KTH ROYAL INSTITUTE
OF TECHNOLOGY

General introduction to Hydrodynamic Instabilities

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KTH Mechanics, November 2015

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Outline

- General introduction to Hydrodynamic Instabilities
- Linear instability of parallel flows
  - Inviscid vs. viscous, temporal vs. spatial, absolute vs. convective instabilities
- Non-modal instabilities
  - Transient growth, resolvent, receptivity, sensitivity, adjoint equations
- Extension to complex flows situations and non-linear instabilities

References

- **Books**:
  - Charru, Hydrodynamic Instabilities, Cambridge Univ. Press
  - Drazin, Introduction to Hydrodynamic Stability, Cambridge Univ. Press
  - Huerre & Rossi, Hydrodynamic Instabilities in Open Flows, Cambridge Univ. Press
  - Schmid & Henningson, Stability and Transition in Shear Flows, Springer-Verlag

- **Articles**:
  - P. Schmid & L. Brandt, *Analysis of fluid systems: stability, receptivity, sensitivity*. Appl. Mech. Rev **66**(2), 024803, 2014.

Introduction

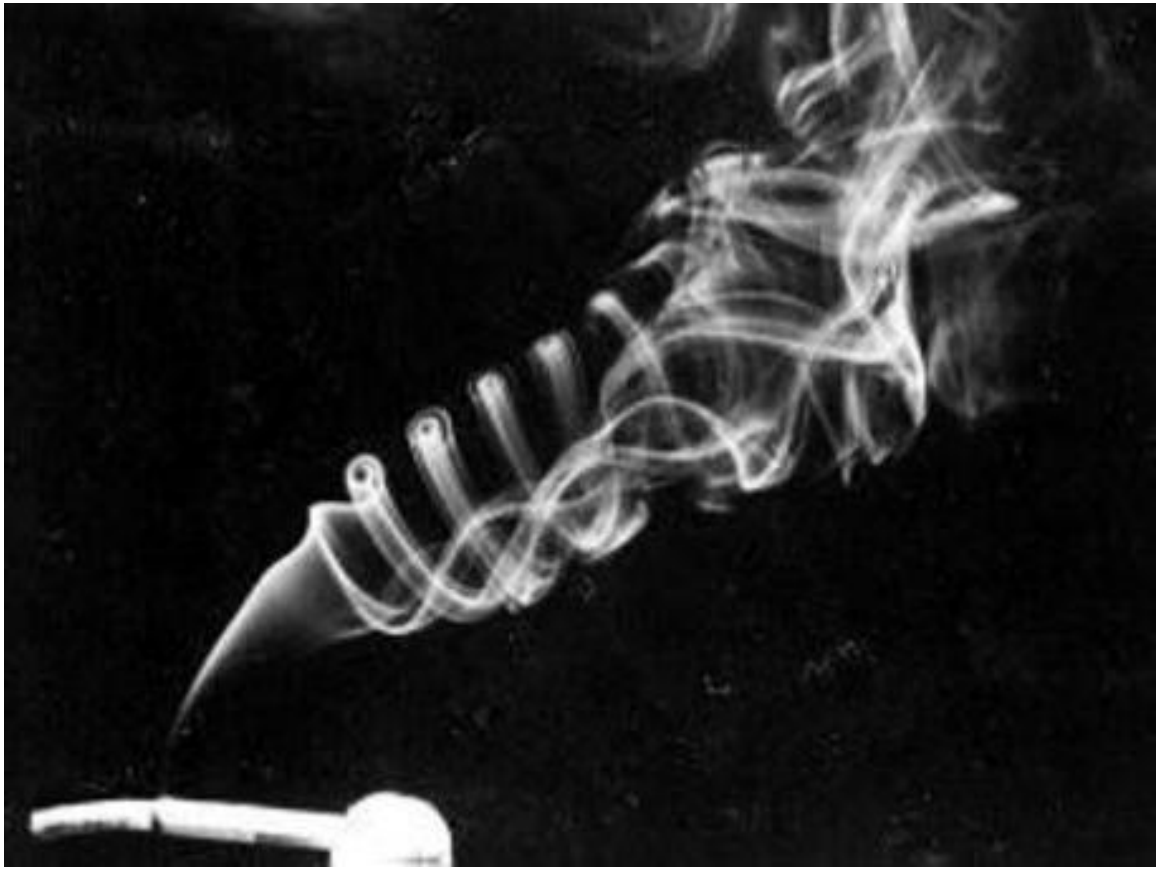
Illustrations, simple examples and local hydrodynamic stability equations.

Illustrations

What are hydrodynamic instabilities?

Transition in pipe flow

Pioneer experiment by Osborn Reynolds in 1883.



Smoke from a cigarette.

Three different flow regimes can easily be identified : laminar, transition and turbulence.

Kelvin-Hemholtz billows

Named after Lord Kelvin and Hermann von Helmholtz.
One of the most common hydrodynamic instabilities.

