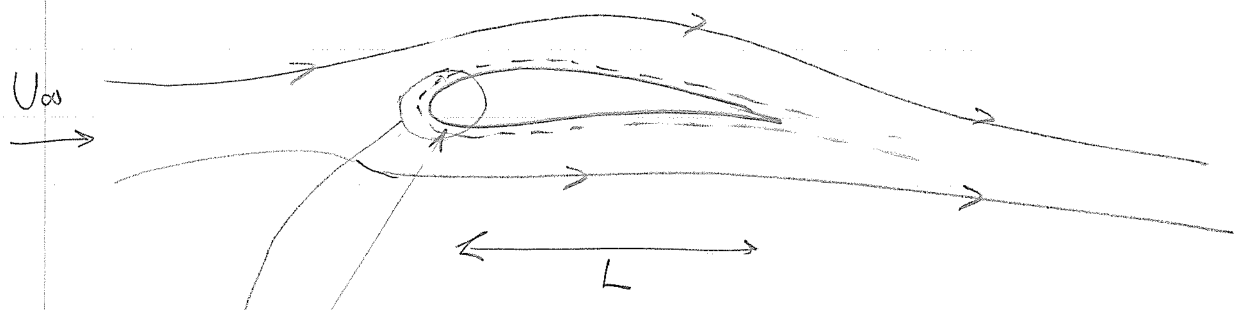
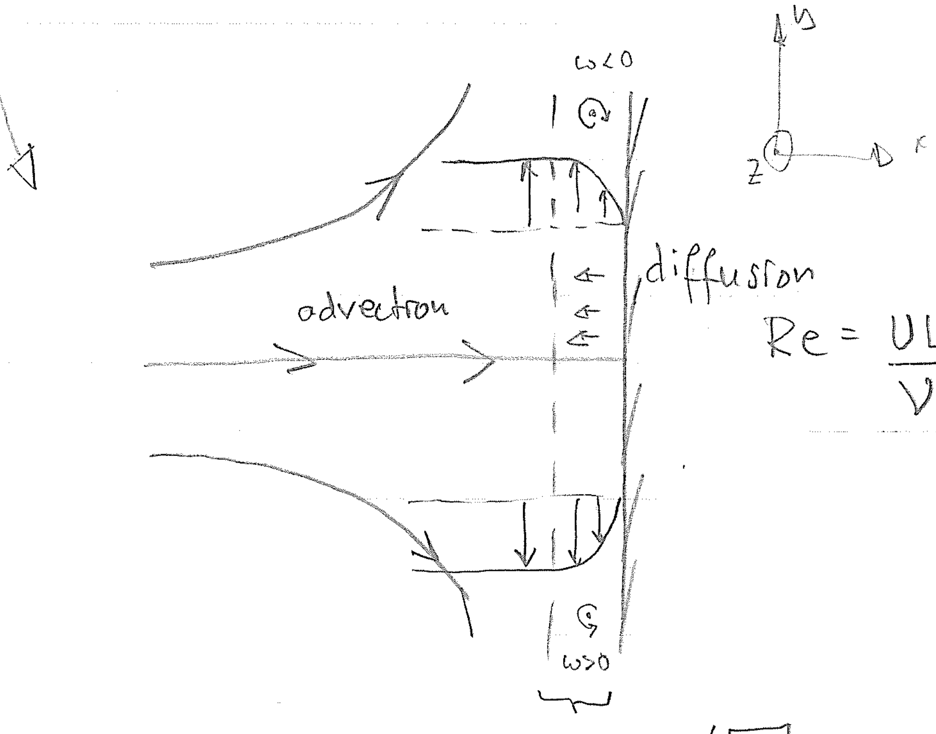
Flows at large Re.

