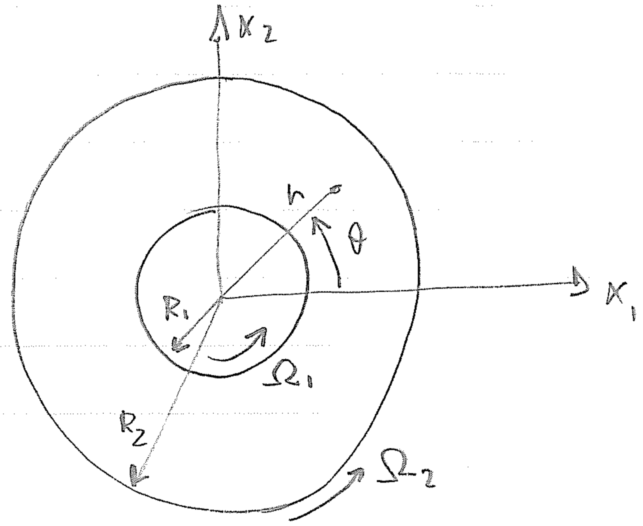
Navier-Stokes equations in curvilinear coordinates.

See Appendix B of Kundu & Cohen.
— " — A.3 of lecture notes
(copies are distributed on written exam.)

Ex. Flow between coaxial rotating cylinders



Boundary conditions: $u_r = 0$ at $r = R_1, R_2$

$u_\theta(r = R_1) = \Omega_1 R_1$

$u_\theta(r = R_2) = \Omega_2 R_2$

Assume: steady flow $\frac{\partial}{\partial t} = 0$

circular symmetry $\frac{\partial}{\partial \theta} = 0$

incompressible fluid $\rho = \text{const}!$

$\mu = \text{const}.$

Continuity equation $\Rightarrow$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \underbrace{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}_{=0} = 0 \quad \Rightarrow \quad r u_r = f(\theta)$$

$$u_r = \frac{f(\theta)}{r} \qquad \text{B.C.} \Rightarrow f = 0 \quad \therefore \quad u_r = 0$$

(In HW 2 you have porous cylinders and f=const.)

r-momentum:

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$ balance between radial pressure gradient and centripetal acceler.

θ-momentum:

$$0 = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u_\theta\right) - \frac{u_\theta}{r^2}\right]$$ net viscous force /unit volume $=0$

$$[\ldots] = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\right) =$$

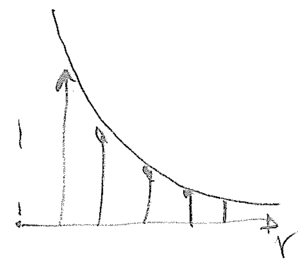
$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = C \quad \Rightarrow \quad r u_\theta = \frac{C r^2}{2} + B$$

$$u_\theta(r) = A r + \frac{B}{r}$$

Ar is a solid body rotation

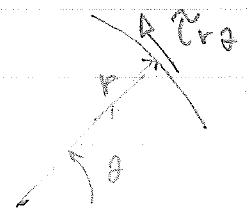
$\frac{B}{r}$ is an irrotational vortex



Vorticity $\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right) - \underbrace{\frac{1}{r}\frac{\partial u_r}{\partial \theta}}_{=0} = \frac{1}{r}\frac{\partial}{\partial r}\left(Ar^2 + B\right)$

$$= 2A + 0$$

Viscous stress $\tau_{r\theta} = 2\mu \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) =$



$$= \mu r \frac{\partial}{\partial r}\left(A + \frac{B}{r^2}\right) =$$

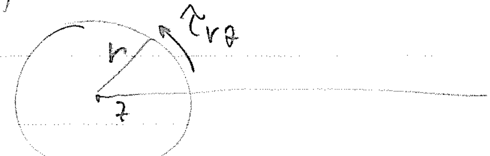
$$= -2\mu \frac{B}{r^2}$$

solid body rotation gives no contrib. to viscous stress since deformation rate is zero.

irrotational part $\Rightarrow \tau_{r\theta}$

Torque $\quad M_z' = \tau_{r\theta}\, r \cdot 2\pi r = -4\pi\mu B$

constant, indep of r



B.C.'s $u_\theta(R_1) = \Omega_1 R_1$, $u_\theta(R_2) = \Omega_2 R_2$

$$-B = \frac{M_z'}{4\pi\mu} = \left(\Omega_2 - \Omega_1\right)\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}, \quad A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$$

