Non-modal stability

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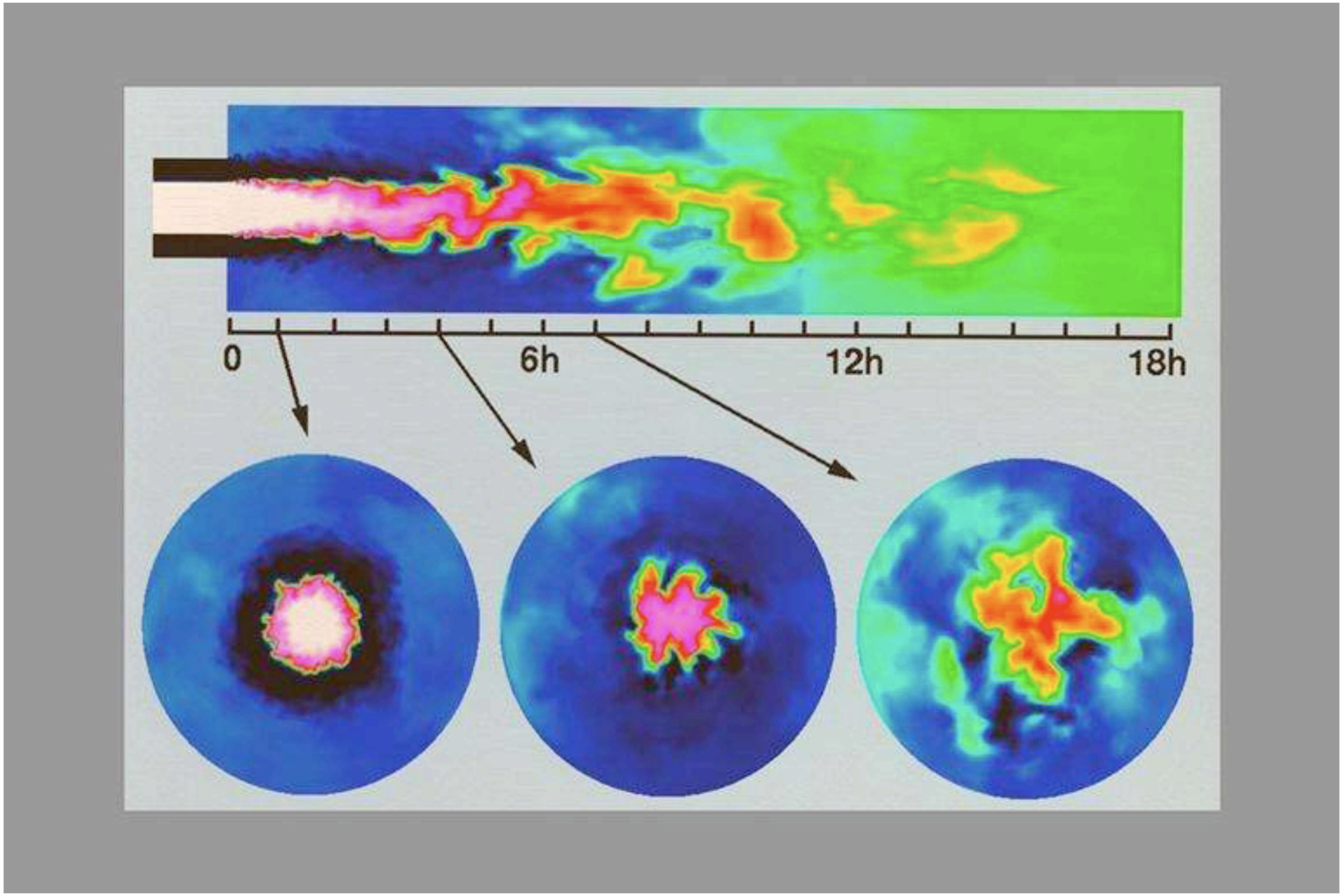
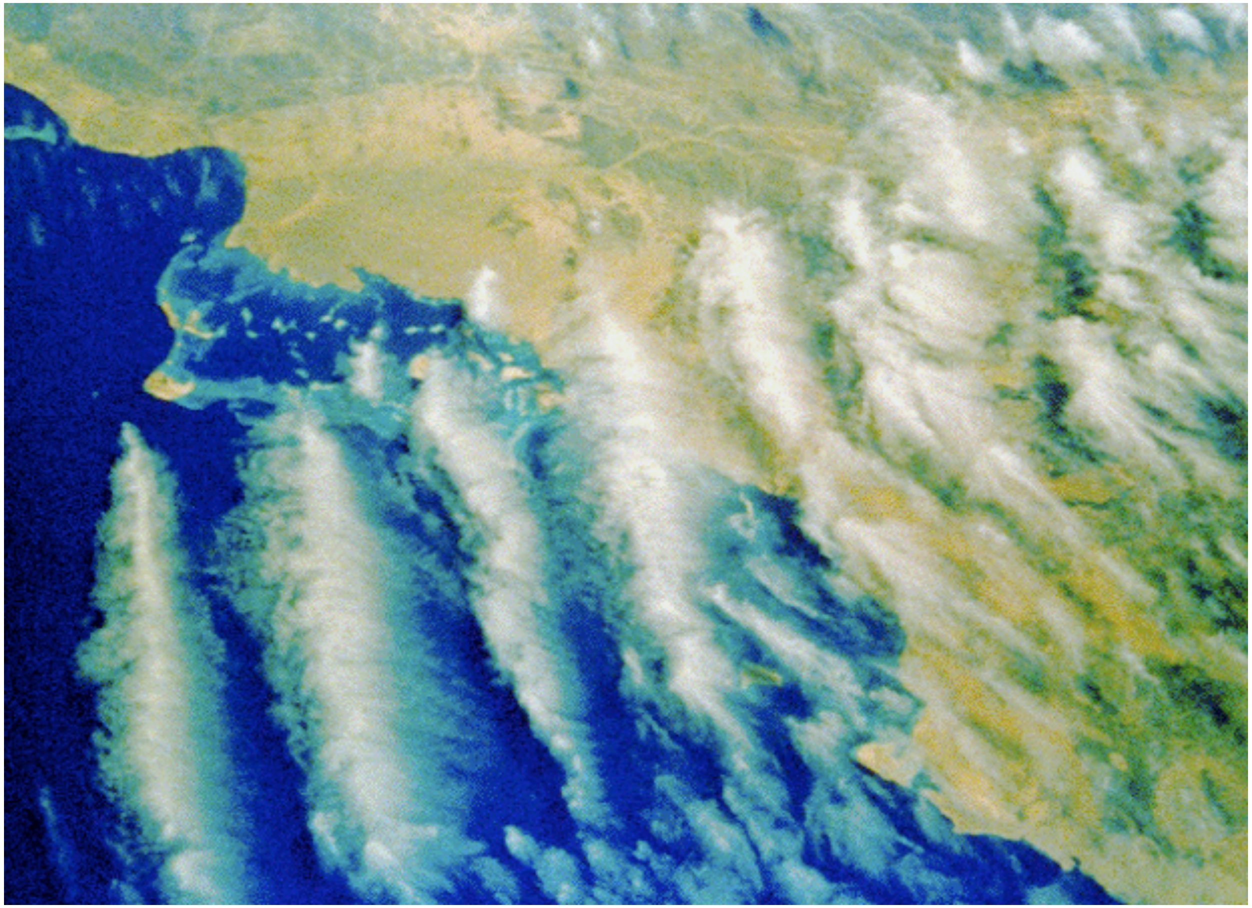
Outline

