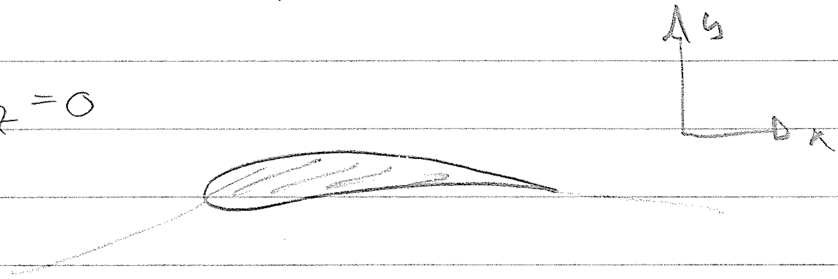
## 2 D irrotational flow.




$\omega_z = 0$

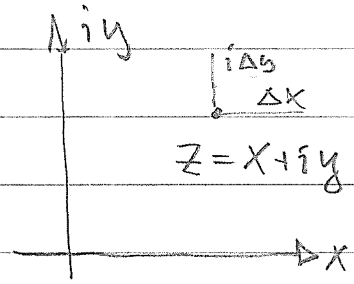
Velocity potential $\underline{u} = \nabla\phi$

Incompressible $\Rightarrow$ $\boxed{\nabla^2\phi = 0}$

Stream function $u = \dfrac{\partial\psi}{\partial y}$ ; $v = -\dfrac{\partial\psi}{\partial x}$

$\omega_z = 0 \Rightarrow$ $\boxed{\nabla^2\psi = 0}$

### Method of solution with analytic functions.




Complex function $F(z)$ is analytic if $\dfrac{dF}{dz}(z)$ exists and is independent of direction.

$$F'(z) = \lim_{\Delta z \to 0} \left( \frac{F(z+\Delta z) - F(z)}{\Delta z} \right)$$

Let $F(z) = \Phi(x,y) + i\,\Psi(x,y)$

$$F' = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{ \phi(x+\Delta x,y) + i\Psi(x+\Delta x,y) - \Phi(x,y) - i\Psi(x,y) \right\} =$$

$$= \frac{\partial \phi}{\partial x} + i\frac{\partial \Psi}{\partial x}$$

$$F' = \lim_{i\Delta y \to 0} \frac{1}{i\Delta y}\left\{ \Phi(x,y+\Delta y) + i\Psi(x,y+\Delta y) - \phi(x,y) - i\Psi(x,y) \right\} =$$

$$= \frac{1}{i}\frac{\partial \phi}{\partial y} + \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$

Real part: $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \Psi}{\partial y}$

Imaginary part: $\dfrac{\partial \Psi}{\partial x} = -\dfrac{\partial \phi}{\partial y}$

Cauchy-Riemann equations

One finds:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

$$\nabla \phi \cdot \nabla \Psi = \frac{\partial \phi}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \Psi}{\partial y} = -\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial x} = 0$$

Iso level surfaces of $\phi$ & $\Psi$ are orthogonal.

Any analytic function $F(z) = \phi + i\psi$ is a candidate to describe an irrotational, incompressible 2D flow field with:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

We define the complex velocity $W(z)$

$$W(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv$$

$$= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial x} = u - iv$$

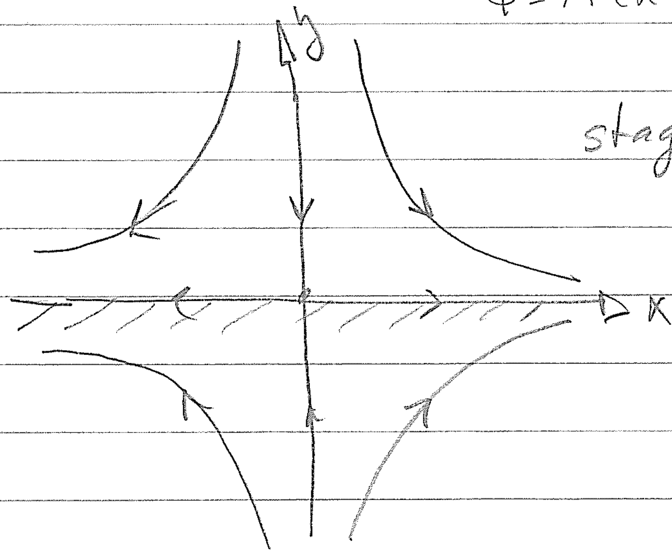
W is complex conjugate of velocity vector.