If $\Omega_2 = \Omega_1 = \Omega \Rightarrow M_z = B = 0$, $A = \Omega$

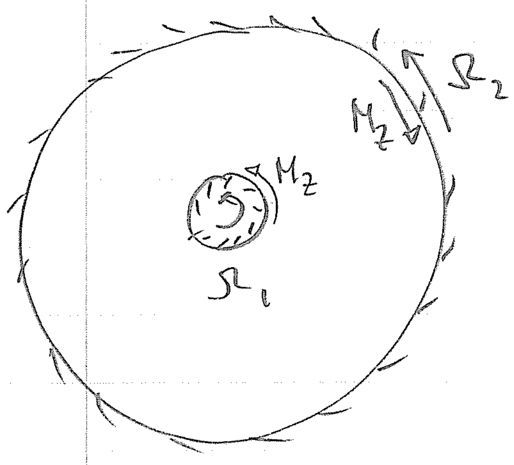
$\Rightarrow u_\theta = \Omega r$ solid body rotation of fluid in a rotating tank

If $\Omega_2 = 0$ $\Omega_1 = \Omega$ and $R_2 \rightarrow \infty$

$\Rightarrow B = \Omega R_1^2 \quad A = 0$

$u_\theta = \frac{\Omega R_1^2}{r}$ irrotational vortex outside rotating cylinder

Work rate carried out by cylinders



work rate / unit length on fluid

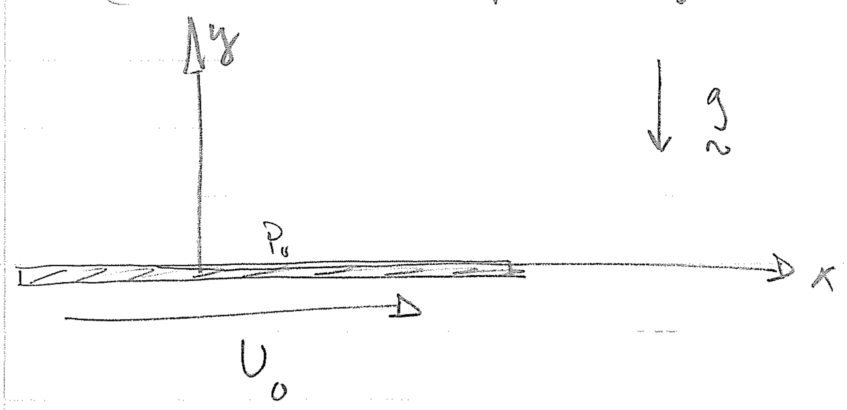
$= M_z(\Omega_2 - \Omega_1) =$

$= \underbrace{4\pi\mu(\Omega_2-\Omega_1)^2 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2)}}_{\geq 0}$

Dissipates into heat

Time-dependent flow with inertial effects.

Instantaneous start of infinite plate
(Stoke's 1st problem)



Assume $\underset{\sim}{u} = (u(y,t), v)$

Continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ + B.C. $\Rightarrow$ $v = 0$

x-momentum: $\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

y-momentum: $0 = -\frac{\partial p}{\partial y} - \rho g \Rightarrow p = p_0 - \rho g y$

indep. of x !

$\Rightarrow$ $\underline{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$ $\quad \nu = \mu/\rho$ diffusion equation

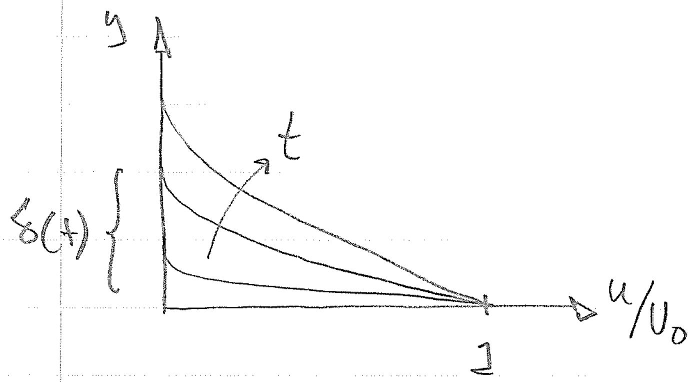
Initial condition $u(y, t=0) = 0 \quad 0 \leq y < \infty$