- Stability of fluid systems
  - modal limit
  - short time dynamics, matrix exponential
- Receptivity
  - resolvent norm
  - Resonance limit
  - Adjoint modes
- Sensitivity
  - Structural sensitivity
  - Base-flow sensitivity

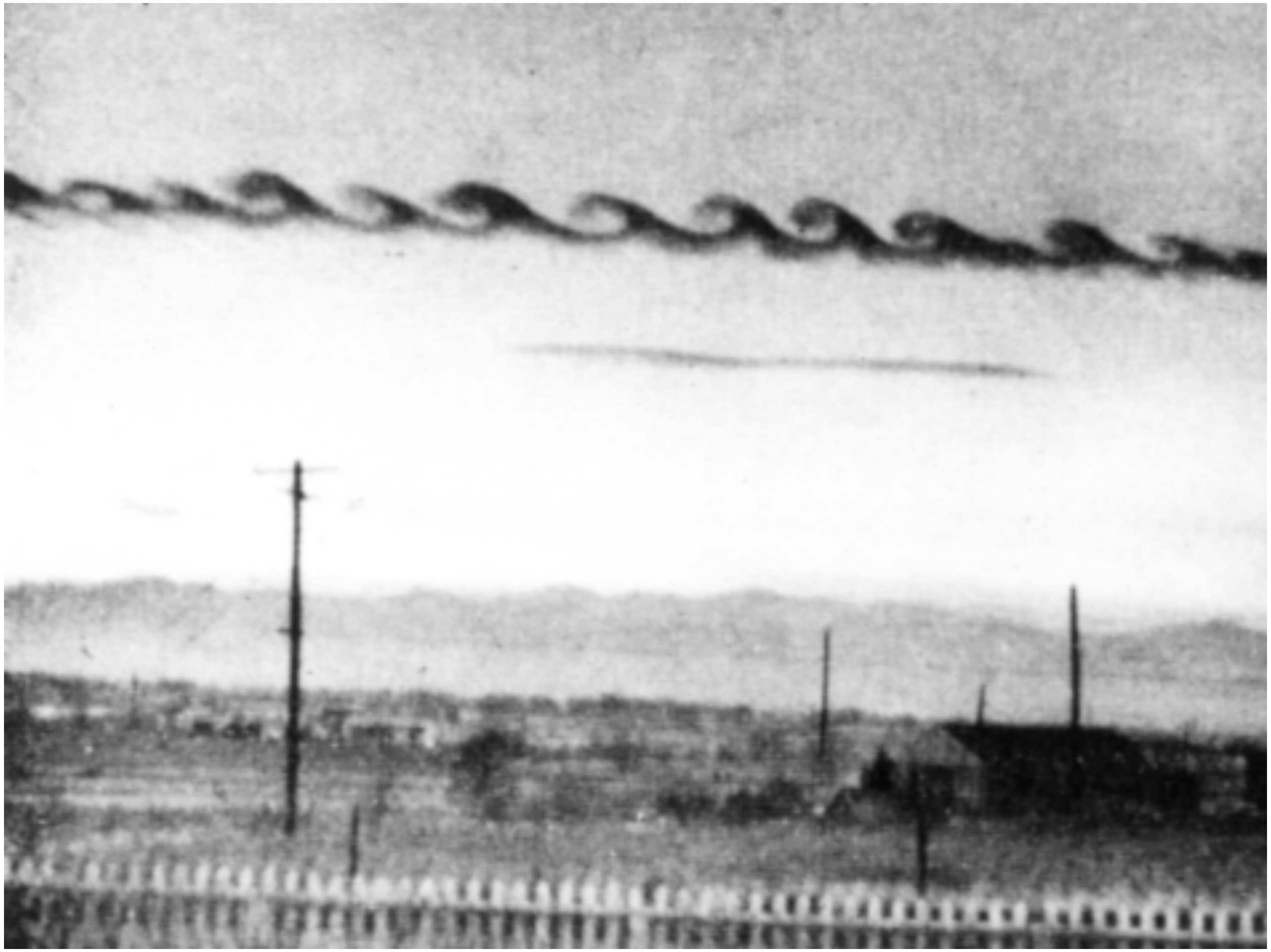
an example of instability and transition that is going to disappear...