<u>Ex.</u> $F = Az^2$ $\quad W = 2Az = 2A(x+iy)$

$$\begin{cases} u = 2Ax \\ v = -2Ay \end{cases}$$

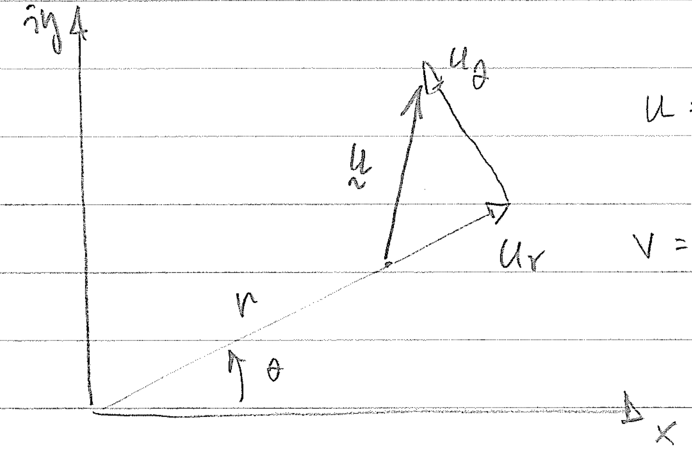
$Az^2 = A(x+iy)^2 = A(x^2-y^2) + 2Aixy$

$\phi = A(x^2-y^2)$, $\psi = 2Axy$




stagnation point flow

Polar coordinates $z = re^{i\theta} = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}$




$u = u_r\cos\theta - u_\theta\sin\theta$

$v = u_r\sin\theta + u_\theta\cos\theta$

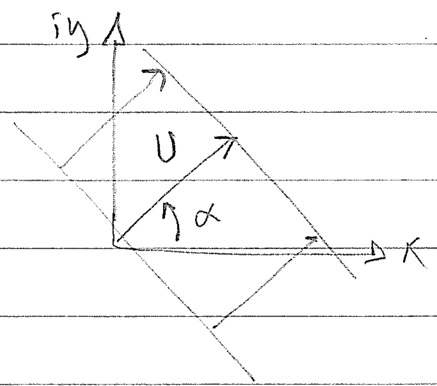
$$W = u - iv = \cos\theta\,(u_r - iu_\theta) + \underbrace{\sin\theta\,(-iu_r - u_\theta)}_{-i\sin\theta(u_r - iu_\theta)}$$

$$= (u_r - iu_\theta)(\cos\theta - i\sin\theta) =$$

$$= (u_r - iu_\theta)\,e^{-i\theta}$$

<u>Ex.</u> $F = Ue^{-i\alpha}z$

$$w = F' = Ue^{-i\alpha} = \underbrace{U\cos\alpha}_{u} - i\underbrace{U\sin\alpha}_{v}$$




$$W = Ue^{-i\alpha + i\theta - i\theta} = Ue^{i(\theta-\alpha)}e^{-i\theta}$$

$$= \left(\underbrace{U\cos(\theta-\alpha)}_{u_r} + i\underbrace{U\sin(\theta-\alpha)}_{-u_\theta}\right)e^{-i\theta}$$

Ex. $F = Az^n \qquad (A \text{ real}, \ n \geq \frac{1}{2})$

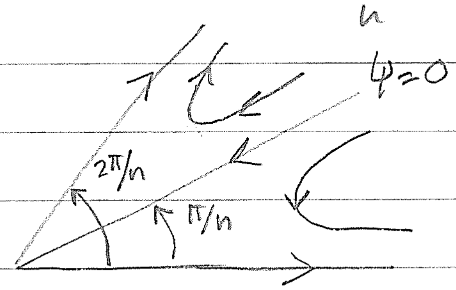
$$w = F' = nAz^{n-1} = nAr^{n-1} e^{in\theta} e^{-i\theta}$$

$$= nAr^{n-1}\left(\cos(n\theta) + i\sin(n\theta)\right) e^{-i\theta}$$

$$\Rightarrow \begin{cases} u_r = nAr^{n-1}\cos(n\theta) \\ u_\theta = -nAr^{n-1}\sin(n\theta) \end{cases}$$

$$F = Ar^n e^{in\theta} = \underbrace{Ar^n\cos(n\theta)}_{\phi(r,\theta)} + i\underbrace{Ar^n\sin(n\theta)}_{\psi(r,\theta)}$$