$\omega \approx 0$



* generation of vorticity at the wall boundary $\omega \neq 0$

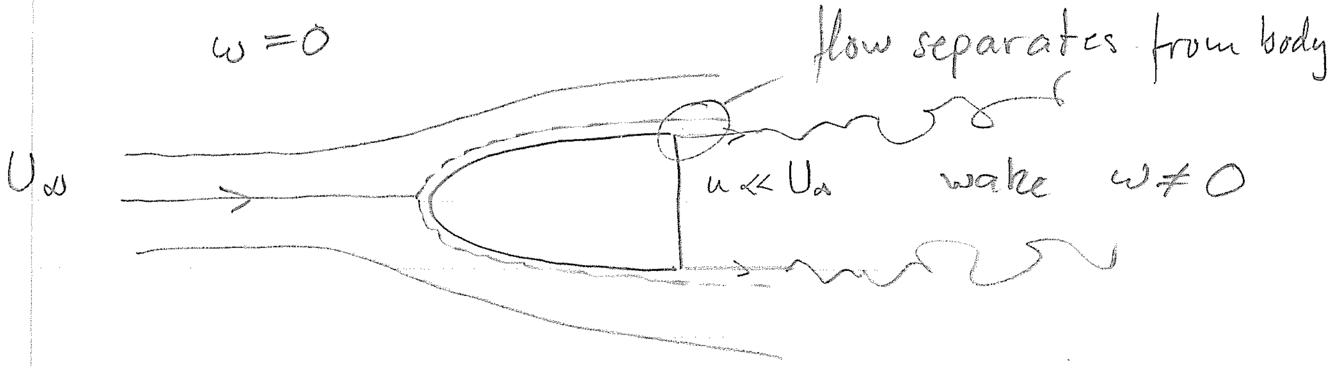
* advection with flow prevents diffusion far away from boundary



$$Re = \frac{UL}{\nu} \gg 1$$

$$\delta \sim L / \sqrt{Re} \ll L$$

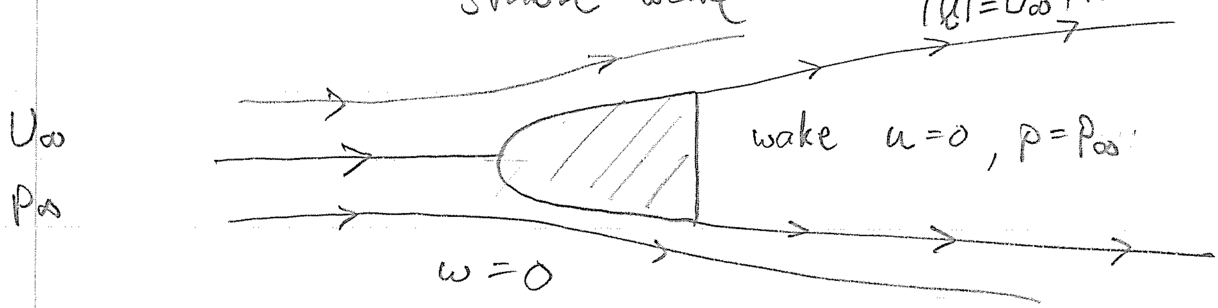
* For streamlined bodies, vorticity is concentrated close to the walls at large Re, generated by viscous effects and no slip-cond.



* For blunt bodies at large Re vorticity is also present in the wake

The assumption of zero vorticity, irrotational flow, is only motivated for flows without "separation".

(* Exception: known separation point
stable wake

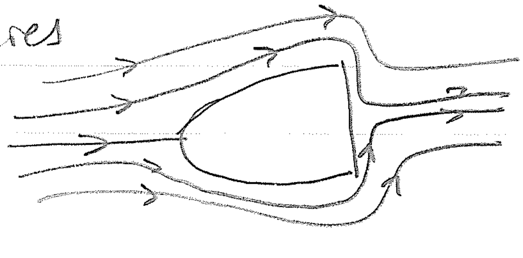


)

Irrotational flow assumption, $\underset{\sim}{\omega} = 0$, for $Re \gg 1$

* the viscous no slip condition <u>cannot</u> be fulfilled (generates vorticity)

* non-physical solutions obtained for blunt bodies



Methods for solving irrotational flow problems.

Assume velocity field is obtained

from potential $\phi(\underset{\sim}{x},t)$: $\underset{\sim}{u} = \nabla\phi$

$\Longrightarrow$ $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} = \nabla \times (\nabla\phi) = 0$

i) Potential flow is irrotational as required.

$$\left(\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} = \underbrace{\varepsilon_{ijk}}_{as} \frac{\partial^2 \phi}{\underbrace{\partial x_j \partial x_k}_{s}} = 0 \right)$$

ii) Potential flow has no net viscous force /unit volume

$$\frac{\partial}{\partial x_j}\tau_{ij} = \mu\nabla^2 u_i = \ldots = -\mu\varepsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} = -\mu(\nabla\times\underset{\sim}{\omega})_i$$