Numerical simulation of a jet combustor: mixing of fuel and air
(Stanford University/NASA Ames)

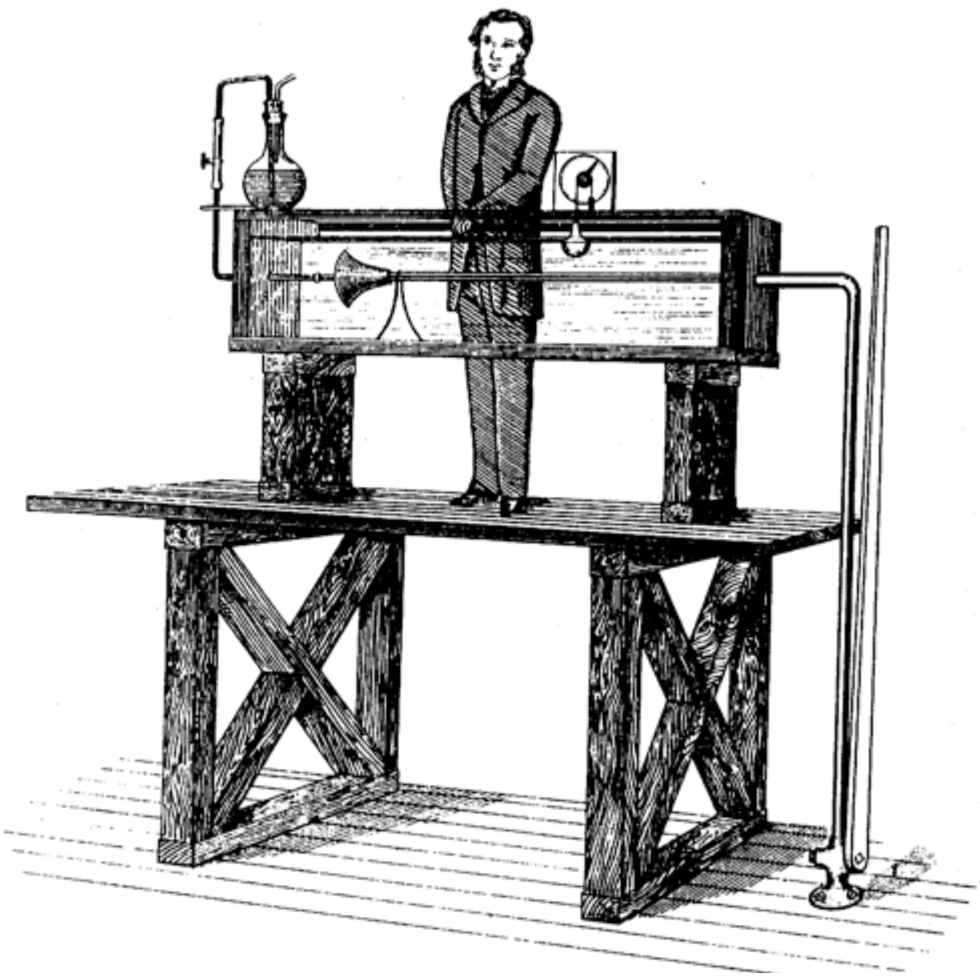


Cirrus clouds developing in a jet stream over Saudi Arabia and the Red Sea. The picture was taken from the Space Shuttle (NASA)



An example of Kelvin-Helmoltz shear instability

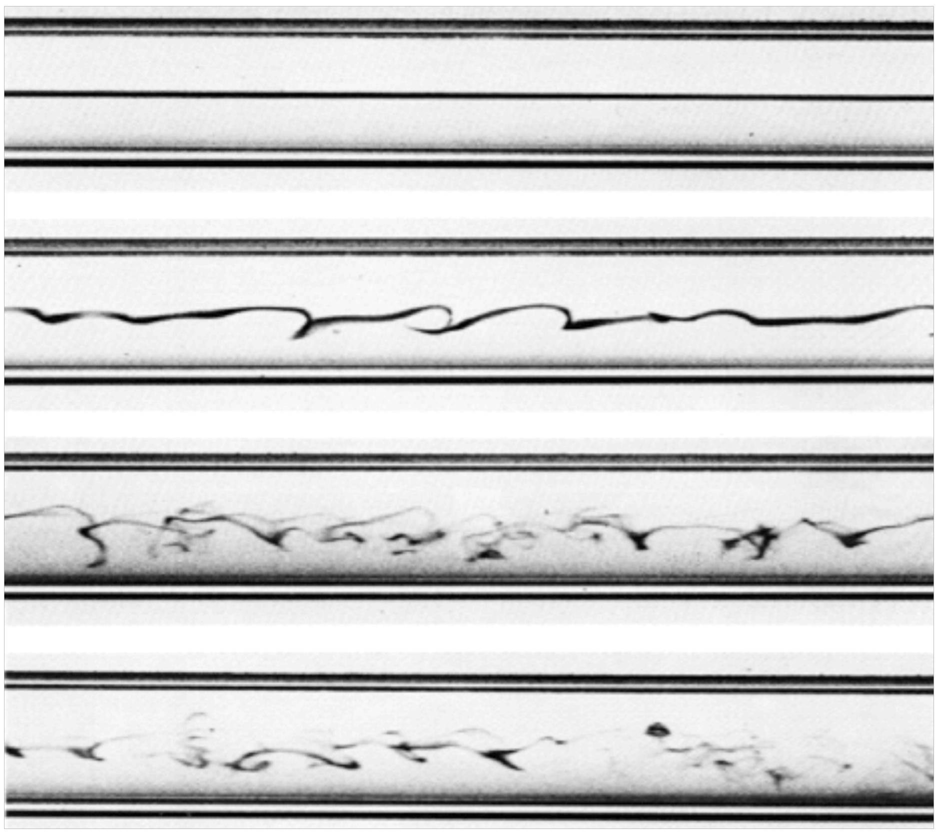
Transition to turbulence



Reynolds pipe flow experiment (1883)