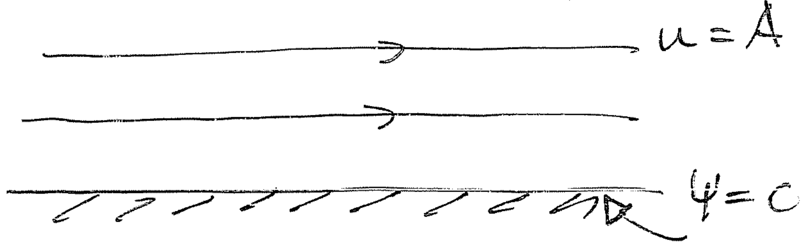
$$\psi = Ar^n\sin(n\theta) = 0 \quad \Rightarrow \quad \theta = \frac{\pi k}{n} \quad k = 0, 1, 2, \ldots$$



$n = 1$ : uniform flow over flat plate

$F = Az$

$W = A$

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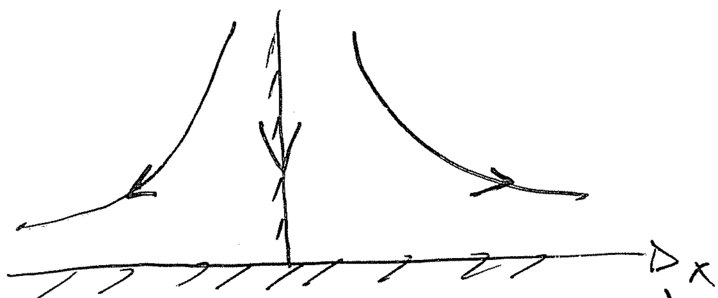
$n = 2$ : stagnation point flow

$F = Az^2 \qquad W = 2Az = 2A(x + iy)$

$$\begin{cases} u = 2Ax \\ v = -2Ay \end{cases}$$




(or flow in a $90°$ corner)

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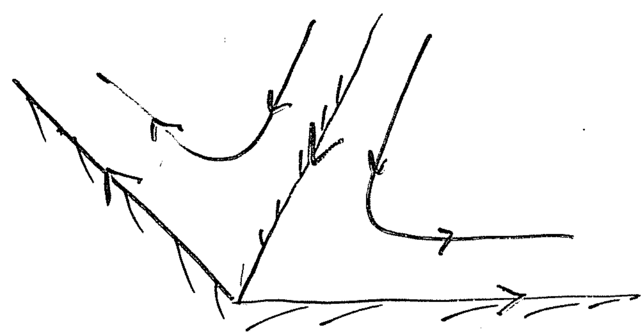
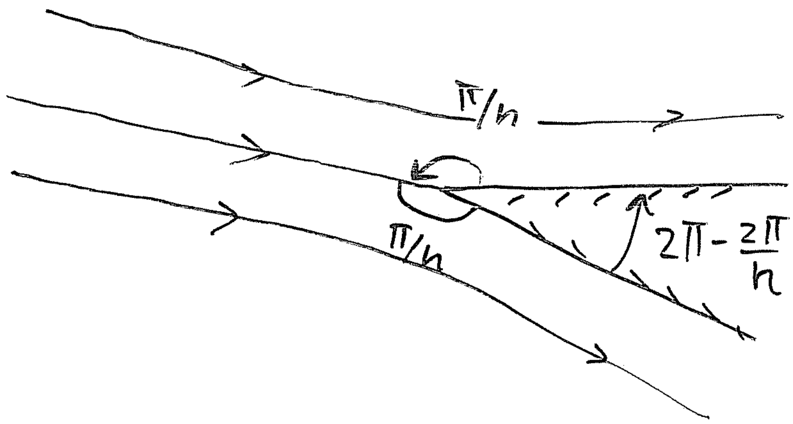
$n > 2$ :