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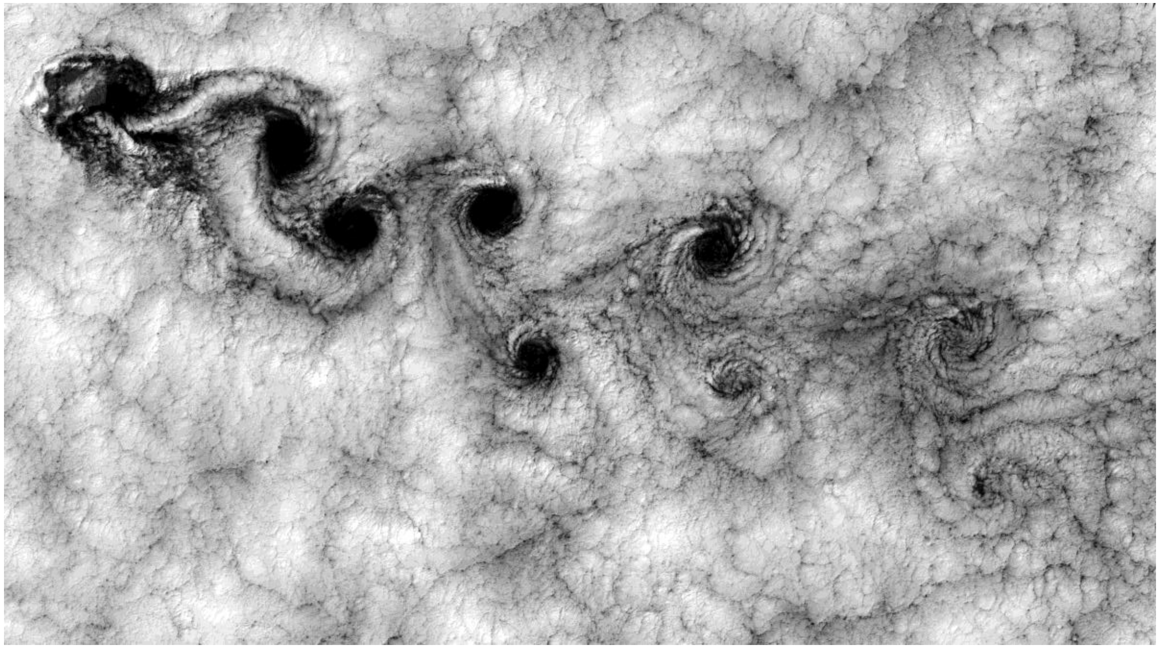
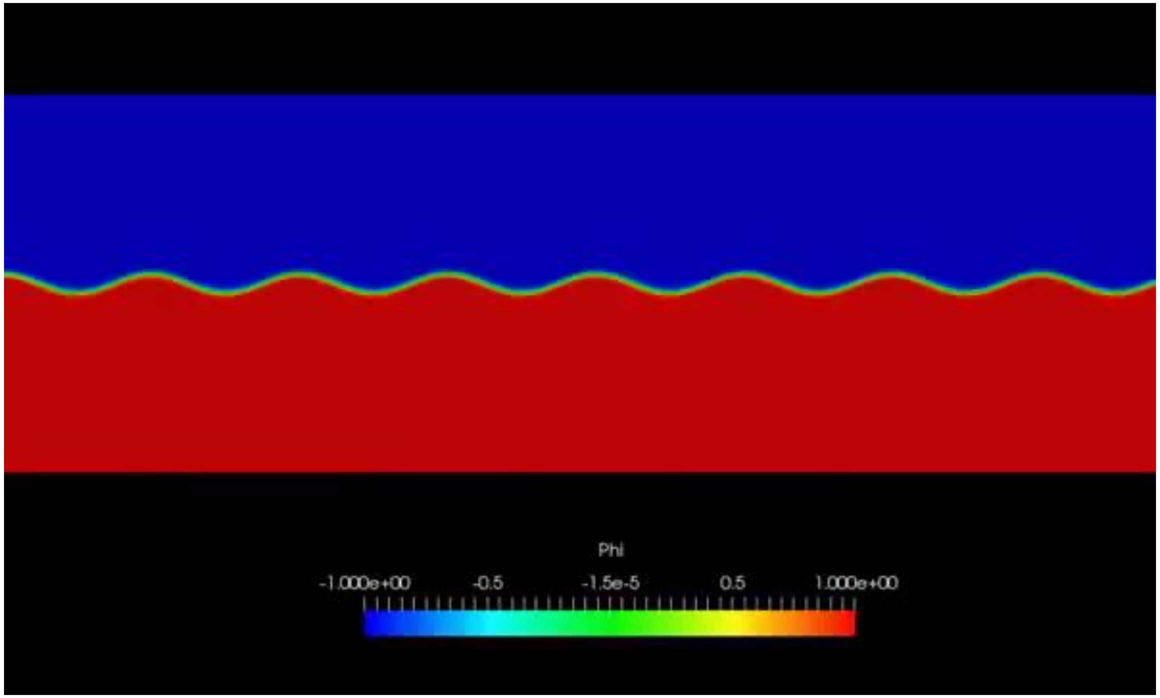


von-Karman vortex street

Named after the engineer and fluid dynamicist Theodore von Karman.



Rayleigh-Taylor instability

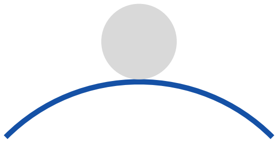
Instability of an interface between two fluids of different densities.

Some definitions, mathematical formulation and a simple example.

How do we study them?

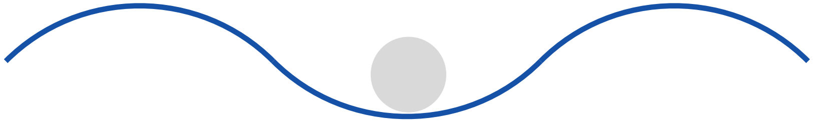
Stability?

Stable



Unstable

Neutral



Conditionaly stable

We will mostly discuss about **linear (in)stability** in this course.

Some definitions

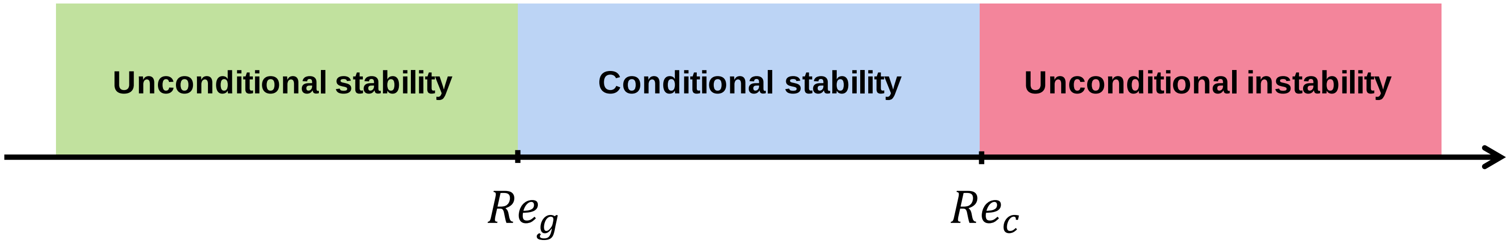
Let us consider a nonlinear dynamical system

$$\frac{d\mathbf{Q}}{dt} = f(\mathbf{Q}, Re)$$

For a given value of the control parameter Re, equilibrium solutions of the system are given by

$$f(\mathbf{Q}_b, Re) = 0$$

Some definitions



Three situations can be encountered depending on the value of Re:

1. $Re < Re_g$: The equilibrium is **unconditionally stable**. Whatever the shape and amplitude of the perturbation, it decays and the system return to its equilibrium position.
2. $Re_c < Re$: The equilibrium is **unconditionally (linearly) unstable**. At least one infinitesimal perturbation will always depart away from it.
3. $Re_g \leq Re \leq Re_c$: The stability of the equilibrium depends on the shape and finite-amplitude of the perturbation. Determining the shape and amplitude of such perturbation usually requires solving a complex nonlinear problem.