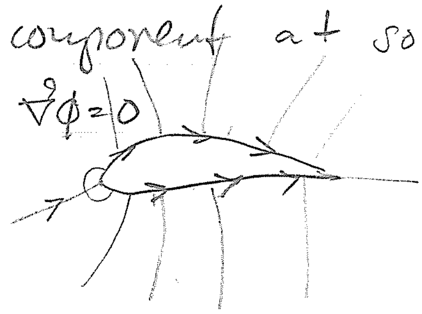
but cannot satisfy no slip - condition at solid wall

iii) Potential flow is incompressible if

$$\nabla\cdot\underset{\sim}{u} = \nabla\cdot(\nabla\phi) = \underline{\underline{\nabla^2\phi = 0}}$$

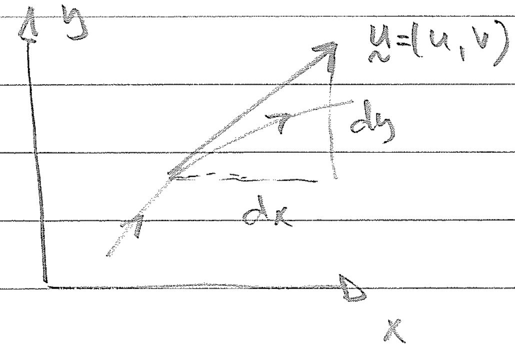
Solve Laplace eq. for $\phi(\underset{\sim}{x},t)$ with condition of no normal velocity component at solid wall:

$$\underset{\sim}{u}\cdot\underset{\sim}{n} = \nabla\phi\cdot\underset{\sim}{n} = 0$$



Stream function

2D



Streamlines : tangent to velocity vector

$$\frac{dy}{dx} = \frac{v}{u} \quad \text{or} \quad \frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad u\,dy - v\,dx = 0$$

$$\left(3D: \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \right)$$

2D: Streamfunction $\psi = \psi(x,y)$

such that $\psi = \text{const.} \Rightarrow$ streamlines

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \left\{ \underbrace{\frac{\partial \psi}{\partial x} = -v}, \; \underbrace{\frac{\partial \psi}{\partial y} = u} \right\} =$$

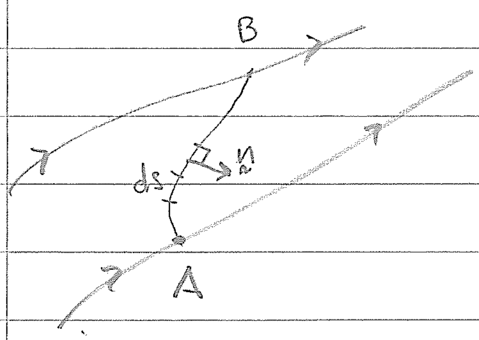
$$= -v\,dx + u\,dy = 0 \quad \text{on streamlines}$$

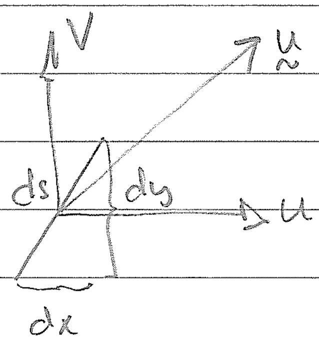
Consistent with continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \quad \text{ok!}$$

Def. $\psi(x,y) = \int^{P(x,y)} (-v\,dx + u\,dy)$

volume flux between two streamlines





$$dQ = u\,dy - v\,dx$$

$$Q_{AB} = \int_A^B dQ = \int_A^B u\,dy - v\,dx = \int_{(\cdot)}^{B} u\,dy - v\,dx - \int_{(\cdot)}^{A} u\,dy - v\,dx$$

$$= \Psi_B - \Psi_A$$

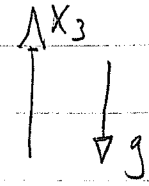
Irrotational flow $\omega_z = 0$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla^2 \psi = 0$$

If flow is irrotational, the streamfunction must satisfy Laplace eq.