- Transition is a complex physical process
- critically depends on the disturbance environment
- is parameter-dependent
- is important for the design of fluid systems

Transition to turbulence



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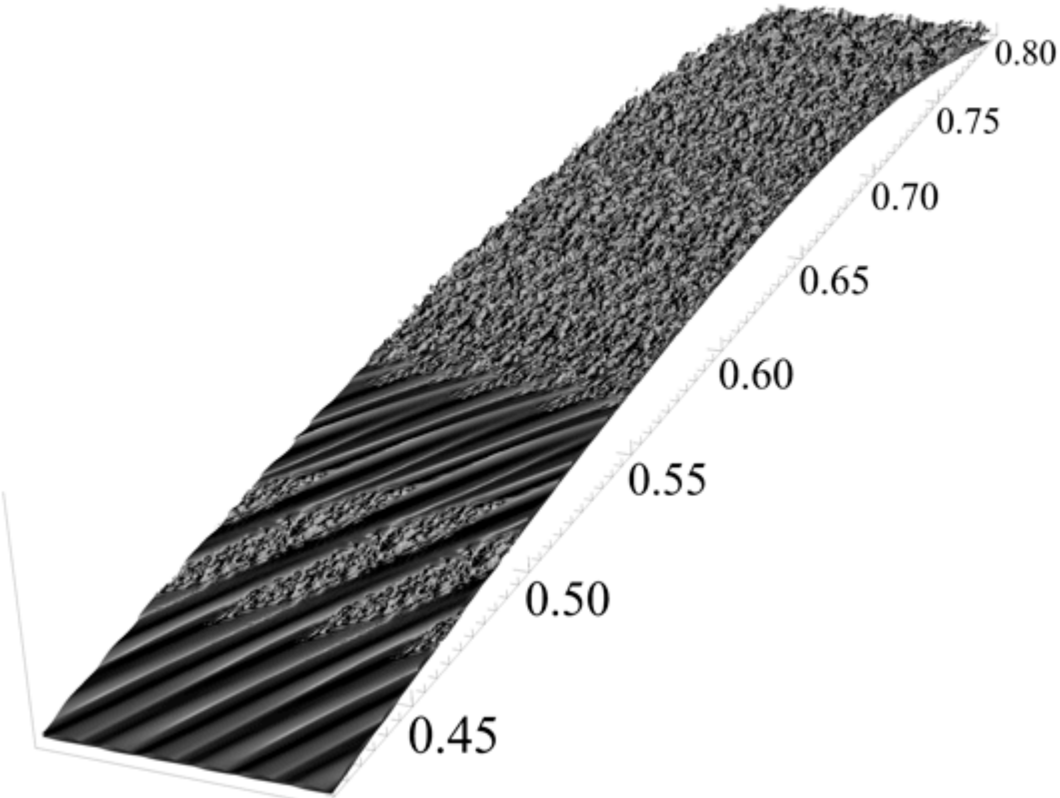
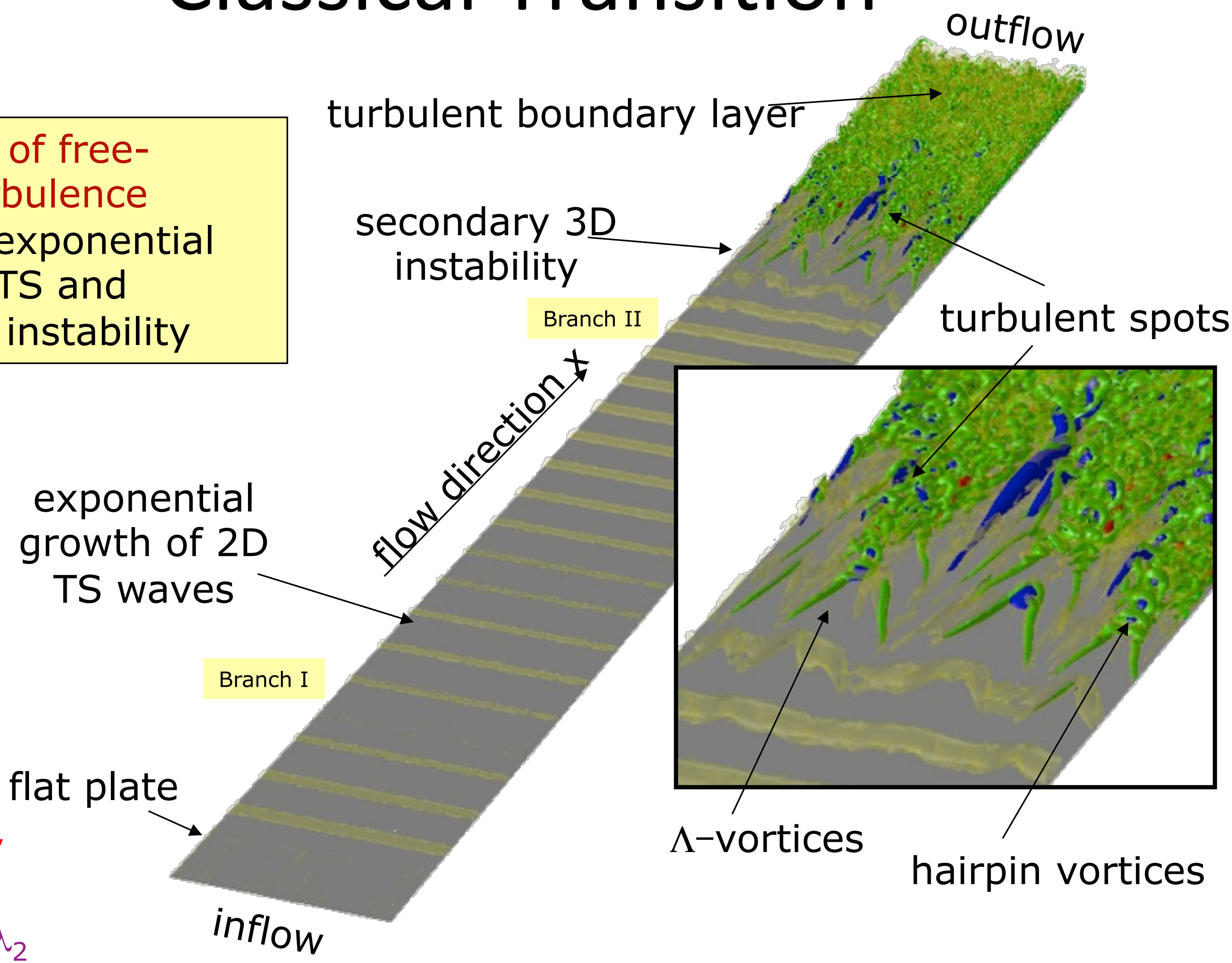


