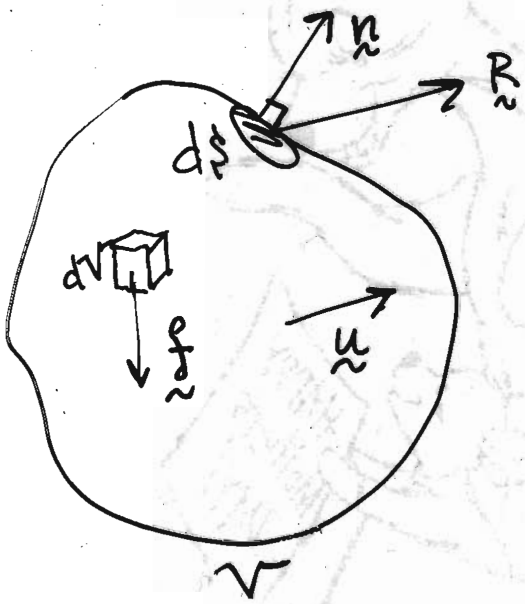
$$\underbrace{\frac{D}{Dt}\int_{V(t)} \rho \underset{\sim}{u} \, dV}_{} = \int_V \rho \underset{\sim}{f} \, dV + \oint\!\!\oint_S \underset{\sim}{T} \cdot \underset{\sim}{n} \, dS$$

Transform to integrals over fixed volume

We need <u>Reynolds transport theorem</u>.

{Show OH of RTT}

Control volume and forces.



body force per unit mass f_i

surface force per unit area $R_i = T_{ij} n_j$

momentum per unit volume ρu_i

Newton's law of motion

Rate of change of momentum of <u>material volume</u> equals sum of forces acting on volume

Reynolds transport theorem

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \frac{d}{dt}\int_{V} F\,dV + \underbrace{\oint\!\!\!\oint_{S} F(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS}_{\text{net outflow of } F \text{ through fixed surface}}$$

$V(t)$: material volume

V: fixed volume

Gauss' theorem

$$\iiint_{V} \nabla\cdot(\underset{\sim}{A})\,dV = \oint\!\!\!\oint_{S} \underset{\sim}{A}\cdot\underset{\sim}{n}\,dS$$

$$\iiint_{V} \frac{\partial}{\partial x_k}\left(A_{ijk}\right) dV = \oint\!\!\!\oint_{S} A_{ijk}\, n_k\, dS$$

OH
$$\frac{D}{Dt}\int_{V(t)} F\,dV = \frac{d}{dt}\int_{V} F\,dV + \underbrace{\oint\!\!\oint_{S} F(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS}_{\text{net outflow of } F \text{ through fix } S.}$$

$V(t)$: material volume

V: fixed volume

Proof of RTT see Kundu & Cohen or alternative derivation:

$$\frac{D}{Dt}\int_{V(t)} F\,dV(t) = \int_{V} \frac{DF}{Dt}\,dV + \int_{V} F\,\frac{D\,dV}{Dt} =$$

$$= \int_{V}\left(\frac{\partial F}{\partial t} + \underset{\sim}{u}\cdot\nabla F\right)dV + \int_{V} F\,\nabla\cdot\underset{\sim}{u}\,dV =$$

$$= \int_{V}\left(\frac{\partial F}{\partial t} + \nabla\cdot(F\underset{\sim}{u})\right)dV = \left\{\begin{array}{l}\text{Gauss theorem}\\ \text{Show OH}\end{array}\right\} =$$

fix volume

$$= \frac{d}{dt}\int_{V} F\,dV + \oint\!\!\oint_{S} F(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS$$

Newtons law of motion in ~~fixed~~ open volume.
(Momentum theorem.)

$$\frac{d}{dt}\int_V \rho \underset{\sim}{u}\, dV + \oiint_S \rho \underset{\sim}{u}\,(\underset{\sim}{u}\cdot\underset{\sim}{n})\, dS = \int_V \rho \underset{\sim}{f}\, dV + \oiint_S \underset{\sim}{T}\cdot\underset{\sim}{n}\, dS$$

$$\underbrace{\frac{d}{dt}\int_V \rho u_i\, dV}_{\text{Rate of change of momentum in fixed control volume}} + \underbrace{\oiint_S \rho u_i (u_k n_k)\, dS}_{\text{Net outflow of momentum per unit time}} = \underbrace{\int_V \rho f_i\, dV}_{\text{body forces}} + \underbrace{\oiint_S T_{ik} n_k\, dS}_{\text{surface forces on } V}$$

Gauss' theorem $\Rightarrow$

$$\int_V \left\{ \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) - \rho f_i - \frac{\partial}{\partial x_k}(T_{ik}) \right\} dV = 0$$

Should hold for arbitrary volume $\Rightarrow \{\dots\} = 0$

Differential form of Newt. law of mot.

