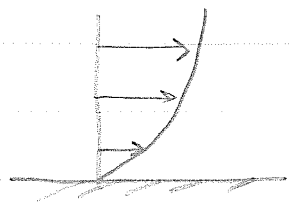
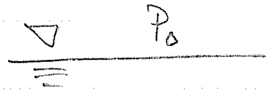
Navier-Stoke's equations.

Boundary conditions

* 

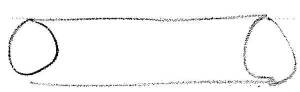
$u_i = 0$ on solid surfaces at rest

"no slip" condition of viscous fluid

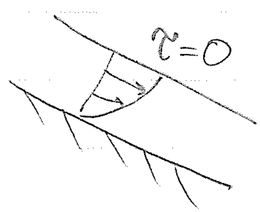
* 

surface force given at boundary

– e.g. stagnant water surface at constant pressure



– pressure given at the inlet /outlet of a pipe flow



– zero shear stress at free surface

Thermal boundary conditions

* $T = T_{wall}$ temperature specified at solid surface

* $-k \nabla T \cdot \underset{\sim}{n} = 0$ thermally isolated surface

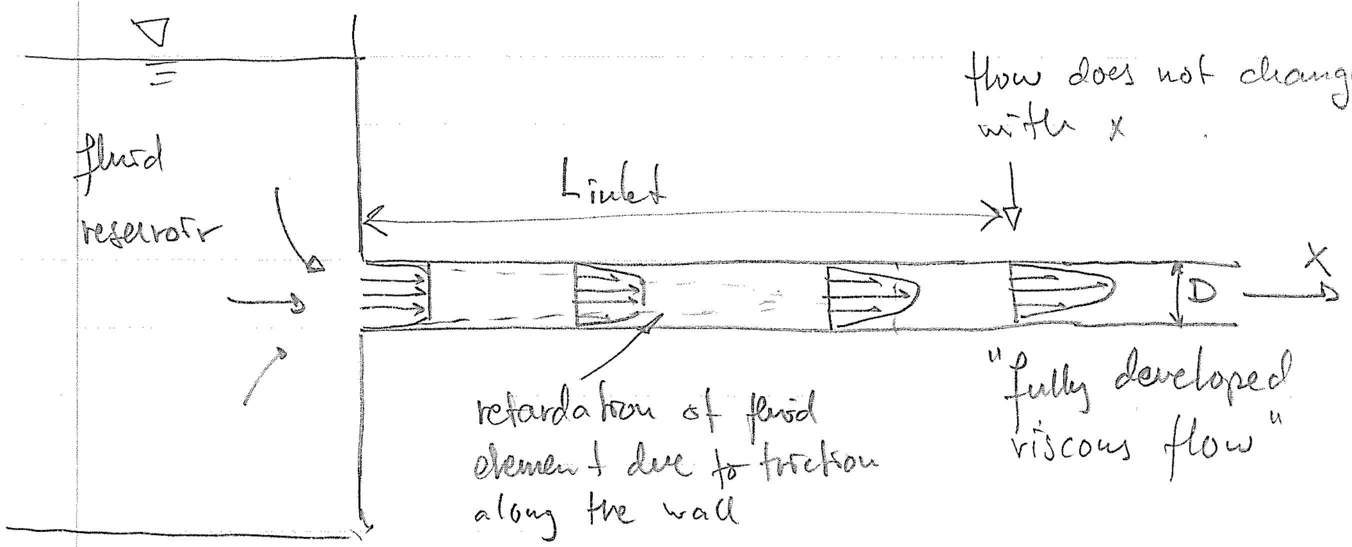
Exact analytical solutions of the N.-S. equations.

Examples are rare, but instructive to learn.