- Transition is a compex physical process
- critically depends on the disturbance environment
- is parameter-dependent
- is important for the design of fluid systems

Transition on a swept wing
Simulations by Hosseini (2013)

Classical Transition

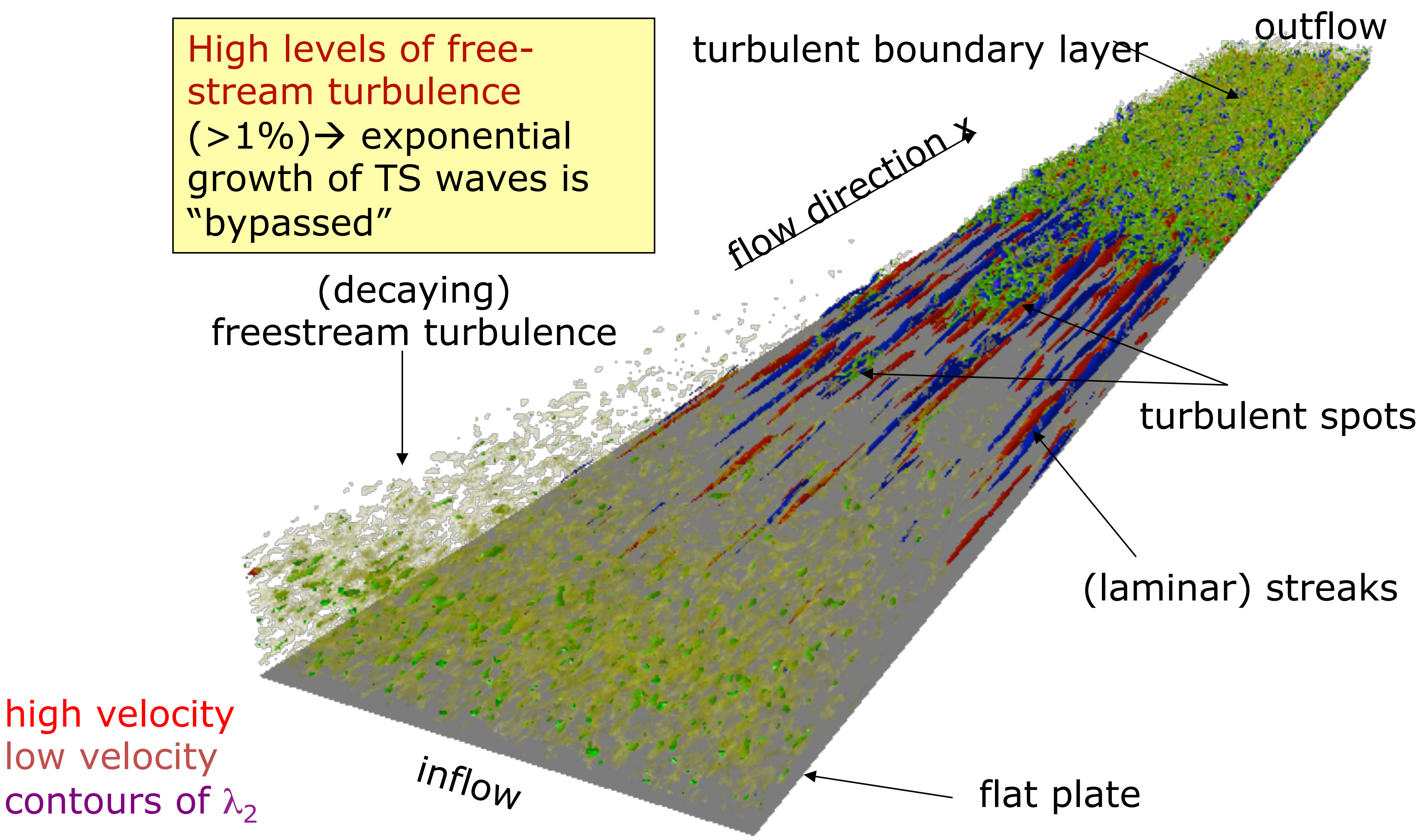
Low levels of free-stream turbulence (<1%) → exponential growth of TS and secondary instability