Mathematical formulation of the problem

- The first part of this course concerns linear stability analysis, that is the determination of the unconditional linear instability threshold Re_c.

- The dynamics of an infinitesimal perturbation $\mathbf{q}$ can be studied by linearizing the system in the vicinity of the equilibrium $\mathbf{Q}_b$

$$\frac{d\mathbf{q}}{dt} = \mathbf{J}\mathbf{q}$$

with $\mathbf{J}$ the Jacobian matrix of the system evaluated at $\mathbf{Q}_b$.

Mathematical formulation of the problem

The ij-th entry of the Jacobian matrix evaluated in the vicinity of $\mathbf{Q}_b$ is given by

$$J_{ij} = \left.\frac{\partial f_i}{\partial Q_j}\right|_{\mathbf{Q}_b}$$

Mathematical formulation of the problem

- This linear dynamical system is autonomous in time. Its solutions can be sought in the form of normal modes

$$\mathbf{q}(t) = \hat{\mathbf{q}} e^{\lambda t} + c.c$$

with $\lambda = \sigma + i\omega$.

- Injecting this form for $\mathbf{q}(t)$ into our linear system yields the following eigenvalue problem

$$\lambda \hat{\mathbf{q}} = \mathbf{J} \hat{\mathbf{q}}$$

Mathematical formulation of the problem

The linear (in)stability of the equilibrium $\mathbf{Q}_b$ then depends on the value of the growth rate $\sigma = \Re(\lambda)$ of the leading eigenvalue :

1. If $\sigma < 0$, the system is linearly stable ($Re < Re_c$).
2. If $\sigma > 0$, the system is linearly unstable ($Re > Re_c$).
3. If $\sigma = 0$, the system is neutrally stable ($Re = Re_c$).

Mathematical formulation of the problem

The value of $\omega = \Im(\lambda)$ characterizes the oscillatory nature of the perturbation :

1. If $\omega \neq 0$, the perturbation oscillates in time.
2. If $\omega = 0$, the perturbation has a monotonic behavior.

Exercise

- Consider the equation of motion of a damped pendulum.

$$\ddot{\theta} = -k\dot{\theta} - \omega_0^2 \sin(\theta)$$

- Introducing $x = \theta$ and $y = \dot{\theta}$, this equation can be rewritten as a 2×2 system of first order ODE's

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ky - \omega_0^2 \sin(x) \end{cases}$$

Exercise

1. Compute the two equilibrium solutions of this system.

2. Derive the linear equations governing the dynamics of an infinitesimal perturbation.

3. Study the linear stability of
   a) the first equilibrium. Is it linearly stable or unstable?
   b) the second one. Is it linearly stable or unstable?

Navier-Stokes, Reynolds-Orr and Orr-Sommerfeld-Squire equations

Local hydrodynamic stability analysis

Navier-Stokes equations

- The dynamics of an incompressible flow of Newtonian fluid are governed by

$$\begin{cases} \dfrac{\partial \mathbf{U}}{\partial t} = -(\mathbf{U}\cdot\nabla)\mathbf{U} - \nabla P + \dfrac{1}{Re}\nabla^2\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \end{cases}$$

Navier-Stokes equations

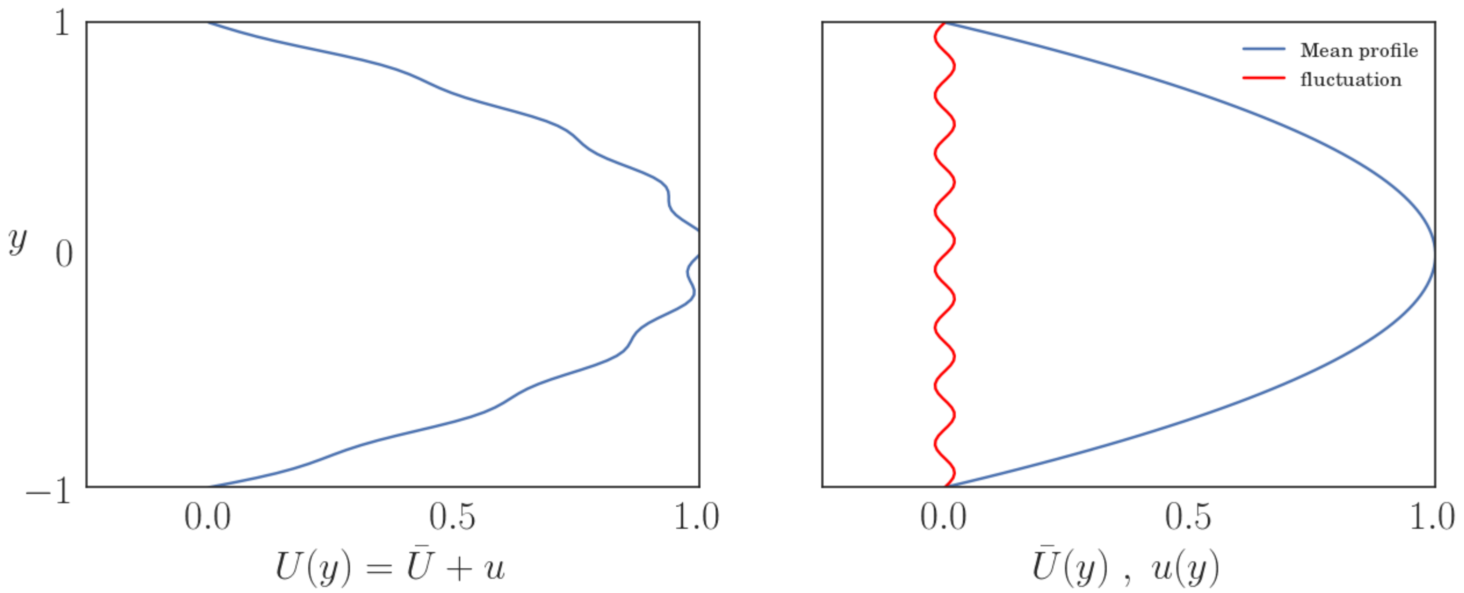
- It is a system of nonlinear partial differential equations (PDE's). The variables depend on both time and space.