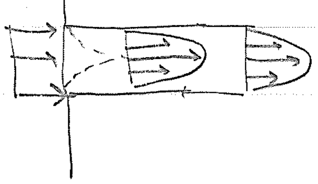
- Geometrical idealizations
  * 2D flow
  * 1 infinite dimension
  * axial symmetry

- Physical assumptions
  * steady state
  * "fully developed flow"
  * incompressible fluid
  * constant viscosity/heat conductivity

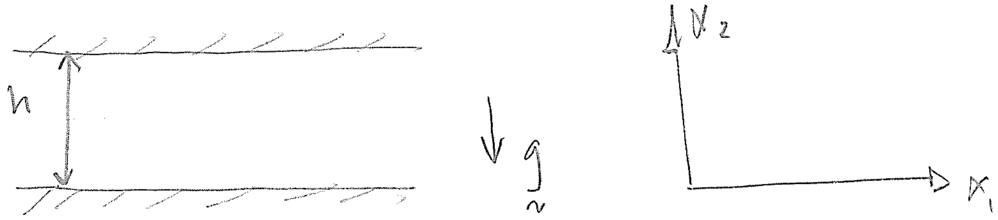
Flow in channel (or tube)



Inlet length for laminar flow $L_{inlet} \sim D\,Re$; $Re = \frac{\overline{U} D}{\nu}$

$Re \ll 1$  $L_{inlet} \sim D$

Fully developed channel flow



$\underset{\sim}{u} = (u_1(x_2), u_2(x_2), 0)$ * 2D flow

* incompressible flow $\underbrace{\frac{\partial u_1}{\partial x_1}}_{=0} + \frac{\partial u_2}{\partial x_2} + \underbrace{\frac{\partial u_3}{\partial x_3}}_{=0} = 0$

$\Rightarrow$ $u_2 = \text{const.} = 0$ from b.c. on solid wall
(In a model
for a porous wall/membrane, we could allow
a flow normal to the walls $u_2 = \text{const.} \neq 0$)

* assume steady state flow
$\frac{\partial}{\partial t}(\cdot) = 0$

2D momentum equation $\Rightarrow$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}\right) = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{\partial^2 u_1}{\partial x_2^2}$$

$$\rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\right) = -\frac{\partial p}{\partial x_2} - \rho g + \mu\frac{\partial^2 u_2}{\partial x_1^2} + \mu\frac{\partial^2 u_2}{\partial x_2^2}$$

$$\Rightarrow \quad 0 = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2 u_1}{\partial x_2^2} \qquad (1)$$

$$0 = -\frac{\partial p}{\partial x_2} - \rho g \qquad (2)$$

$$(2) \Rightarrow \quad p = -\rho g x_2 + P(x_1)$$

$$\Rightarrow \quad \frac{\partial p}{\partial x_1} = \frac{dP}{dx_1}(x_1)$$

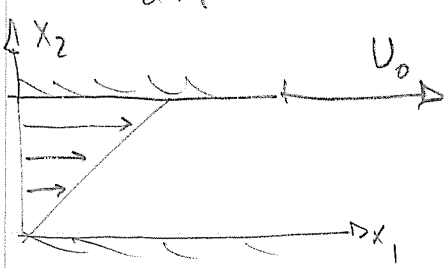
$$(1) \Rightarrow \quad 0 = \underbrace{-\frac{dP}{dx_1}(x_1)}_{\text{cannot vary with } x_1 \text{ since}} + \mu \frac{\partial^2 u_1(x_2)}{\partial x_2^2}$$

indep. of x_1

$$\therefore \quad \frac{dP}{dx_1} \text{ is constant} \Rightarrow \quad \frac{d^2 u_1}{dx_2^2} = \frac{1}{\mu}\frac{dP}{dx_1} = \text{const.} \quad (3)$$