$1 \le n \le 2$ : flow towards a wedge (or in a corner)




at surface $\theta = 0$ : $u_r = n A r^{n-1}$

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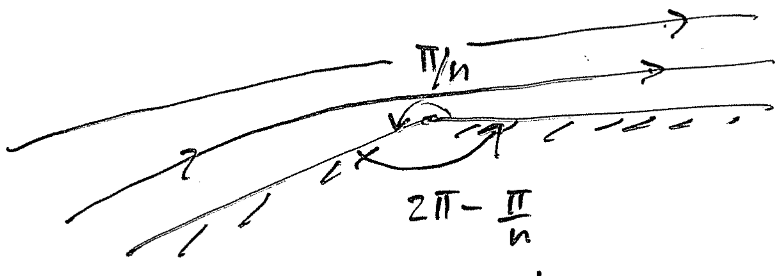
$n < 1$ : flow around corner




at surface $\theta = 0$ : $u_r = \dfrac{nA}{r^{1-n}}$

$\theta = \dfrac{\pi}{n}$ : $u_r = -\dfrac{nA}{r^{1-n}}$

infinite velocity at corner!

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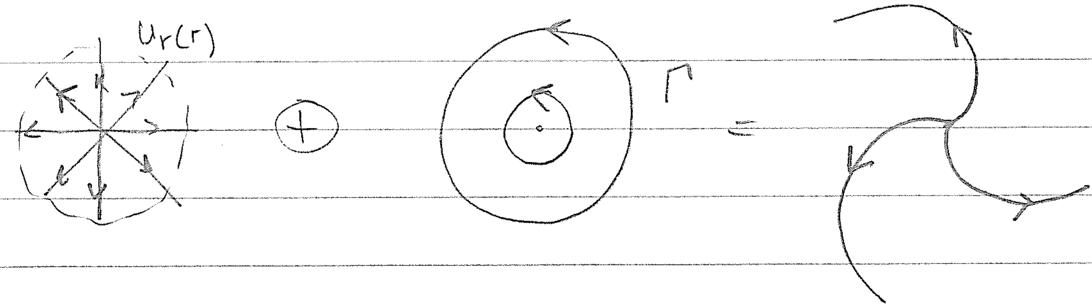
$n = \frac{1}{2}$ : flow around edge

Ex. Line source & line vortex

$$F(z) = \left(\frac{m - i\Gamma}{2\pi}\right) \ln z \qquad z = re^{i\theta}$$

$$w = \frac{m - i\Gamma}{2\pi}\frac{1}{z} = (m - i\Gamma)\frac{1}{2\pi r} e^{-i\theta} \triangleq (u_r - i u_\theta) e^{-i\theta}$$

$$\begin{cases} u_r = \dfrac{m}{2\pi r} & m = 2\pi r\, u_r(r) \text{ is volume flux/width} \\ u_\theta = \dfrac{\Gamma}{2\pi r} & \Gamma \text{ is circulation} \end{cases}$$



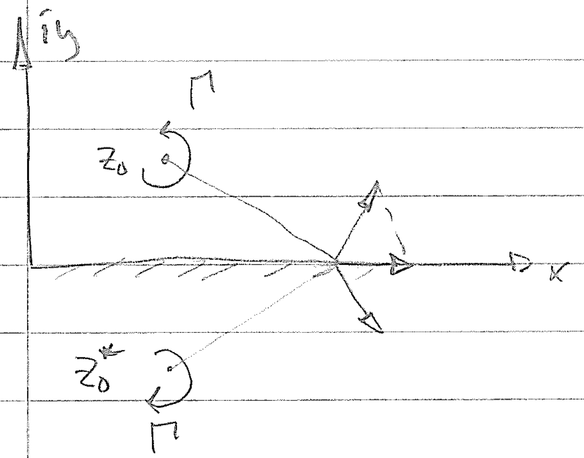

$$F = \left(\frac{m - i\Gamma}{2\pi}\right)(\ln r + i\theta) \hat{=} \phi + i\psi$$

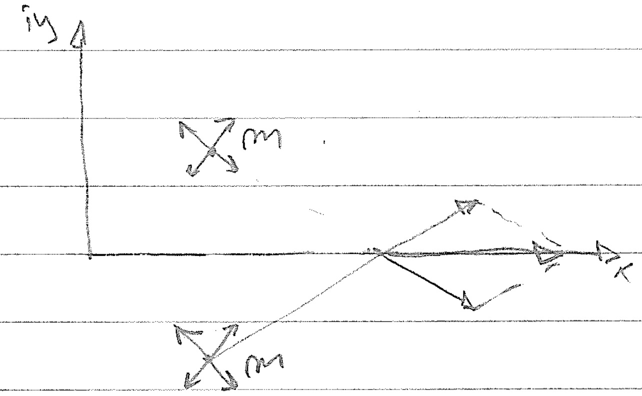
$$\phi = \frac{m}{2\pi}\ln r + \frac{\Gamma}{2\pi}\theta$$

$$\psi = \frac{m}{2\pi}\theta - \frac{\Gamma}{2\pi}\ln r$$

Superpositions of analytic functions give new analytic solution.
Laplace equation is linear!