- In the rest of this course, we will assume that the equilibrium solution (or base flow) $\mathbf{Q}_b = (\mathbf{U}_b, P_b)^T$ is given.

Reynolds-Orr equation

- Assume a velocity field of the form

$$\mathbf{U} = \bar{\mathbf{U}} + \mathbf{u}$$



Reynolds-Orr equation

- Plugging it into the Navier-Stokes equations, the governing equations for the fluctuation $\mathbf{u}$ read

$$
\begin{cases}
\nabla \cdot \mathbf{u} = 0 \\
\dfrac{\partial \mathbf{u}}{\partial t} = \underbrace{-(\mathbf{u} \cdot \nabla)\bar{\mathbf{U}} - (\bar{\mathbf{U}} \cdot \nabla)\mathbf{u} - \nabla p + \dfrac{1}{Re}\nabla^2 \mathbf{u}}_{\text{Linear terms}} \underbrace{- (\mathbf{u} \cdot \nabla)\mathbf{u}}_{\text{Nonlinear term}}
\end{cases}
$$

Reynolds-Orr equation

- Multiplying from the left by $\mathbf{u}$ gives an evolution equation for the local kinetic energy

$$\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{u}) = -\mathbf{u}\cdot\big((\mathbf{u}\cdot\nabla)\bar{\mathbf{U}}\big) - \mathbf{u}\cdot\big((\bar{\mathbf{U}}\cdot\nabla)\mathbf{u}\big) + \mathbf{u}\cdot\left(\frac{1}{Re}\nabla^2\mathbf{u}\right)$$
$$-\mathbf{u}\cdot(\nabla p) - \mathbf{u}\cdot\big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)$$

Reynolds-Orr equation

- After integrating over the whole volume, the Reynolds-Orr equation governing the evolution of the total kinetic energy of the perturbation finally reads

$$\frac{dE}{dt} = \underbrace{-\int_V \mathbf{u} \cdot \left((\mathbf{u} \cdot \nabla) \overline{\mathbf{U}} \right) dV}_{\textbf{Production}} \underbrace{- \frac{1}{Re} \int_V \nabla \mathbf{u} : \nabla \mathbf{u} \, dV}_{\textbf{Dissipation}}$$

- The evolution of the perturbation's kinetic energy results from a competition between production and dissipation.

Reynolds-Orr equation

- These two terms only involve linear mechanisms whether or not we initially considered the nonlinear term in the momentum equation.

- The non-linear term is energy-conserving. It only scatters the energy along the different velocity components and length scales.

Reynolds-Orr equation

- In the case of the Navier-Stokes equations, investigating the dynamics of infinitesimal perturbation allows one to:

  1. Identify the critical Reynolds number beyond which the steady equilibrium flow is unconditionally unstable. (Linear stability analysis)

  2. Highlight the underlying physical mechanisms through which any kind of perturbation (linear or nonlinear) relies to grow over time. (Reynolds-Orr equation)