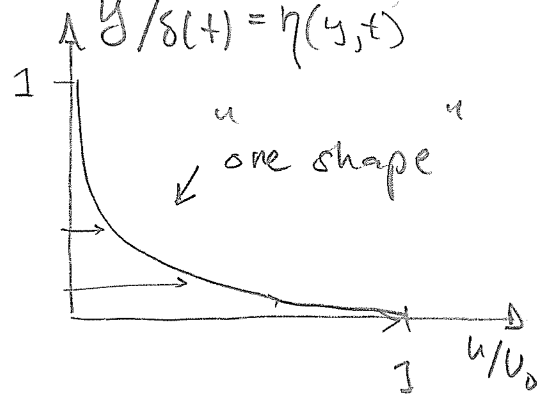
Boundary cond. $\begin{cases} u(y=0,t) = U_0 & t > 0 \\ u(y \to \infty, t) = 0 & t \geq 0 \end{cases}$

Methods of solution
- * separation of variables
- * Laplace transformation
- ⟶ * similarity ansatz

Similarity solution of the form

$$u(y,t) = U_0\, f\left(y/\delta(t) \right) = U_0\, f(\eta(y,t))$$





$\delta(t)$ is the diffusion length (penetration length of b.c.)

$$\left(\frac{\partial u}{\partial t}\right)_y = U_0 \frac{df}{d\eta}\left(\frac{\partial \eta}{\partial t}\right)_y = U_0 \frac{df}{d\eta}\left(\frac{-y}{\delta^2}\right)\frac{d\delta}{dt} = -U_0 \frac{df}{d\eta}\,\eta\,\frac{1}{\delta}\frac{d\delta}{dt}$$

$$\left(\frac{\partial u}{\partial y}\right)_t = U_0 \frac{df}{d\eta}\left(\frac{\partial \eta}{\partial y}\right)_t = U_0 \frac{df}{d\eta}\frac{1}{\delta} \;;\; \left(\frac{\partial^2 u}{\partial y^2}\right)_t = U_0 \frac{d^2 f}{d\eta^2}\frac{1}{\delta^2}$$

$$\Rightarrow \quad -\frac{df}{d\eta}\,\eta\,\underbrace{\frac{1}{\delta}\frac{d\delta}{dt}} = \nu \frac{d^2 f}{d\eta^2}\underbrace{\frac{1}{\delta^2}}$$

both depend of t, must be same function of t

$$\Rightarrow \frac{1}{\delta(t)}\frac{d\delta}{dt} = C' \frac{1}{\delta^2(t)}$$

orbitrary constant

$$\Rightarrow \frac{d}{dt}\left(\frac{\delta^2}{2}\right) = C' = 2\nu \Rightarrow \underline{\underline{\delta^2 = 4\nu t}}$$

any choice will do $\neq 0$

O.D.E. $\quad \frac{d^2f}{d\eta^2} + 2\eta\frac{df}{d\eta} = 0 \quad$ (ν cancel because of choice of C'.)

B.C. $\begin{cases} \frac{u(0,t)}{U_0} = f(0) = 1 \\ \frac{u(\infty,t)}{U_0} = f\left(\eta = \frac{y}{\delta} \to \infty\right) = 0 \end{cases}$

compatible

I.C. $\quad \frac{u(y,0)}{U_0} = f(\eta \to \infty) = 0$

ODE $\Rightarrow \quad \frac{f''}{f'} = -2\eta \; ; \quad \underbrace{\int_0^\eta \frac{f''}{f'}\,d\eta}_{\ln|f'(\eta)/f'(0)|} = -\eta^2$

$$f'(\eta) = f'(0)\,e^{-\eta^2}$$

$$f(\eta) = f'(0)\int_0^\eta e^{-z^2}dz + \underbrace{f(0)}_{=1}$$

B.C. $\quad f(\eta \to \infty) = f'(0)\underbrace{\int_0^\infty e^{-z^2}dz}_{\sqrt{\pi}/2} + 1 = 0$

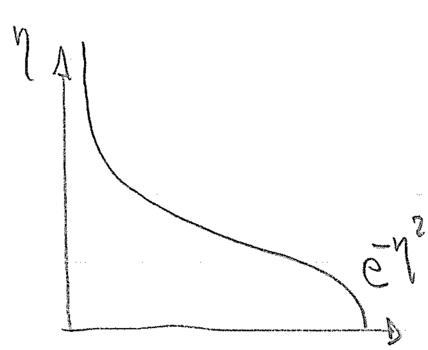
$$\Rightarrow f'(0) = -\frac{2}{\sqrt{\pi}}$$

