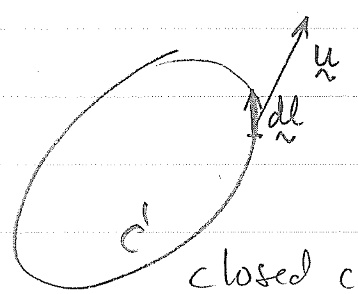
Vorticity dynamics.

Basic vortex flows

$u_\theta = \frac{1}{2}\omega r$ solid body rotation
vorticity $\omega_z = \omega$
(homogeneous distribution of vorticity)

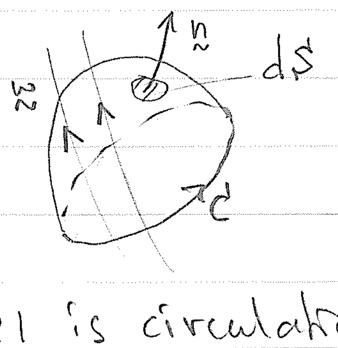
$u_\theta = \frac{\Gamma}{2\pi r}$ irrotational vortex / line vortex
vorticity $\omega_z = 0$
(infinite vorticity at origin)

Definition: Circulation $\Gamma = \oint_C \underset{\sim}{u} \cdot \underset{\sim}{dl}$



closed curve C

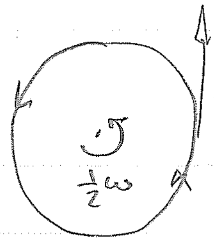
Stoke's theorem: $\oint_C \underset{\sim}{u} \cdot \underset{\sim}{dl} = \iint_S \nabla \times \underset{\sim}{u} \cdot \underset{\sim}{n}\, dS = \iint_S \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, dS$



$$= \iint_S |\underset{\sim}{\omega}| \underbrace{n_\omega dS}_{\text{projected } dS \perp \text{ to } \underset{\sim}{\omega}}$$

$|\underset{\sim}{\omega}|$ is circulation / unit area

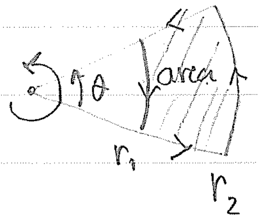
Solid b.r.



$u_\theta = \frac{1}{2}\omega r$

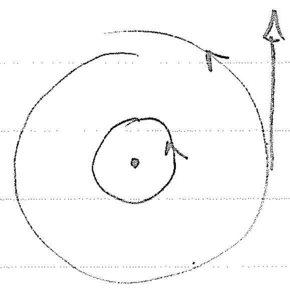
$$\Gamma = \oint u_\theta \, r \, d\theta = \frac{1}{2}\omega r^2 \, 2\pi$$

$$= \omega \underbrace{\pi r^2}_{\text{area}}$$



$$\Gamma = \frac{1}{2}\omega r_2^2 \, \theta - \frac{1}{2}\omega r_1^2 \, \theta = \omega \underbrace{\frac{\theta}{2}(r_2^2 - r_1^2)}_{\text{area}}$$

Line vortex

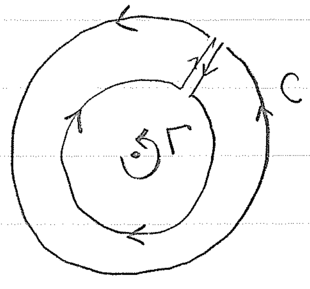


$u_\theta = \frac{\Gamma}{2\pi r}$ $\quad \Gamma = \oint \frac{\Gamma}{2\pi r} \, r \, d\theta = \Gamma$

Γ is the strength of line vortex.

Although $\omega_z = 0$, $\Gamma \neq 0$ for all C enclosing the vortex.

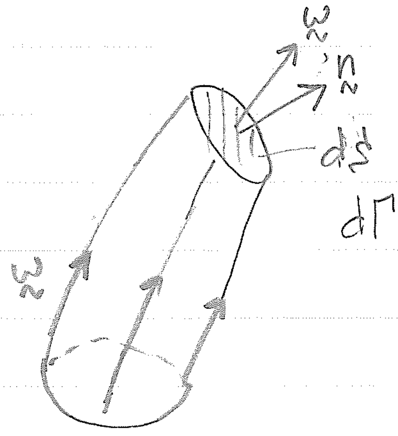
Curve not enclosing (ex.)



$$\Gamma_C = \frac{\Gamma}{2\pi r_2} \cdot r_2 \, 2\pi - \frac{\Gamma}{2\pi r_1} r_1 \, 2\pi = 0$$

All vorticity concentrated to vortex centre.