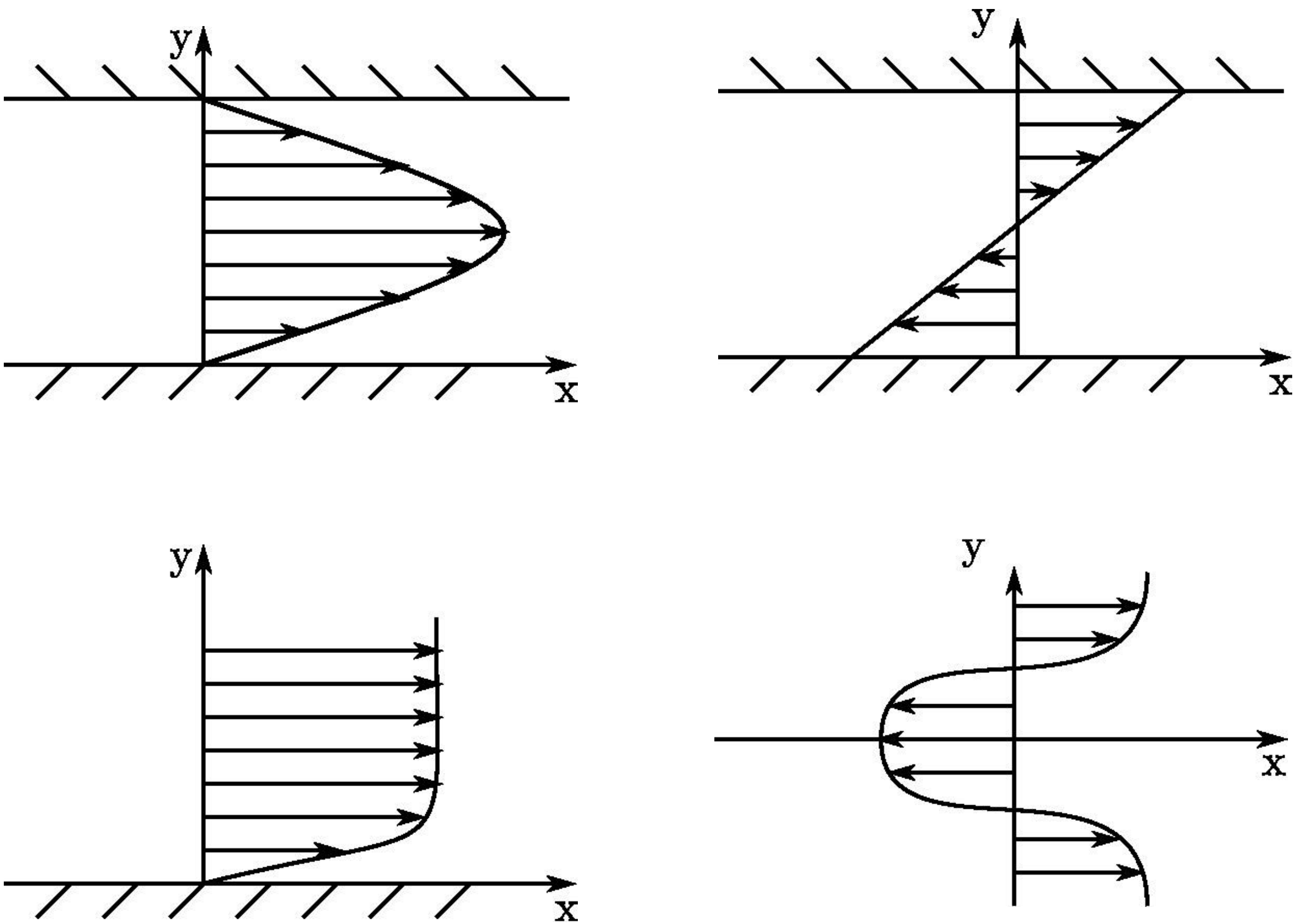
Parallel flow assumption

- For the sake of simplicity, in the rest of the course we will assume a base flow of the form

$$\mathbf{U}_b = (U(y), 0,0)$$

- The base flow only depends on the cross-stream coordinate. We neglect the streamwise evolution of the flow.

Examples of parallel flows



Linearized Navier-Stokes equations

- In the most general case (3D base flow and perturbation), the linearized Navier-Stokes equations read

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = -(\mathbf{U}_b \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{U}_b - \nabla p + \dfrac{1}{Re}\nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Linearized Navier-Stokes equations

- Based on the parallel flow assumption used for $\mathbf{U}_b$, these equations simplify to

$$\begin{cases} \dfrac{\partial u}{\partial t} = -U_b \dfrac{\partial u}{\partial x} - U_b' v - \dfrac{\partial p}{\partial x} + \dfrac{1}{Re} \nabla^2 u \\ \dfrac{\partial v}{\partial t} = -U_b \dfrac{\partial v}{\partial x} - \dfrac{\partial p}{\partial y} + \dfrac{1}{Re} \nabla^2 v \\ \dfrac{\partial w}{\partial t} = -U_b \dfrac{\partial w}{\partial x} - \dfrac{\partial p}{\partial z} + \dfrac{1}{Re} \nabla^2 w \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

Orr-Sommerfeld equation

- Taking the divergence of the momentum equations gives

$$\nabla^2 p = -2U_b' \frac{\partial v}{\partial x}$$

- One can now eliminate the pressure in the v-equation

$$\left[\left(\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x}\right)\nabla^2 - U_b'' \frac{\partial}{\partial x} - \frac{1}{Re}\nabla^4\right] v = 0$$

- This is the **Orr-Sommerfeld equation**. It governs the dynamics of the wall-normal velocity component of the perturbation.

Squire equation

- The normal vorticity is given by

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

- Its governing equation is

$$\left[\frac{\partial}{\partial t} + U_b \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2\right] \eta = -U_b' \frac{\partial v}{\partial z}$$

- This is the **Squire equation**. It governs the dynamics of the horizontal flow (u, w).

Orr-Sommerfeld-Squire equations

- The Orr-Sommerfeld-Squire (OSS) equations read

$$\begin{cases} \dfrac{\partial v}{\partial t} = \left(-U_b \dfrac{\partial}{\partial x} \nabla^2 + U_b'' \dfrac{\partial}{\partial x} + \dfrac{1}{Re} \nabla^4\right) v \\[2ex] \dfrac{\partial \eta}{\partial t} = \left(-U_b \dfrac{\partial}{\partial x} + \dfrac{1}{Re} \nabla^2\right) \eta - U_b' \dfrac{\partial v}{\partial z} \end{cases}$$

- In matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{L}}_{OS} & 0 \\ \mathbf{C} & \boldsymbol{\mathcal{L}}_{S} \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix}$$

Orr-Sommerfeld-Squire equations

- The dynamics of the cross-stream velocity v are decoupled from the dynamics of the normal vorticity η.

- The linear stability of the Squire equation is dictated by the linear stability of the Orr-Sommerfeld one.

- As a consequence, to determine the asymptotic time-evolution $(t \rightarrow \infty)$ of an infinitesimal perturbation, it is sufficient to consider the Orr-Sommerfeld equation only.

Linear stability of the OS equation

- The OS equation is autonomous in time t, and in the space coordinates x and z.

- Its solutions can be sought in the form of normal modes

$$
\begin{aligned}
v(x,y,z,t) &= \hat{v}(y)e^{i(\alpha x+\beta z-\omega t)} + c.c \\
&= \Re\left\{\hat{v}(y)\ e^{i[\alpha x+\beta z-\alpha(c_r+ic_i)t]}\right\} \\
&= |\hat{v}(y)|\ \cos[\alpha(x-c_r t)+\beta z]\ e^{\alpha c_i t}
\end{aligned}
$$

with

α	the streamwise wavenumber of the perturbation,
β	its spanwise wavenumber,
ω	the complex angular frequency,
c_r	the phase speed,
αc_i	the temporal growth rate.

Linear stability of the OS equation

Introducing the normal mode ansatz into the OS equation yields

$$\left[(U-c)(D^2-k^2)-U''-\frac{1}{i\alpha Re}(D^2-k^2)^2\right]\hat{v}=0$$

with

$$k^2=\alpha^2+\beta^2$$

and

$$D^2=\frac{\partial^2}{\partial y^2}$$

Squire transformation

- In 1933, Squire proposed a change of variables to reduce the 3D problem to an equivalent 2D one.