high velocity
low velocity
contours of λ_2

Bypass Transition

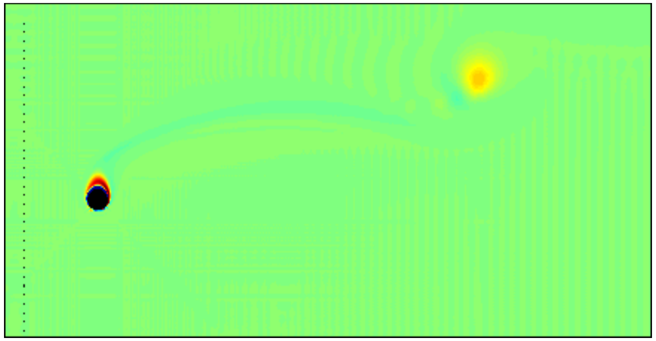
High levels of free-stream turbulence (>1%)→ exponential growth of TS waves is "bypassed"



Oscillators vs. noise amplifiers

Open flows: global instability and transient growth

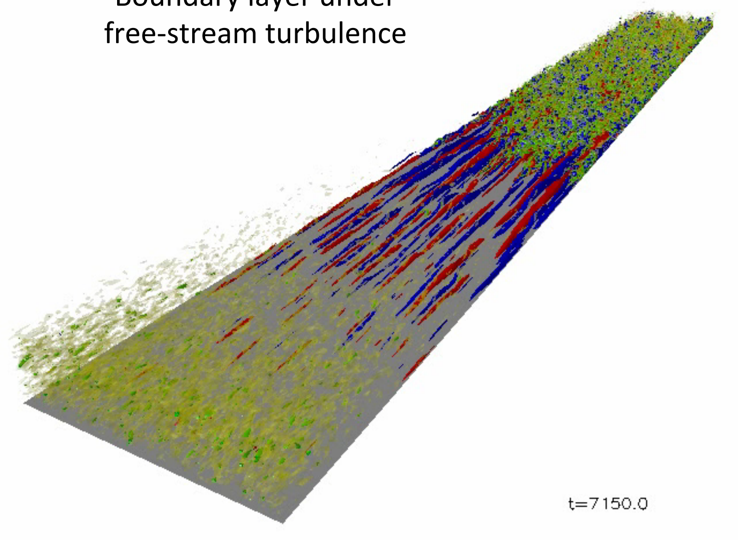
Rotating cylinder



Hydrodynamic oscillators:

Global instability
Intrinsic frequency
Local absolute instability (WKB)

Boundary layer under
free-stream turbulence



Noise amplifiers:

Globally stable
Broad-band frequency spectrum
Local convective instability (WKB)

Globally transient growth of perturbations!

Two concepts of stability

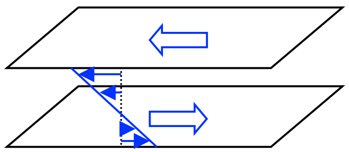
- **Linear stability: we are interested in the minimum critical parameter above which a specific initial condition of infinitesimal amplitude grows exponentially**

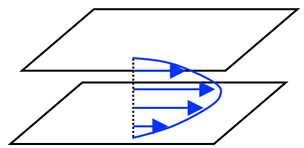
- **Energy stability: we are interested in the maximum critical parameter below which a general initial condition of finite amplitude decays monotonically**

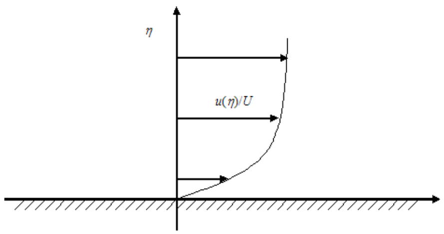
Stability analysis

Hydrodynamic stability

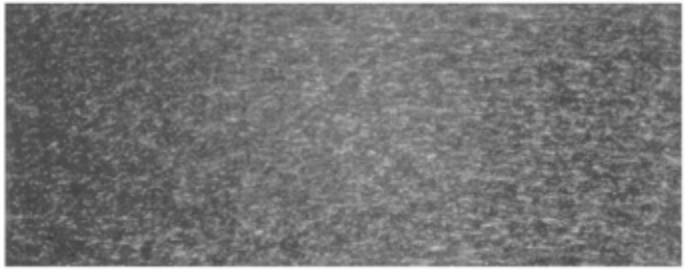
- Solutions of Navier-Stokes: Couette, Poiseuille, boundary layer, Jet…



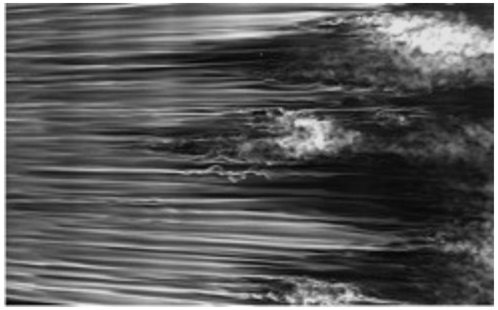




- Can we observe them in the lab?