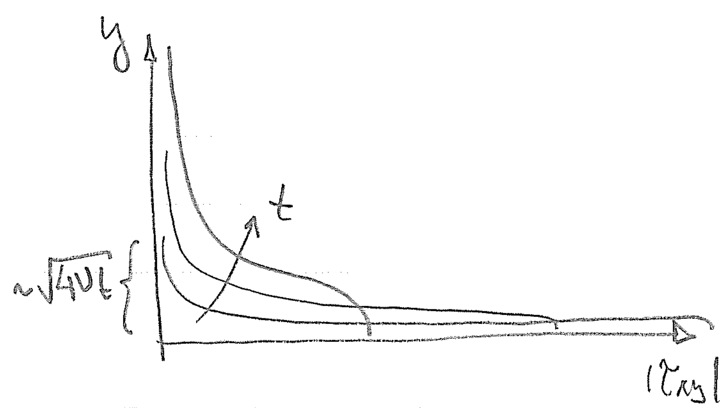
$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\zeta^2}\, d\zeta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{4\nu t}} e^{-\xi^2}\, d\xi$$

$$= 1 - \mathrm{erf}\left(y/\sqrt{4\nu t} \right) \quad \text{error function}$$

Shear stress $\tau_{xy} = \mu \frac{\partial u}{\partial y} = \mu U_0 \frac{df}{d\eta} \frac{1}{\delta(t)} =$

$$= -\mu U_0 \frac{2}{\sqrt{\pi}} e^{-\eta^2} \frac{1}{\sqrt{4\nu t}} = -\frac{\mu U_0}{\sqrt{\pi \nu t}} e^{-(y/4\nu t)^2}$$

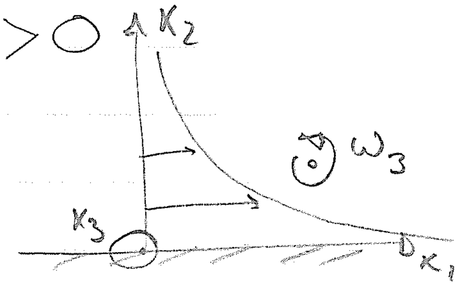




$t \to 0 \Rightarrow |\tau_{xy}| \to \infty$ as infinite force is required to instantaneously accelerate the fluid.

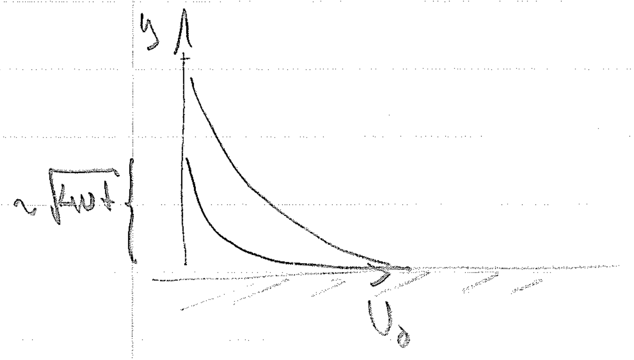
Vorticity $\omega_z = \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = -\frac{\partial u}{\partial y} = -\frac{U_0}{\delta} \frac{df}{d\eta}$

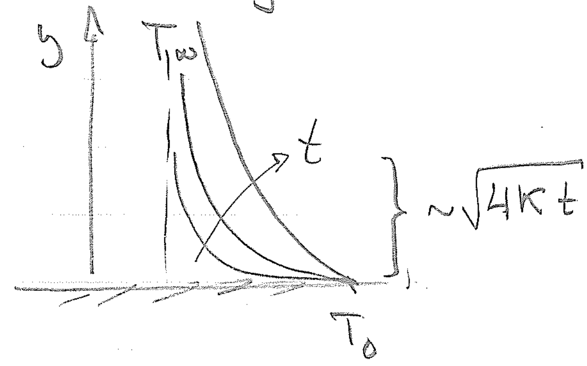
$$= -\frac{\tau_{xy}}{\mu} > 0$$



Analogy with diffusion of heat.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad\qquad \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial y^2}$$





$$\frac{T - T_0}{T_0 - T_\infty} = 1 - \operatorname{erf}\left\{ y/\sqrt{4Kt} \right\}$$