- Assuming that

$$\tilde{\alpha} = \sqrt{\alpha^2 + \beta^2}\,, \tilde{\omega} = \frac{\tilde{\alpha}}{\alpha}\omega\,, \tilde{\alpha} Re_{2D} = \alpha Re \text{ and } \tilde{v} = \hat{v},$$

the OS equation reduces to

$$\left[(U - \tilde{c})(D^2 - \tilde{\alpha}^2) - U'' - \frac{1}{i\tilde{\alpha} Re_{2D}}(D^2 - \tilde{\alpha}^2)^2\right]\tilde{v} = 0$$

with $\tilde{\omega} > \omega$ and $Re_{2D} < Re$.

Squire theorem (1933)

Theorem: For any three-dimensional unstable mode (α, β, ω) of temporal growth rate ω_i there is an associated two-dimensional mode $(\tilde{\alpha}, \widetilde{\omega})$ of temporal growth rate

$$\widetilde{\omega}_i = \sqrt{\alpha^2 + \beta^2}\,\frac{\omega_i}{\alpha}$$

which is more unstable since $\widetilde{\omega}_i > \omega_i$. Therefore, when the problem is to determine an instability condition, it is sufficient to consider only two-dimensional perturbations.

Summary

Summary

- Hydrodynamic instabilities are ubiquitous in nature.

- Though the Navier-Stokes equations are nonlinear PDE's, the kinetic energy transfer from the base flow to the perturbation is governed by a linear equation (Reynolds-Orr equation).

- Hence, as a first step toward our understanding of transition to turbulence, investigating the dynamics of infinitesimally small perturbations governed by the linearized Navier-Stokes equations can prove useful.

Summary

- Investigating the linear stability of a given system is a four-step procedure:
  1. Compute an equilibrium solution $\mathbf{Q}_\mathrm{b}$ of the original nonlinear system.
  2. Linearize the equations in the vicinity of $\mathbf{Q}_b$.
  3. Use the normal mode ansatz to formulate the problem as an eigenvalue problem.
  4. Solve the eigenvalue problem.

- The linearly stable or unstable nature of $\mathbf{Q}_b$ is governed by the eigenspectrum of the Jacobian matrix.

Summary

- For the Navier-Stokes equations, using the parallel flow assumption greatly reduces the complexity of the perturbation's governing equations.

- Making use of the Orr-Sommerfeld-Squire equations decreases the dimension of the problem from $\mathbb{R}^{4n}$ to $\mathbb{R}^{2n}$.

- The linear (in)stability of the flow is solely governed by the Orr-Sommerfeld equation, thus further reducing the dimension of the problem down to $\mathbb{R}^{n}$.

- Thanks to the Squire theorem, it is sufficient to investigate the linear stability of two-dimensional perturbations to determine the instability condition.

Inviscid instability of parallel flows

Rayleigh equation, Rayleigh, Fjørtotf and Howard theorems and the vortex sheet instability

Rayleigh equation

- The linear (in)stability of a viscous parallel flow is governed by the Orr-Sommerfeld equation

$$\left[(U-c)(D^2-\alpha^2)-U''-\frac{1}{i\alpha Re}(D^2-\alpha^2)^2\right]\hat{v}=0$$

- In the inviscid limit ($Re \to \infty$), it reduces to the **Rayleigh equation**

$$[(U-c)(D^2-\alpha^2)-U'']\hat{v}=0$$

Rayleigh equation

- It can be useful to introduce the stream function ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

- The Rayleigh equation for the normal mode $\hat{\psi}$ then reads

$$[(U - c)(D^2 - \alpha^2) - U'']\hat{\psi} = 0$$

with appropriate boundary conditions.

Rayleigh inflection point theorem

Theorem: The existence of an inflection point in the velocity profile of a parallel flow is a necessary (but not sufficient) condition for linear instability.

Rayleigh inflection point theorem

Demonstration: Let us assume the flow is unstable so that

$c_i \neq 0$ and $U - c \neq 0$

Dividing the Rayleigh equation by $U - c$, multiplying by $\hat{\psi}^*$ and integrating from $y = -1$ to $y = 1$ gives

$$\int_{-1}^{1} \left(\left| D\hat{\psi} \right|^2 + \alpha^2 \left| \hat{\psi} \right|^2 \right) dy + \int_{-1}^{1} \frac{U''}{U - c} \left| \hat{\psi} \right|^2 dy = 0$$