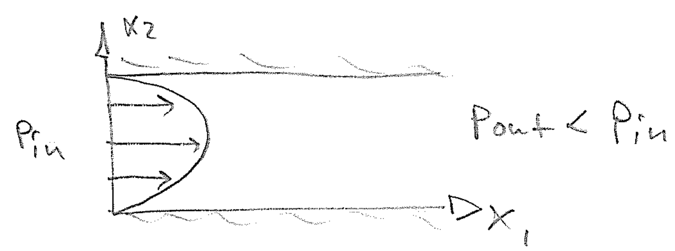
We will study two cases

A) $\frac{dP}{dx_1} = 0$



Forcing by upper wall

B) $\frac{dP}{dx_1} < 0$



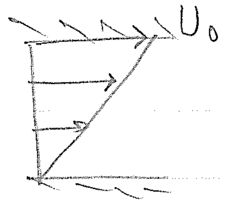
Forcing by pressure difference

A) Integrate (3) $\Rightarrow$ $u_1 = A x_2 + B$

Boundary conditions : $u_1(x_2=0)=0$, $u_1(x_2=h)=U_0$

$$\Rightarrow B=0 \quad , \quad A = U_0/h$$

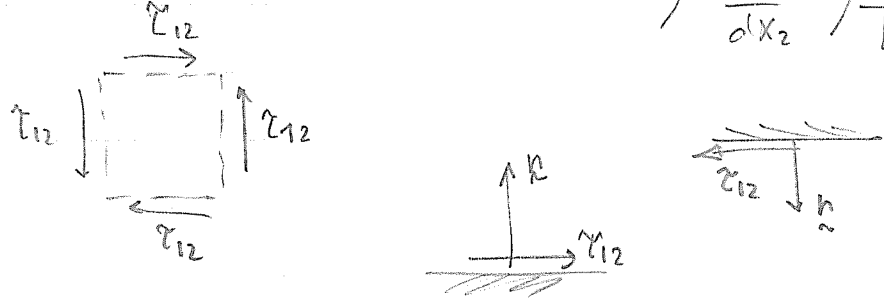
$u_1(x_2) = U_0 \dfrac{x_1}{h}$

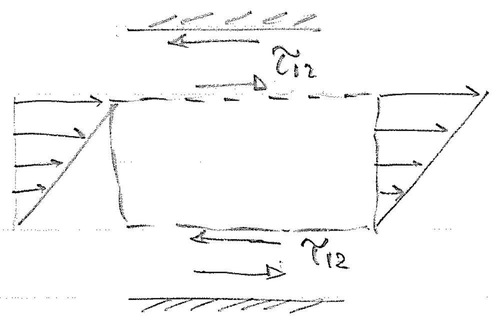


Plane Couette flow

calculate shear stress $\tau_{12} = 2\mu \, \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) =$

$$= \mu \frac{du_1}{dx_2} = \mu \frac{U_0}{h} \quad \text{(constant)}$$





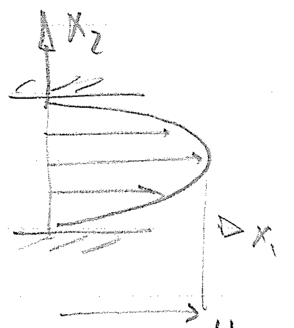
$$\underbrace{\frac{d}{dt}\int_V \rho \underset{\sim}{u}\, dV}_{=0} + \underbrace{\oint\!\!\oint_S \rho \underset{\sim}{u}\, \underset{\sim}{u}\cdot\underset{\sim}{n}\, dS}_{=0} =$$

$$= \underbrace{\int_V \rho \underset{\sim}{g}\, dV + \oint\!\!\oint_S (-p)\underset{\sim}{n}\, dS}_{=0} + \underbrace{\oint\!\!\oint_S \underset{\sim}{\tau}\cdot\underset{\sim}{n}\, dS}_{=0}$$

B) Integrate (3) $\Rightarrow$ $u_1 = \frac{1}{\mu}\frac{dp}{dx_1}\frac{x_2^2}{2} + C_1 x_2 + C_2$

Boundary conditions: $u_1(x_2=0) = u_1(x_2=h) = 0$

$$\Rightarrow \quad u_1(x_2) = -\frac{h^2}{2\mu}\frac{dp}{dx_1}\frac{x_2}{h}\left(1-\frac{x_2}{h}\right) = u_{max}\, 4\frac{x_2}{h}\left(1-\frac{x_2}{h}\right)$$