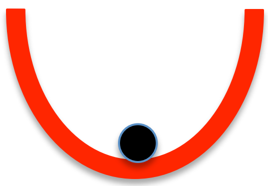
Couette flow, Re=400



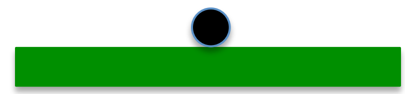
Boundary layer

Turbulent chaotic motions

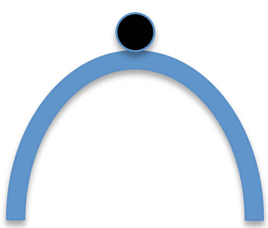
Stability



Stable



Neutral



Unstable

Conditionally
(nonlinearly) stable

We will talk about linear stability mostly

Overview

- Stability analysis
  - Baseflow and disturbances
  - Linearised equations
  - Normal modes, waves
- Thermal instability
  - Benard’s problem, natural convection
  - Analytical solution
- Instability of shear flows (jet, wake, channel, boudnary layers)
  - Parallel flow assumption
  - Viscous and inviscid instability
  - Non-modal stability and transient growth
  - Sensitivity analysis
- Laminar-turbulent transition

Stability analysis

1. Linear stability: *steady* base flow, solution of Navier-Stokes ekv.

$$\mathbf{U} = (U, V, W)\,, \quad T = T(x_i)$$

2. Disturbances $u(x_i, t), \quad T'(x_i, t)$

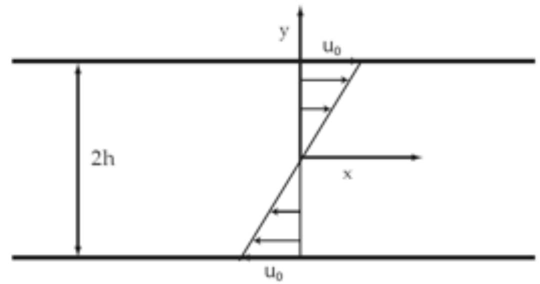
3. Derive ekv for the disturabnce evolution $\dfrac{\partial u}{\partial t} = F(u; \mathbf{U})$

4. Assume small amplitudes, linear ekv. $\dfrac{\partial u}{\partial t} = A(\mathbf{U})u$

5. Assume $u(x_i, t) = \tilde{u}\, e^{-i\omega t}$

   egienvalue problem $-i\omega\tilde{u} = A(\mathbf{U})\tilde{u}$

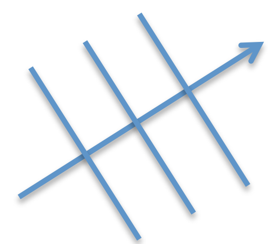
Linear stability: normal modes



- Homogeneous base flow in x och z $\quad \mathbf{U} = (U(y), 0, 0)\,, \quad T = T(y)$

- Perturbations $u(x_i, t) = \hat{u}(y) e^{ikx + imz - i\omega t}$

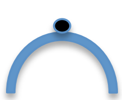
Wave-vector (k, m)



$\omega = \omega_r + i\omega_i$	Complex frequency
k	Wavenumber in x-direction
m	Vavenumber in z-direction