Ex. Mirror images in a plane



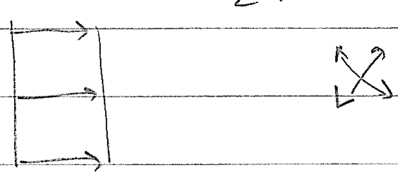




mirror images superposed

$$F = \frac{(m - i\Gamma)}{2\pi} \ln(z - z_0) + \frac{(m + i\Gamma)}{2\pi} \ln(z - z_0^*)$$

Ex. Line source ⊕ uniform stream

$$F = Uz + \frac{m}{2\pi} \ln z = Ure^{i\theta} + \frac{m}{2\pi}(\ln r + i\theta)$$

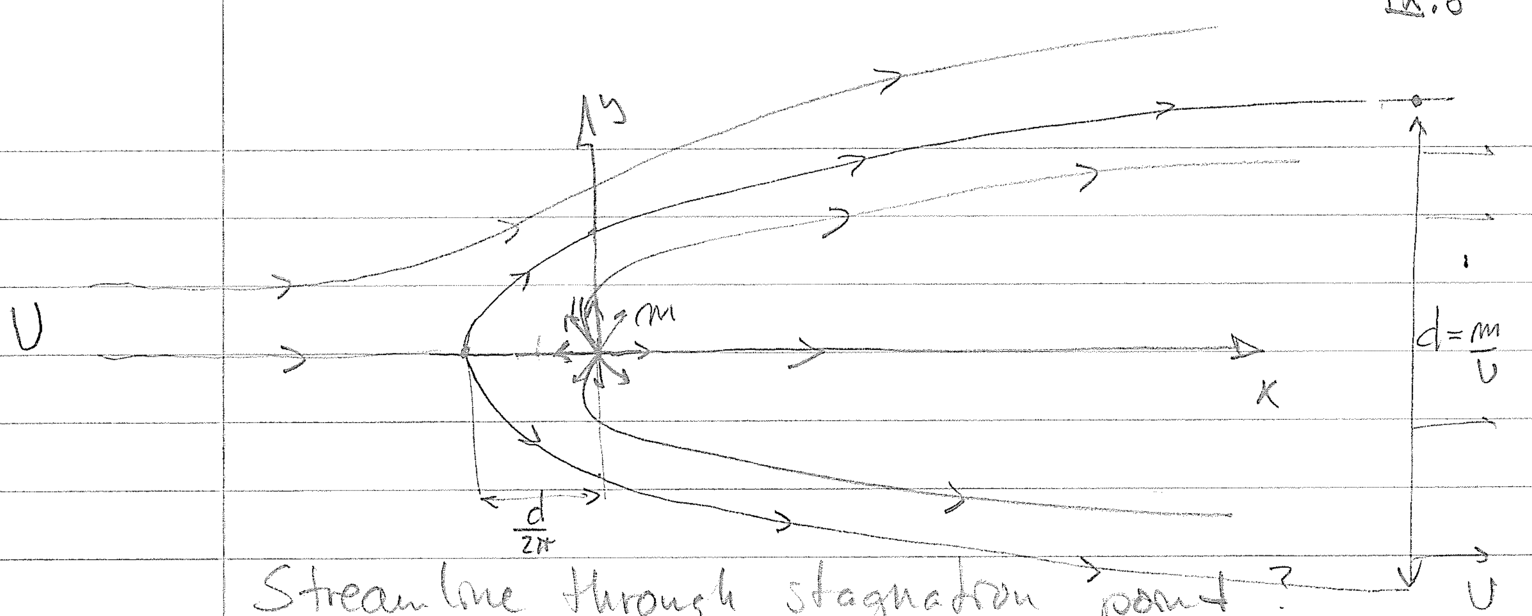


Stream function $\psi = Ur\sin\theta + \frac{m}{2\pi}\theta$

$$w = F' = U + \frac{m}{2\pi z}$$

Stagnation point if $w = 0$

$$U + \frac{m}{2\pi z_s} = 0 \Rightarrow z_s = -\frac{m}{2\pi U}$$




Streamline through stagnation point?