Plane Poiseuille flow

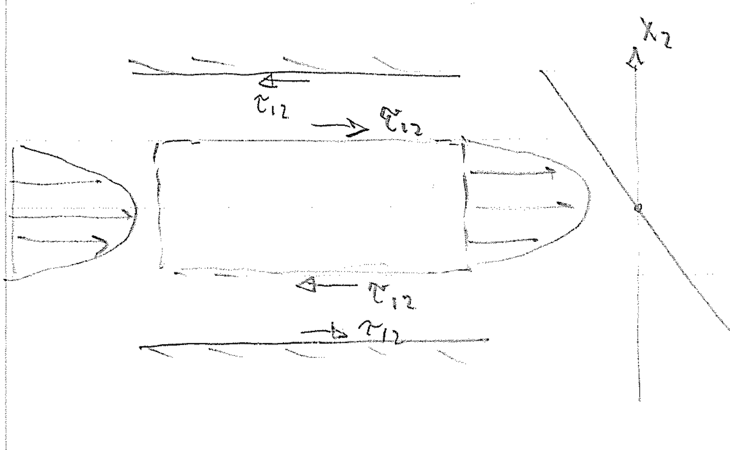
$$u_{max} = -\frac{h^2}{8\mu}\frac{dp}{dx_1}$$

Volumetric flow rate / unit width $Q = \int_0^h u_1(x_2)\,dx_2$

$$= u_{max}\int_0^h 4\frac{x_2}{h}\left(1-\frac{x_2}{h}\right)dx_2 = u_{max}\frac{4h}{6} = -\frac{h^3}{12\mu}\frac{dp}{dx_1}$$

Average velocity $\bar{U} = \frac{Q}{h} = \frac{2}{3}u_{max}$

$$u_1(x_2) = \frac{3}{2}\bar{U}\; 4\frac{x_2}{h}\left(1-\frac{x_2}{h}\right) \qquad \tau_{12} = \mu\frac{du_1}{dx_2} = \mu\frac{3}{2}\frac{\bar{U}}{h}4\left(1-\frac{2x_2}{h}\right)$$



$\tau_{12}(h) = -6\frac{\mu\bar{U}}{h}$

$\tau_{12}(h/2) = 0$

$\tau_{12}(0) = \frac{6\mu\bar{U}}{h}$

$$\underbrace{\frac{d}{dt}\int \varrho \underset{\sim}{u}\, dV}_{=0} + \underbrace{\oiint_{\underset{\sim}{S}} \varrho \underset{\sim}{u} (\underset{\sim}{u}\cdot\underset{\sim}{n})\, d\underset{\sim}{S}}_{=0} = \underbrace{\int_V \varrho \underset{\sim}{g}\, dV + \iint_{\underset{\sim}{S}_{horizontal}} (-p)\underset{\sim}{n}\, d\underset{\sim}{S}}_{=0} +$$

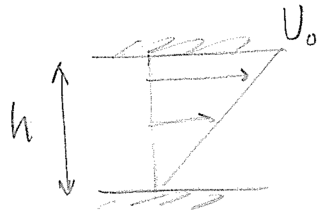
$$\underbrace{+ \iint_{\underset{\sim}{S}_{vertical}} (-p)\underset{\sim}{n}\, d\underset{\sim}{S} + \iint_{\underset{\sim}{S}_{horizontal}} \tau_{12}\, n_2\, d\underset{\sim}{S}\, \underset{\sim}{e}_x}_{=0}$$

$$-\frac{dP}{dx_1} L \cdot h - 2 \cdot \frac{6\mu \bar{U}}{h} L = 0$$

$$\Rightarrow \quad \bar{U} = -\frac{dP}{dx_1}\,\frac{h^2}{12\mu}$$

Exact solution to <u>energy equation</u>.

Plane Couette flow



$$u_1 = U_0 \frac{x_2}{h}$$

$\tau_{12} = \mu \frac{U_0}{h}$ shear stress executes work rate / unit area of plate

$$\underline{\underline{U_0 \tau_{12}}} = \mu \frac{U_0^2}{h}$$

Mechanical energy eq.: $\quad 0 = u_1 \frac{d}{dy} \tau_{12}$

$$\frac{d}{dy}\left(u_1 \tau_{12}\right) - \tau_{12}\frac{du_1}{dy} = \frac{d}{dy}\left(u_1 \tau_{12}\right) - \underbrace{\mu\left(\frac{du_1}{dy}\right)^2}_{\Phi} = 0$$