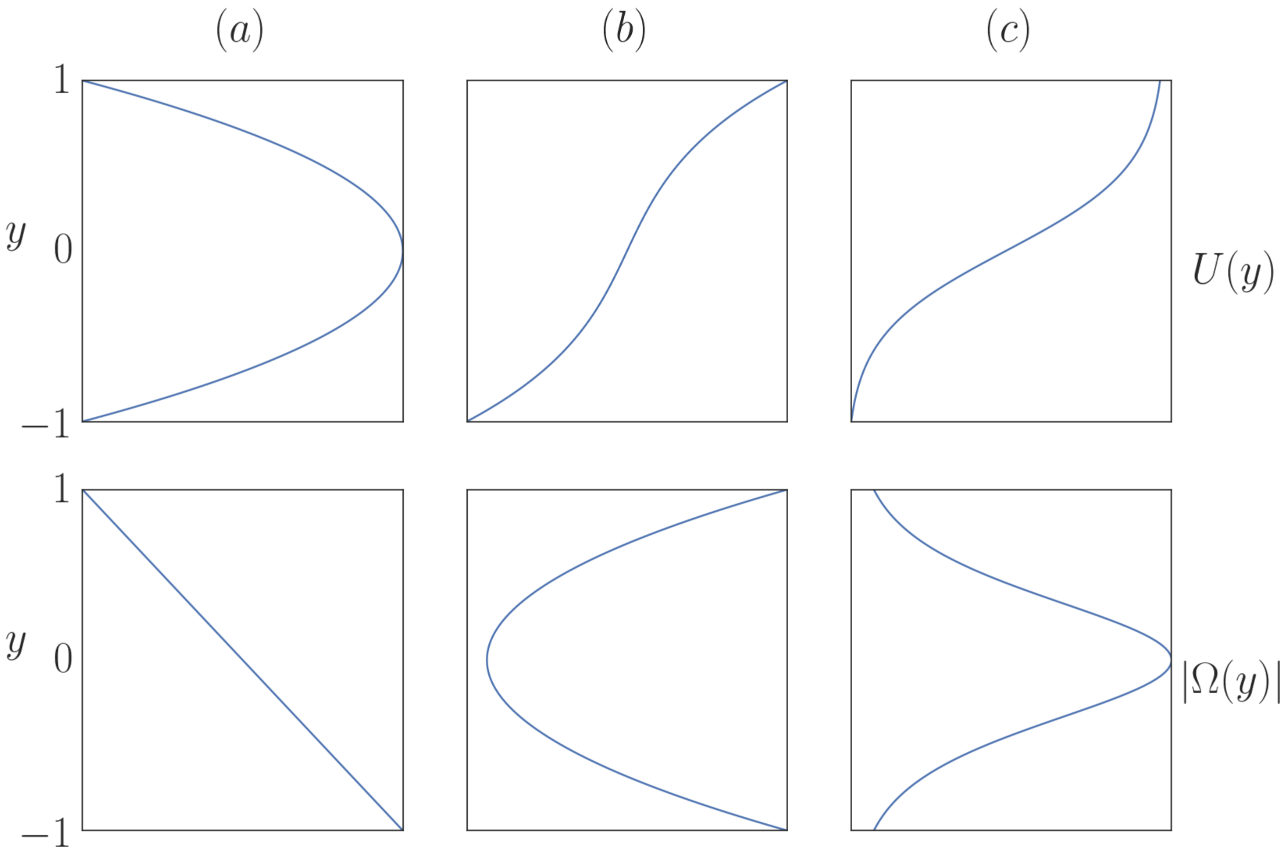
Rayleigh inflection point theorem

Let us consider only the imaginary part of this integral

$$\Im\left(\int_{-1}^{1}\frac{U''}{U-c}\left|\hat{\psi}\right|^{2}dy\right)=\int_{-1}^{1}\frac{c_iU''}{|U-c|^2}\left|\hat{\psi}\right|^{2}dy=0$$

By assumption, we have $c_i \neq 0$ and this integral must vanish. As a consquence $U''(y)$ must change sign, i.e., the velocity profile must have an inflection point.

Rayleigh inflection point theorem



Rayleigh inflection point theorem

- According to the Rayleigh theorem :
  - Velocity profile (a) is stable (in the inviscid limit).
  - Vecolity profiles (b) and (c) can **potentially** be unstable.

- One important conclusion of this theorem is that, **if the effect of viscosity on the pertubation is neglected**, both the Poiseuille flow and the Blasius boundary layer flow are stable (see profile (a))..

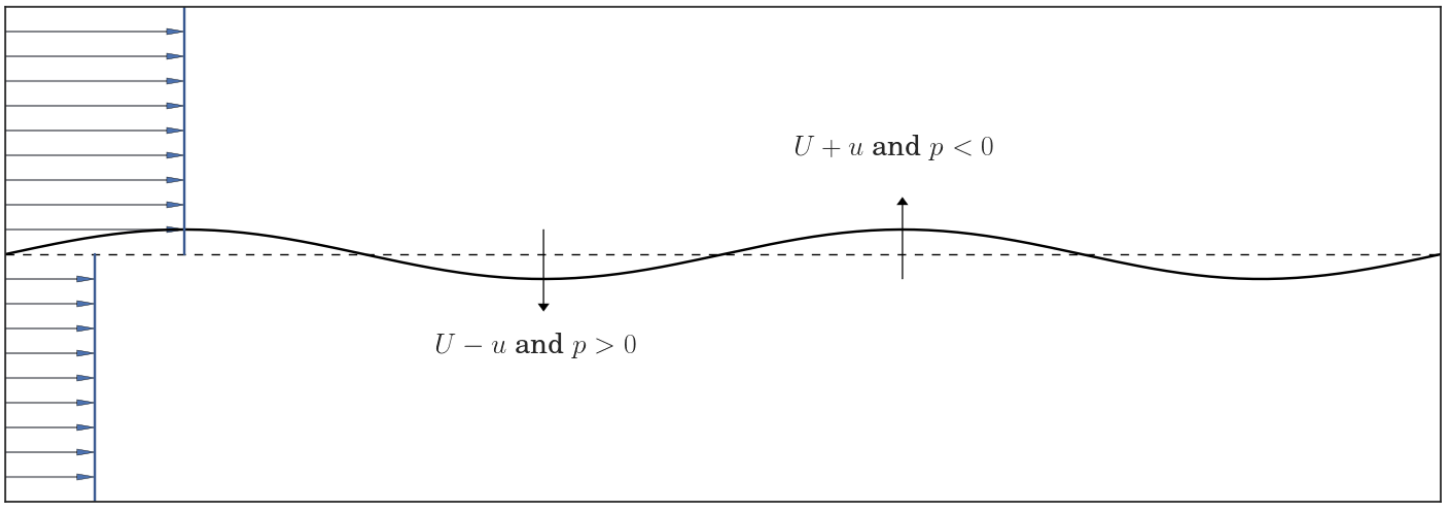
Fjørtoft theorem

Theorem: For a monotonic velocity profile, a necessary (but still not sufficient) condition for instability is that the inflection point corresponds to a vorticity maximum.