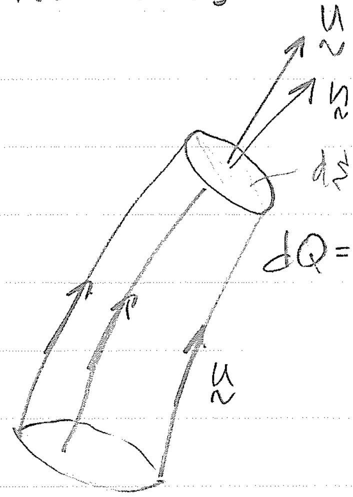
Vortex lines



$d\Gamma = \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, d\underset{\sim}{S}$

vortex tube

Streamlines



$dQ = \underset{\sim}{u} \cdot \underset{\sim}{n}\, d\underset{\sim}{S}$

stream tube

$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$

$\Rightarrow \nabla \cdot \underset{\sim}{\omega} = 0$ analogous to $\nabla \cdot \underset{\sim}{u} = 0$ for incompressible flow

<u>Viscous stress</u> $\tau_{ij} = 2\mu\, \bar{e}_{ij} + \mu_B\, \bar{\bar{e}}_{ij}$

solid body rotation $\bar{e}_{ij} = \bar{\bar{e}}_{ij} = 0$ no deformation
$\Rightarrow$ no viscous stress

irrotational flow deformation rate $\neq 0 \Rightarrow \tau_{ij} \neq 0$

<u>Viscous force/unit volume</u> $\frac{\partial}{\partial x_j}(\tau_{ij}) = -\mu(\nabla \times \underset{\sim}{\omega})_i$

$= 0$ for $\underset{\sim}{\omega} = \text{const.}$ <u>and</u> for irrotational flow.

Kelvin's circulation theorem.

If: a) inviscid flow $\tau_{ij} = 0$ or $Re \to \infty$

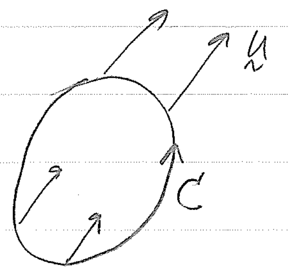
b) conservative body force $\underset{\sim}{f} = -\nabla \Psi$

e.g. $\Psi = -\underset{\sim}{g} \cdot \underset{\sim}{x}$

c) barotropic flow $\rho(p)$

e.g. $\rho = \text{const.}$ or isentropic flow $p/\rho^{\gamma} = \text{const.}$

Then:



material curve C

$$\Gamma = \oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l}$$

$$\boxed{\frac{D}{Dt}\Gamma = 0}$$

Proof:

$$\frac{D}{Dt}\Gamma = \frac{D}{Dt}\oint u_i\, dx_i = \underbrace{\oint \frac{Du_i}{Dt} dx_i}_{I} + \underbrace{\oint u_i \frac{D}{Dt} dx_i}_{II}$$

$$II = \oint_C u_i\, du_i = \oint_C d\left(\frac{u_i u_i}{2}\right) = 0$$

$$I = \oint_C \left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + f_i + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\tau_{ij}\right)}_{=0 \text{ for } \mu = 0} \right) dx_i$$

$$\oint_C \frac{1}{\varrho(p)} dp = \oint \frac{dF}{dp}(\varrho)\, dp = \oint dF = 0$$

$$\oint -\frac{\partial \Pi}{\partial x_i} dx_i = -\oint d\Pi = 0$$

$$\Rightarrow \frac{D}{Dt}\Gamma = 0$$

Irrotational flow at $t=0$ $\quad \underset{\sim}{\omega}(t=0) = 0$
(except at singular points of line vortex)

$\Gamma(t=0) = \iint \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, dS = 0$ for any curve

C (not enclosing the line vortex) and surface S.