$\Psi = \text{constant} = ?$ $\quad z_s = -\frac{m}{2\pi U} \Rightarrow r_s = \frac{m}{2\pi U}, \vartheta_s = \pi$

$$\Psi_s = 0 + \frac{m}{2} = U \underbrace{r \sin\theta}_{=y} + \frac{m}{2\pi}\vartheta$$

Parametric representation

$$y = \frac{1}{2}\frac{m}{U} - \frac{m}{U}\frac{\vartheta}{2\pi}$$

$\vartheta \to 0 \quad y \to \frac{1}{2}\frac{m}{U}$

$\vartheta \to 2\pi \quad y \to -\frac{1}{2}\frac{m}{U}$

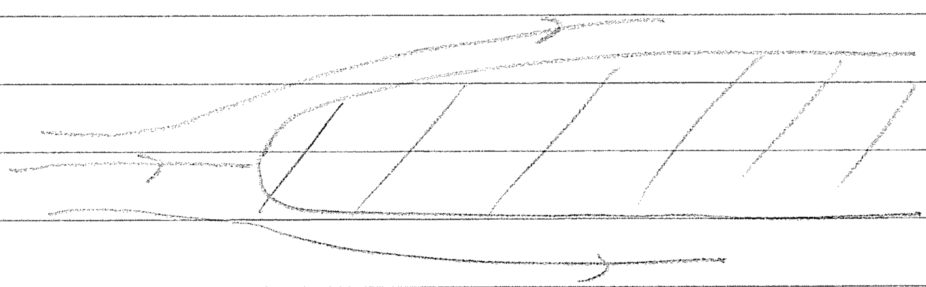
$d = \frac{m}{U}$

$$= d\left(\frac{1}{2} - \frac{\vartheta}{2\pi}\right)$$

$\vartheta = \pi \quad y = 0, \quad r_s = \frac{d}{2\pi}$

$\vartheta = \pm\pi/2, \; y = \pm d/4$

Flow past half-infinite body.



Ex. Dipole (doublet)

$$F = \frac{\mu}{2\pi z} \qquad \mu \text{ is dipole strength}$$

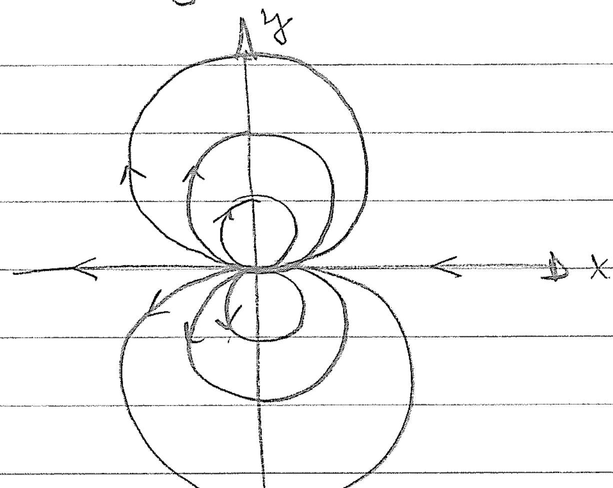
$$= \frac{\mu z^*}{2\pi z z^*} = \frac{\mu}{2\pi r^2}(x - iy)$$

$$= \underbrace{\frac{\mu}{2\pi}\frac{x}{x^2+y^2}}_{\phi} - i\underbrace{\frac{\mu}{2\pi}\frac{y}{x^2+y^2}}_{\psi}.$$

$$\psi = -\frac{\mu}{2\pi}\frac{y}{x^2+y^2} = C \Rightarrow \text{streamlines}$$

$$\frac{y}{x^2+y^2} = \frac{1}{2R} \Rightarrow x^2+y^2-2Ry = 0$$

$$x^2+(y-R)^2-R^2 = 0$$




One finds:

$$u_r = -\frac{\mu}{2\pi r^2}\cos\theta \ , \quad u_\theta = -\frac{\mu}{2\pi r^2}\sin\theta$$