Fjørtoft theorem

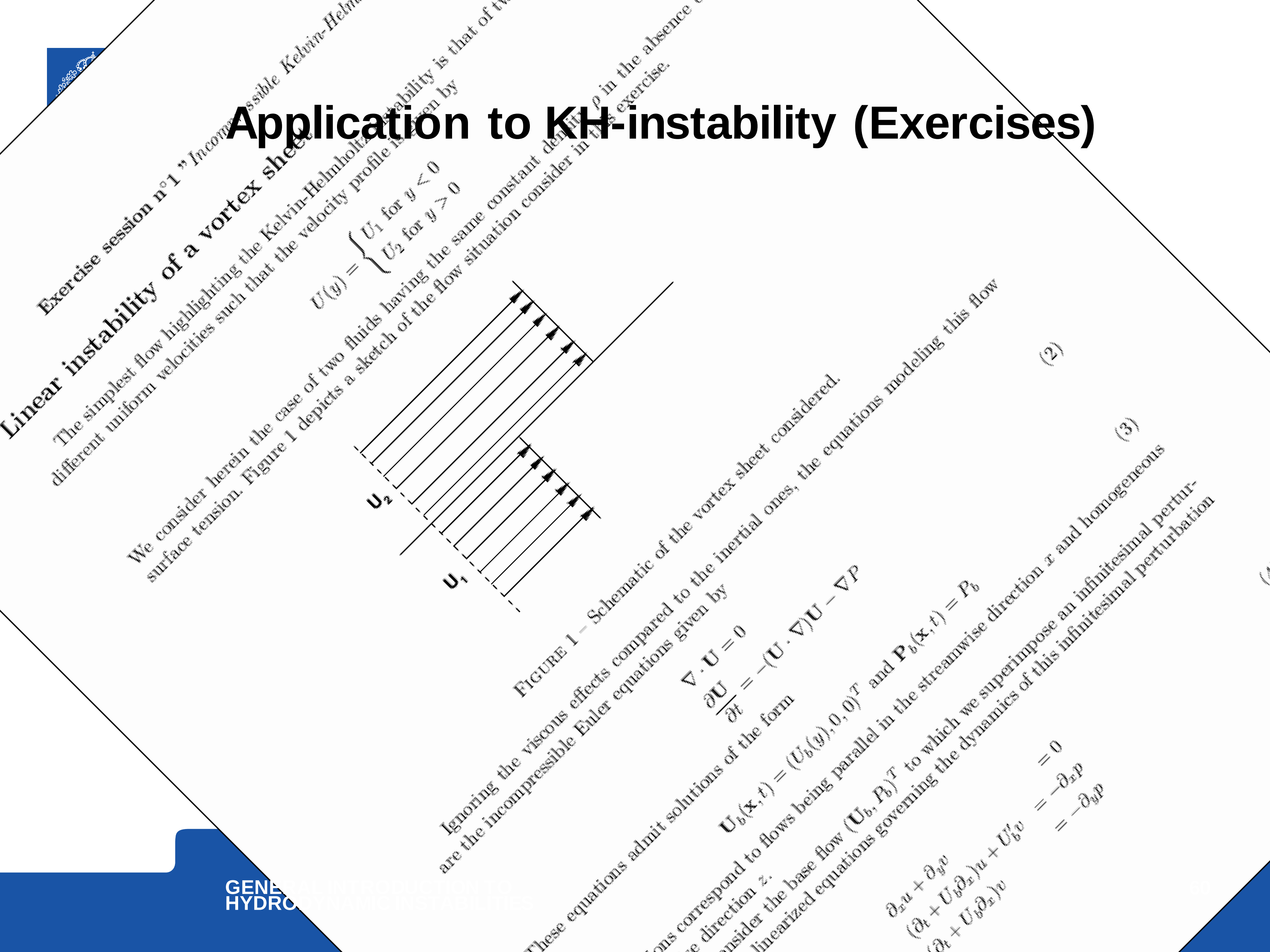
- In the inviscid limit, according to the Fjørtoft theorem :
  - Velocity profiles (a) and (b) are stable.
  - Vecolity profile (c) can **potentially** be unstable.

Application to KH-instability



''Bernoulli effect''-like explanation of the Kelvin-Helmholtz instability.

Application to KH-instability (Exercises)



Exercise session n°1 " *Incompressible Kelvin-Helm...*

Linear instability of a vortex sheet

The simplest flow highlighting the Kelvin-Helmholtz instability is that of two different uniform velocities such that the velocity profile is given by

$$U(y) = \begin{cases} U_1 \text{ for } y < 0 \\ U_2 \text{ for } y > 0 \end{cases}$$

We consider herein the case of two fluids having the same constant density ρ in the absence of surface tension. Figure 1 depicts a sketch of the flow situation consider in this exercise.

FIGURE 1 – Schematic of the vortex sheet considered.

Ignoring the viscous effects compared to the inertial ones, the equations modeling this flow are the incompressible Euler equations given by

$$\nabla \cdot \mathbf{U} = 0 \quad (2)$$

$$\frac{\partial \mathbf{U}}{\partial t} = -(\mathbf{U} \cdot \nabla)\mathbf{U} - \nabla P \quad (3)$$

These equations admit solutions of the form

$$\mathbf{U}_b(\mathbf{x}, t) = (U_b(y), 0, 0)^T \text{ and } \mathbf{P}_b(\mathbf{x}, t) = P_b$$

...ions correspond to flows being parallel in the streamwise direction x and homogeneous ...e direction z.

...nsider the base flow $(\mathbf{U}_b, P_b)^T$ to which we superimpose an infinitesimal pertur... linearized equations governing the dynamics of this infinitesimal perturbation

$$\begin{aligned} \partial_x u + \partial_y v &= 0 \\ (\partial_t + U_b \partial_x) u + U_b' v &= -\partial_x p \\ (\partial_t + U_b \partial_x) v &= -\partial_y p \end{aligned} \quad (4)$$

Summary

Summary

- If we assume that the inertial effects are much larger than the viscous ones ($Re \to \infty$), the Orr-Sommerfeld equation reduces to the Rayleigh equation.

- Rayleigh theorem states that an inflection point in the velocity profile is a necessary (but not sufficient) condition for inviscid instability.

- Fjørtoft theorem states that this inflection point needs to correspond to a maximum in the vorticity distribution. This is however still just a necessary (but not sufficient) condition for inviscid instability.

Summary

- Ignoring all effects of viscous diffusion leads to an unbounded growth rate at large wave numbers (small wavelength).
- Despite this limitation, the vortex sheet problem enable a relatively good understanding of the Kelvin-Helmholtz instability process.
- More realistic models, as the broken line velocity profile, avoids the divergence of the growth rate at large wave numbers while retaining the invisicid approximation.