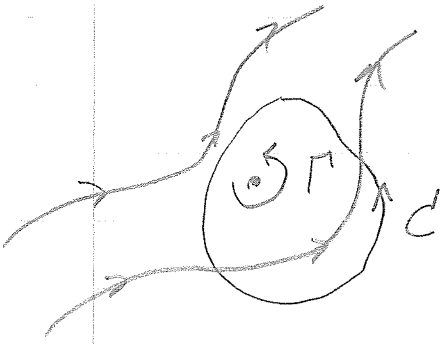
then since $\frac{D}{Dt}\Gamma = 0$ for any curve

$\Rightarrow \Gamma = \iint_{\underset{\sim}{S}} \underset{\sim}{\omega}(t) \cdot \underset{\sim}{n}\, dS = 0$ for any surface S.

thus $\underline{\underset{\sim}{\omega}(t) = 0.}$

Irrotational flow remains irrotational
under the conditions of Kelvin's theorem.

Irrotational flow with line vortex Γ.



$$\frac{D}{Dt}\Gamma_c = 0\,, \quad \Gamma_c = \Gamma = \text{constant}$$

the material curve will enclose the vortex at all times

Material curves not enclosing the vortex

$$\frac{D}{Dt}\Gamma_c = 0\,, \quad \Gamma_c = 0$$

Helmholtz theorems (same conditions as Kelvin's)

- vortex lines are material lines and move with the fluid
- the strength of a line vortex, i.e. Γ, remains constant in time
- the strength of a line vortex is constant along its length
- a line vortex cannot end within the fluid (closed loop or end at solid wall)

Derivation of the vorticity equation.

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\dfrac{1}{\rho}\nabla p + \mathbf{f} + \nu\nabla^2\mathbf{u} \\[2em] \nabla\cdot\mathbf{u} = 0 \end{cases}$$

allow for small variations of ρ

$\boldsymbol{\omega} = \nabla\times\mathbf{u}$ vorticity vector

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

use : $\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$

$= \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + \boldsymbol{\omega}\times\mathbf{u}$

[since: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,\mathbf{a}\cdot\mathbf{c} - \mathbf{c}\,\mathbf{a}\cdot\mathbf{b}$ then

$$\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\frac{(\mathbf{u}\cdot\mathbf{u})}{2} - \mathbf{u}\cdot\nabla\mathbf{u} \quad]$$

$$\nabla\times\left[\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + \boldsymbol{\omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f} + \nu\nabla^2\mathbf{u}\right]$$

note $\nabla\times\nabla(\cdot) = 0$

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \nabla\times(\dot{\boldsymbol{\omega}}\times\mathbf{u}) + \nabla\times(\boldsymbol{\omega}\times\dot{\mathbf{u}}) =$$

$$= \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \mathbf{u}\underbrace{\nabla\cdot\boldsymbol{\omega}}_{=0} + \boldsymbol{\omega}\underbrace{\nabla\cdot\mathbf{u}}_{=0} - \boldsymbol{\omega}\cdot\nabla\mathbf{u}$$

$$\frac{\partial\omega_k}{\partial x_k} = \frac{\partial\varepsilon_{kij}\frac{\partial u_j}{\partial x_i}}{\partial x_k} = \varepsilon_{kij}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = 0$$

[good exercise is to derive this with tensor notation]

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + \underbrace{\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}}_{\text{advection of } \boldsymbol{\omega}}}_{\frac{D\boldsymbol{\omega}}{Dt}} = \underbrace{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}}_{} + \underbrace{\frac{\nabla\rho\times\nabla p}{\rho^2}}_{\substack{\text{baroclinic} \\ \text{generation of } \boldsymbol{\omega} \\ (=0 \text{ if } \rho(p) \\ \text{(barotropic)})}} + \underbrace{\nabla\times\boldsymbol{f}}_{\substack{\text{body force} \\ \text{generation of } \boldsymbol{\omega} \\ (=0 \text{ if} \\ \boldsymbol{f}=-\nabla\Pi)}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\substack{\text{diffusion} \\ \text{of } \boldsymbol{\omega}}}$$

show : stretching & tilting of vortex lines

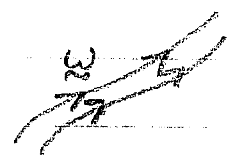
Non-dimensional form (barotropic & cons. body force)

$$\Rightarrow \frac{D\omega_i}{Dt} = \underbrace{\omega_j \frac{\partial u_i}{\partial x_j}}_{} + \frac{1}{Re}\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

compare with eq. for material line element

$$\frac{D}{Dt} dl_i = \frac{\partial u_i}{\partial x_j} dl_j$$

For $Re \to \infty$ $\quad \frac{D}{Dt}\omega_i = \frac{\partial u_i}{\partial x_j}\omega_j$