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) = \rho f_i + \frac{\partial T_{ik}}{\partial x_k}$$

$$\underbrace{u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_k)}{\partial x_k}\right)} + \rho\frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial}{\partial x_k} u_i = \rho f_i + \frac{\partial T_{ik}}{\partial x_k}$$

$$u_i\Bigg(\underbrace{\frac{\partial \rho}{\partial t} + u_k\frac{\partial \rho}{\partial x_k}}_{\frac{D\rho}{Dt}} + \underbrace{\rho\frac{\partial u_k}{\partial x_k}}_{\rho\frac{1}{dV}\frac{D}{Dt}dV}\Bigg)$$

$$\frac{D\rho}{Dt} + \rho\frac{1}{dV}\frac{D}{Dt}dV = \frac{1}{dV}\underbrace{\left(\frac{D\rho}{Dt}dV + \rho\frac{D}{Dt}dV\right)}_{\frac{D}{Dt}(\underbrace{\rho\, dV}_{dm}) = 0}$$

dm fluid mass element

Remains Cauchy's eq. of motion

$$\rho\frac{D u_i}{Dt} = \rho f_i + \frac{\partial T_{ik}}{\partial x_k}$$

net surface force / unit volume

body force / unit volume

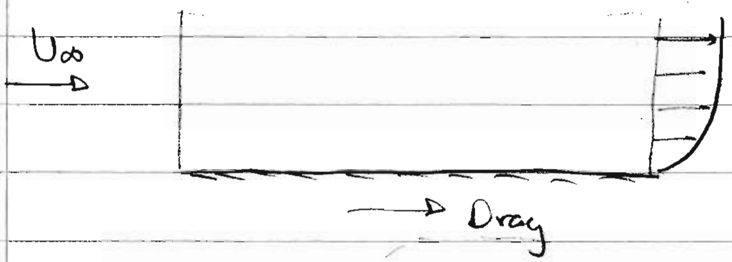
acc. of fluid element

material

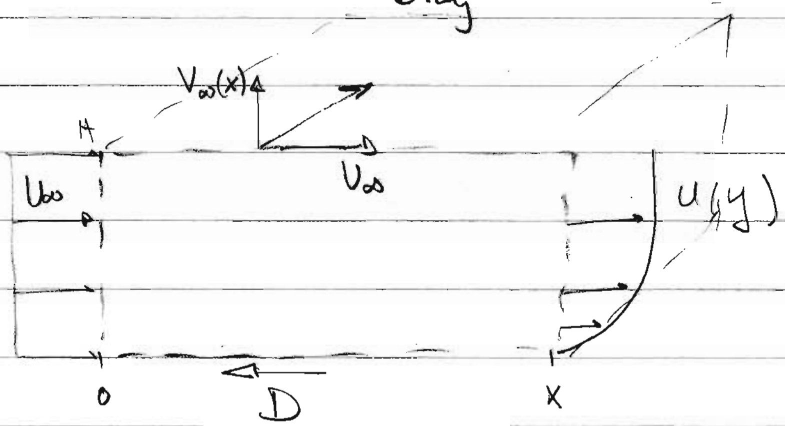
Navier–Stokes eq. are obtained if we insert

$$T_{ij} = -p\,\delta_{ij} + 2\mu\,\bar{e}_{ij} + \mu_B\,\bar{\bar{e}}_{ij}$$

Ex. Application of momentum theorem.



2 D



No loss of mass (volume): $\int_0^H (U_\infty - u(y))\,dy = \int_0^X V_\infty(x)\,dx$

Net outflow of x-momentum:

$$\int_0^H \left(\rho u(y)\, u(y) + \rho U_\infty(-U_\infty)\right) dy + \int_0^X \rho U_\infty V_\infty(x)\,dx =$$

$$= \int_0^H \left(\rho u^2(y) - \cancel{\rho U_\infty^2} + \rho U_\infty(\cancel{U_\infty} - u(y))\right) dy = \int_0^H \rho u(y)\,(u(y) - U_\infty)\,dy$$

$$= -D$$

Drag per unit length $D = \int_0^H \rho u(y)\,(U_\infty - u(y))\,dy$