Dissipation function $\Phi = 2\mu\, e_{ij} e_{ij} \quad [J/m^3 s]$

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_1}\right) \\ (\quad) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\frac{du_1}{dx_2} \\ \frac{1}{2}\frac{du_1}{dx_2} & 0 \end{bmatrix}$$

$$\Phi = 2\mu\left(0^2 + \left(\frac{1}{2}\frac{du_1}{dx_2}\right)^2 + \left(\frac{1}{2}\frac{du_1}{dx_2}\right)^2 + 0^2\right) = \mu\left(\frac{du_1}{dx_2}\right)^2 = \mu\left(\frac{U_0}{h}\right)^2$$

$$\int_0^h \frac{d}{dy}\left(u_1 \tau_{12}\right) dy - \int_0^h \Phi\, dy = 0$$

$$\underline{\underline{U_0 \tau_{12}}} - \mu\left(\frac{U_0}{h}\right)^2 h = 0$$

Work goes into heat by viscous dissipation.

Thermal energy equation, incompressible 2D flow

$$\varrho c_p\left(\frac{\partial T}{\partial t} + u_1\frac{\partial T}{\partial x_1} + u_2\frac{\partial T}{\partial x_2}\right) = k\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right) + \Phi$$

Assume fully developed temp. in Couette flow $T(x_2)$

$$0 = k\frac{d^2T}{dx_2^2} + \Phi = k\frac{d^2T}{dx_2^2} + \mu\left(\frac{U_0}{h}\right)^2$$

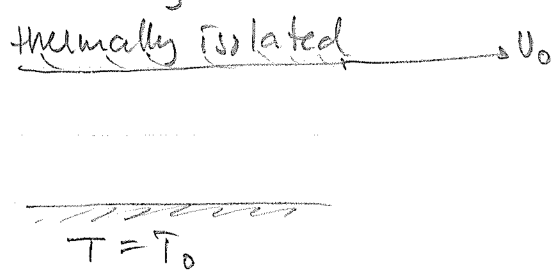
$$\Rightarrow \quad T = -\frac{\mu}{k}\left(\frac{U_0}{h}\right)^2\frac{x_2^2}{2} + A x_2 + B$$

$\mu = \varrho\nu$ $\quad k = \varrho c_p \kappa$ $\quad \Rightarrow \dfrac{\mu}{k} = \dfrac{\nu}{\kappa c_p}$

ν: kinematic viscosity $\quad \kappa$: thermal diffusivity

Prandtl number $Pr = \nu/\kappa$; $\dfrac{\mu}{k} = \dfrac{Pr}{c_p}$

Boundary conditions:

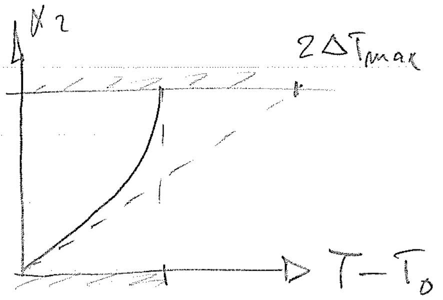


$$q_2(x_2 = h) = -k\frac{dT}{dx_2}(x_2 = h) = 0$$

$$T(x_2 = 0) = T_0$$

$$\Rightarrow \quad B = T_0 \ , \quad A = \frac{Pr}{c_p}\left(\frac{U_0}{h}\right)^2 h$$

$$\Rightarrow \quad T - T_0 = Pr\frac{U_0^2}{c_p}\left(\frac{x_2}{h} - \frac{1}{2}\left(\frac{x_2}{h}\right)^2\right) = Pr\frac{U_0^2}{2c_p}\frac{x_2}{h}\left(2 - \frac{x_2}{h}\right)$$

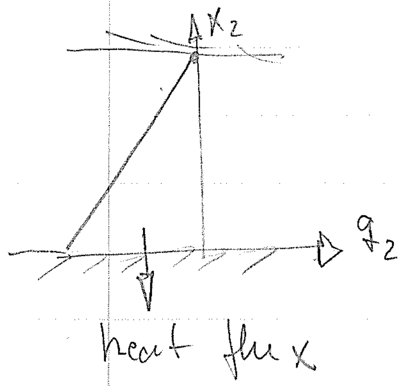


$$T_{max} - T_0 = Pr \frac{U_0^2}{2c_p}$$

maximum heat content/unit mass $= c_p T_{max} = c_p T_0 + Pr \underbrace{\frac{U_0^2}{2}}$