Helmholtz theorem for $Re \to \infty$:

- vortex lines are material lines

i) stretching of vortex line produce ω_i like stretching of dl_i produce length

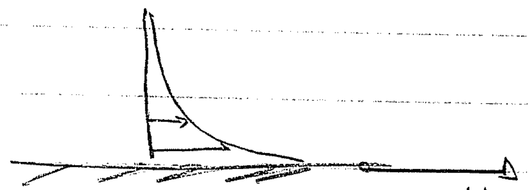
ii) tilting of vortex line produce ω_i in one direction at expense of ω_i in another direction

Study terms in the vorticity equation.

2D:
$$\frac{\partial}{\partial t}\omega_3 + u_1 \frac{\partial \omega_3}{\partial x_1} + u_2 \frac{\partial \omega_3}{\partial x_2} = \nu \nabla^2 \omega_3$$

no tilting & stretching in 2D

ex. Diffusion of vorticity: Stoke's problem

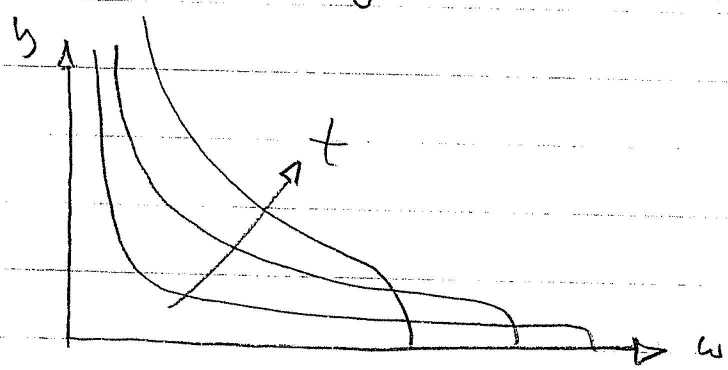


$\omega_3 = \omega_z = \omega(y,t)$

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$$

$$\omega = \frac{U_0}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t} \qquad \left(\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \right)$$

diffusion length scale $\delta(t) = \sqrt{4\nu t}$



vorticity is generated at the wall and diffuses into fluid above (but $-\nu \left.\frac{\partial \omega}{\partial y}\right|_{y=0} = 0$; $t>0$)

note: $\int_0^\infty \omega \, dy = \int_0^\infty -\frac{\partial u}{\partial y} dy = U_0 =$ constant

ω concentrated at the wall for $t = 0^+$

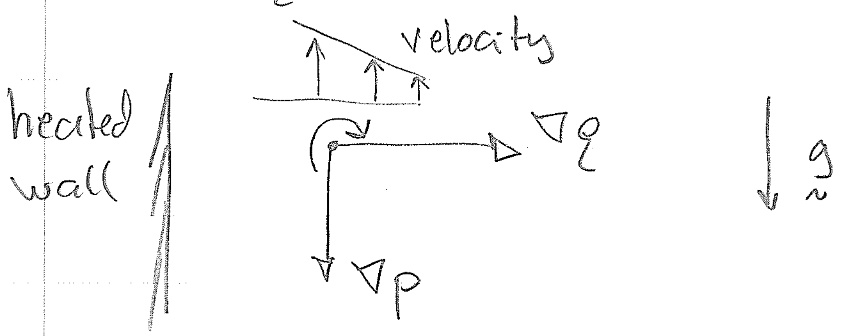
Generation of vorticity.

- if flow is viscous and $\underset{\sim}{\omega}(t=0)=0$, then vorticity is generated at solid boundaries and diffuses into flow.

- if $\underset{\sim}{f}$ is non-conservative, then vorticity is generated by $\nabla \times \underset{\sim}{f}$

- if flow is not barotropic, $\rho(p, T)$, then vorticity is generated by

$$\frac{\nabla \rho \times \nabla p}{\rho^2}$$



Re >> 1