Bernoulli's equation (incompressible)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i - g\delta_{i3}$$



rewrite using:

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j u_j\right) - \varepsilon_{ijk} u_j \omega_k$$

or

$$\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \nabla\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2}\right) - \underset{\sim}{u}\times\underset{\sim}{\omega}$$

and $\nabla^2 u_i = \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} = \ldots = \varepsilon_{ijk}\dfrac{\partial}{\partial x_j}\omega_k = -(\nabla\times\underset{\sim}{\omega})_i$

$$\Rightarrow \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j u_j + \frac{p}{\rho} + g x_3\right) = \varepsilon_{ijk} u_j \omega_k - \nu\varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}$$

or

$$\frac{\partial}{\partial t}\underset{\sim}{u} + \nabla\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + g x_3\right) = \underset{\sim}{u}\times\underset{\sim}{\omega} - \nu\nabla\times\underset{\sim}{\omega}$$

(1): Assume irrotational flow $\underset{\sim}{\omega} = 0$
thus we have a velocity potential ϕ
$u_i = \dfrac{\partial \phi}{\partial x_i}$ or $\underset{\sim}{u} = \nabla\phi$

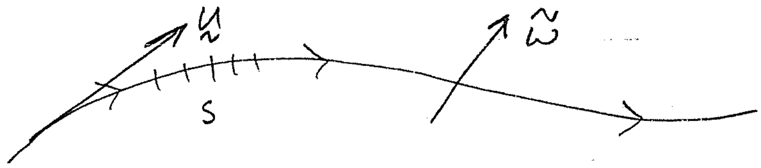
$$\frac{\partial}{\partial x_i}\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}u_j u_j + \frac{p}{\rho} + g x_3\right\} = 0$$

and
$$\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}u_j u_j + \frac{p}{\rho} + g x_3 = f(t)}$$

Require $Re \gg 1$, otherwise $\underset{\sim}{\omega}$ will diffuse from boundaries

(2): Assume $\boldsymbol{\omega} \neq 0$, inviscid flow $Re \gg 1$, and stationary flow $\frac{\partial}{\partial t} = 0$

Integrate momentum eq. along a streamline



$$\mathbf{e}_s = \mathbf{u}/|\mathbf{u}|$$

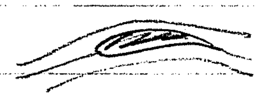
$$\int^s \underbrace{\mathbf{e}_s \cdot \nabla}_{\frac{d}{ds}} \left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\varrho} + g x_3 \right) ds = \underbrace{\int^s \underbrace{\mathbf{e}_s}_{\parallel \mathbf{u}} \cdot \underbrace{(\mathbf{u} \times \boldsymbol{\omega})}_{\perp \mathbf{u}} \, ds}_{= 0}$$

$$\Rightarrow \boxed{\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\varrho} + g x_3 = \text{constant along streamline}}$$

differs between streamlines if $\boldsymbol{\omega} \neq 0$

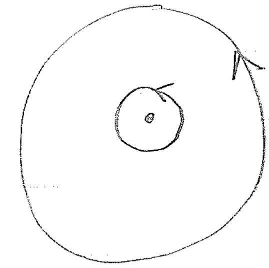
Bernoulli's eq. is often used to calculate p when $\mathbf{u}$ is known.

i) solve $\nabla^2 \phi = 0$ or $\nabla^2 \psi = 0$



ii) calculate p from Bernoulli's eq

Ex. Calculate the pressure, $p(r)$, of an irrotational vortex (vortex line) if the pressure at large distances is p_∞.



$$u_\theta = \frac{\Gamma}{2\pi r} \; ; \; \omega_z = 0 \; ; \; \frac{\partial}{\partial t} = 0$$

$$\frac{1}{2}\rho u_\theta^2 + p(r) = \text{constant} = 0 + p_\infty$$

$$p(r) = p_\infty - \frac{1}{2}\rho\left(\frac{\Gamma}{2\pi r}\right)^2 \; ; \text{ singularity at } r=0.$$

(Viscous core exists as $r \to 0$ such that $u_\theta \to 0$, but still low pressure at $r=0$.)