Result is independent of h ! (mechanical energy/unit mass at $x_2 = h$)

Heat flux density $\quad q_2 = -k \frac{dT}{dx_2} = -k \, Pr \frac{U_0^2}{2c_p} \frac{2}{h}\left(1 - \frac{x_2}{h}\right)$



$$q_2(x_2 = 0) = -k \frac{Pr U_0^2}{2c_p} \frac{2}{h}$$

$$= -k \frac{2\Delta T_{max}}{h}$$

$$= -\mu \frac{U_0^2}{h}$$

At $x_2 = 0 \quad \underset{\sim}{q} \cdot (-\underset{\sim}{e}_2) = \mu \frac{U_0^2}{h} = U_0 \tau_{12}$

compare

$$\underbrace{\frac{d}{dt}\int_V \rho\left(e + \frac{u_k u_k}{2}\right) dV}_{=0} + \underbrace{\oint\!\!\oint_S \rho\left(e + \frac{u_k u_k}{2}\right) \underset{\sim}{u} \cdot \underset{\sim}{n}\, dS}_{=0 \text{ no net flow of energy}} = -\oint\!\!\oint_S \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS + \underbrace{\oint\!\!\oint_S \underset{\sim}{u} \cdot \underset{\sim}{T} \underset{\sim}{n}\, dS}_{= \iint_{S_{upper}} U_0 \tau_{12}\, dS}$$

Liquids $c_p \sim 1000\ J/kgK$ $\quad c_{p_{H_2O}} = 4200\ J/kg\,K$

$Pr_{H_2O} = 8.1 \qquad Pr_{air} = 0.71$

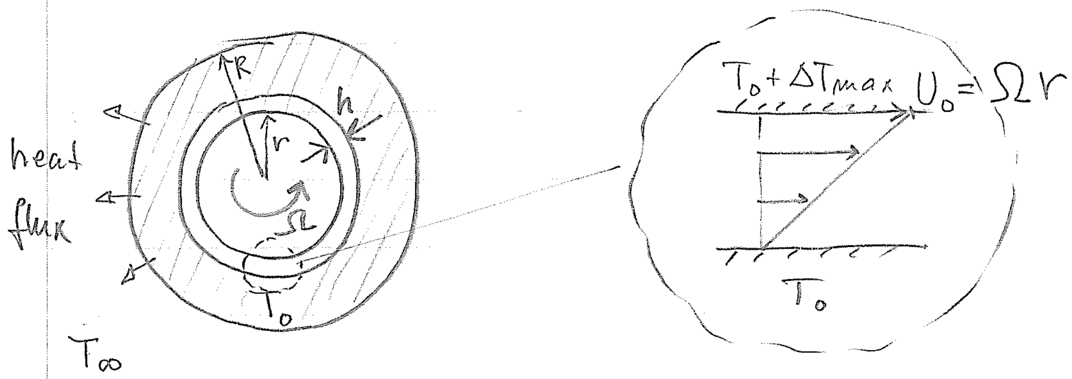
$\{OH\} \Rightarrow Pr_{motor\ oil}\,(T = 60°C) \approx 1000$

Bearing at say $U_0 = 5\,m/s$

$$\Rightarrow \Delta T_{max} = \frac{Pr\,U_0^2}{2c_p} = \frac{1000 \cdot 25}{2 \cdot 2050}\,K = \frac{25000}{4100}\,K \sim 8\,K$$

not so much!

Consider lubricating oil gap around axis.



Cooling by surrounding air at T_∞

Heat flux $\quad 2\pi R\,\alpha_T (T_0 - T_\infty) = 2\pi r\,\frac{2k}{h}\,\Delta T_{max}$

heat transfer coefficient

$$T_0 - T_\infty = \frac{r}{R}\,\frac{2k/h}{\alpha_T}\,\Delta T_{max}$$

If gap small and cooling poor $\Rightarrow \frac{2k/h}{\alpha_T} \gg 1 \Rightarrow T_0 - T_\infty \gg \Delta T_{max}$