Velocity potential $\phi = \frac{\Gamma}{2\pi}\theta + \text{const.}$

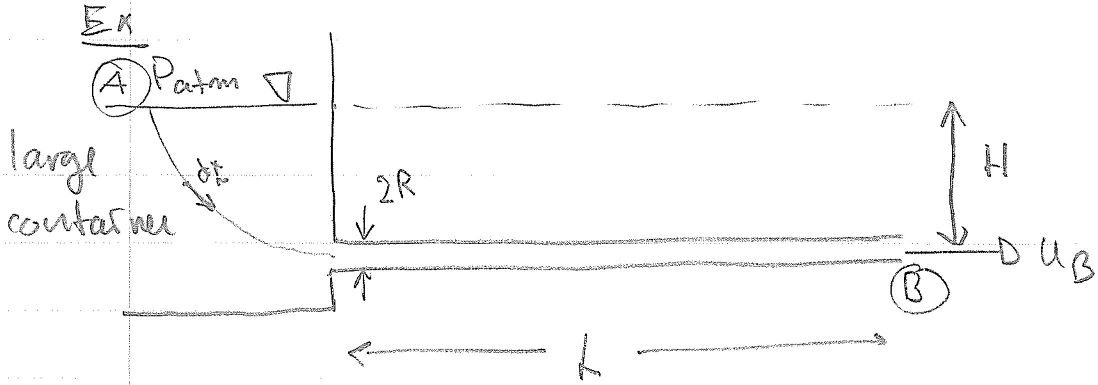
Stream function $\psi = -\frac{\Gamma}{2\pi}\ln r + \text{const.}$

$$\begin{cases} u_r = \dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = 0 \\[2ex] u_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r} = \dfrac{\Gamma}{2\pi r} \end{cases}$$

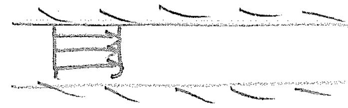
One finds :

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \; ; \; \nabla^2\psi = 0$$

Ex



potential flow assumed



$$\frac{\partial}{\partial t}\Phi_A + \frac{1}{2}U_A^2 + \frac{P_{atm}}{\rho} + gH = \frac{\partial}{\partial t}\Phi_B + \frac{1}{2}U_B^2 + \frac{P_{atm}}{\rho} + gH$$

$u_B(t=0)=0$ is initial condition

$$\Phi_B - \Phi_A = \int_A^B \nabla\phi \cdot d\underset{\sim}{x} = \int_A^B \underset{\sim}{u}\cdot d\underset{\sim}{x} = \underbrace{\int_A^{\text{tube inlet}} \underset{\sim}{u}\cdot d\underset{\sim}{x}}_{\sim U_B R \ll U_B L} + U_B \cdot L$$

in container $u \ll U_B$ (except close to inlet)

$$\frac{\partial}{\partial t}\Phi_B - \frac{\partial}{\partial t}\Phi_A = \frac{d}{dt}U_B \cdot L$$

$$\Rightarrow \quad gH = \frac{d}{dt}U_B \cdot L + \frac{1}{2}U_B^2 - \underbrace{\frac{1}{2}U_A^2}_{\ll \frac{U_B^2}{2}}$$

Let $U_B = \sqrt{2gH}\, F(t)$

$$\Rightarrow \quad gH = \sqrt{2gH}\,\frac{dF}{dt}\, L + \frac{1}{2}\, 2gH\, F^2(t)$$

$$\frac{1}{2} = \frac{L}{\sqrt{2gH}} \frac{dF}{dt} + \frac{1}{2} F^2(t) \Rightarrow \frac{1}{2} = \frac{1}{2}\frac{dF}{d\tilde{t}} + \frac{1}{2}F^2$$

Let $t = \frac{2L}{\sqrt{2gH}} \tilde{t}$ $\qquad [\tilde{t}] = 1$

$$\Rightarrow \frac{dF}{d\tilde{t}} = 1 - F^2(\tilde{t}) = (1+F)(1-F)$$

$$\frac{dF}{1-F^2} = \frac{dF}{(1+F)(1-F)} = d\tilde{t} \Rightarrow \frac{1}{2}\frac{dF}{(1+F)} + \frac{1}{2}\frac{dF}{(1-F)} = d\tilde{t}$$

integrate $\Rightarrow \frac{1}{2}\ln|1+F| - \frac{1}{2}\ln|1-F| = \tilde{t}$

$F(\tilde{t}=0) = 1$

$$\ln\left(\frac{1+F}{1-F}\right) = 2\tilde{t} \qquad \frac{1+F(\tilde{t})}{1-F(\tilde{t})} = e^{2\tilde{t}} = \text{"}a\text{"}$$

$$1 + F = (1-F)a \qquad F(1+a) = a - 1$$

$$F = \frac{a-1}{a+1} = \frac{e^{2\tilde{t}}-1}{e^{2\tilde{t}}+1} = \frac{e^{\tilde{t}}-e^{-\tilde{t}}}{e^{\tilde{t}}+e^{-\tilde{t}}} = \underline{\underline{\tanh(\tilde{t})}}$$