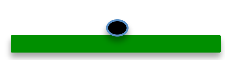
$\omega_i > 0$

Unstable



$\omega_i = 0$

Neutral

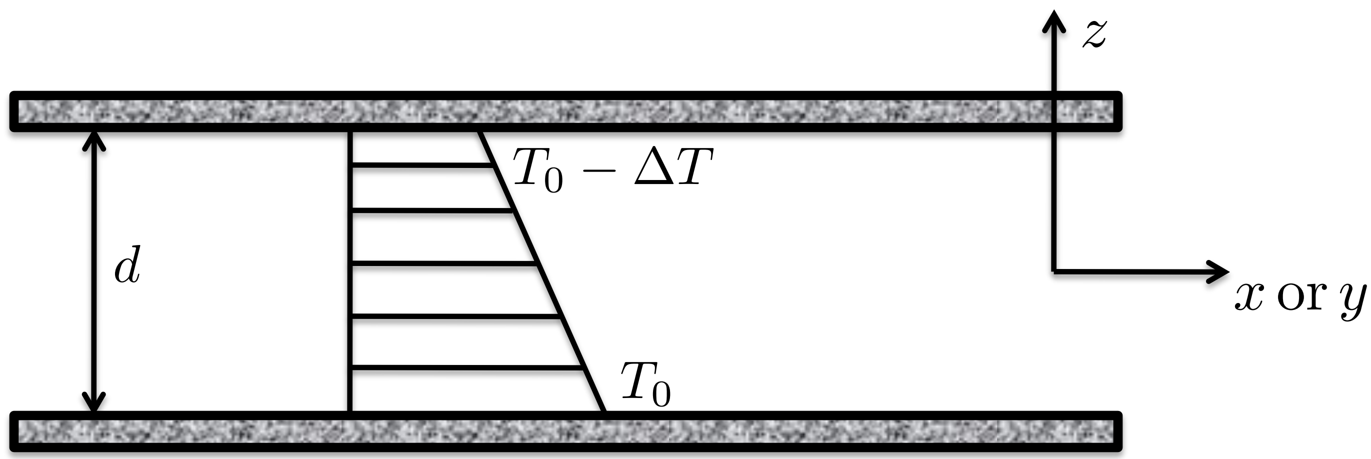


$\omega_i < 0$

Stable

Thermal instability: Benard's problem

Success story of linear stability



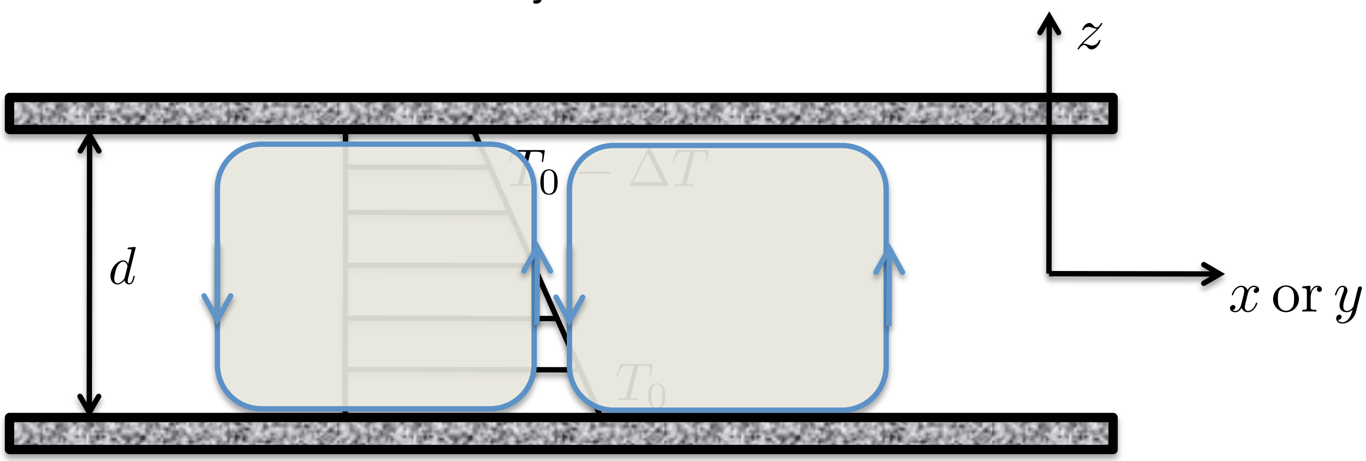
Steady solution with linear temperature profile and zero velocity

Warmer lighter particle below

Cold heavier particles above

Termiskt instabilitet: Benards problem

Triumf av linjär stabilitetsteori

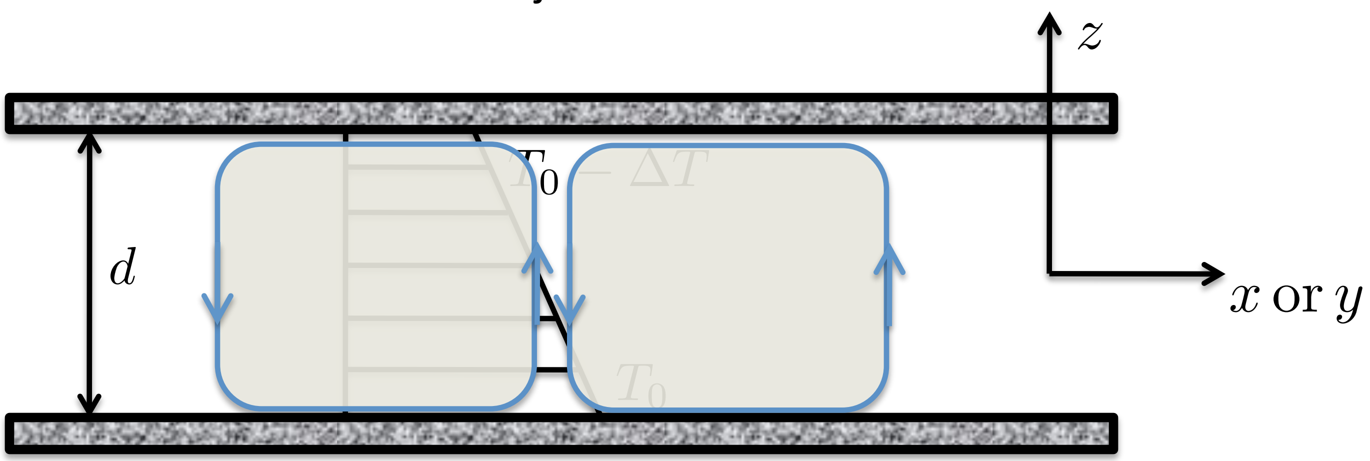


Destabilising force: Buoyancy

Restoring force: viscous forces and thermal diffusion

Termiskt instabilitet: Benards problem

Triumf av linjär stabilitetsteori



Destabilising force: Buoyancy

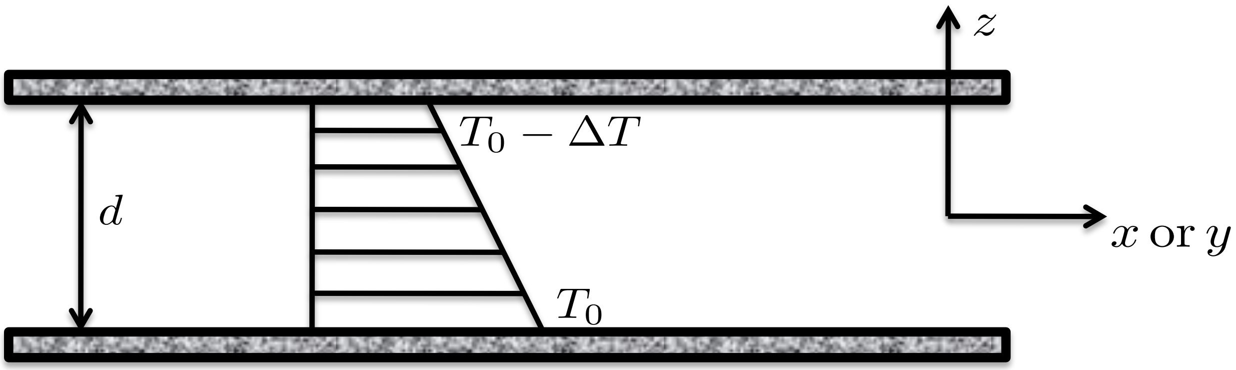
Restoring force: viscous forces and thermal diffusion

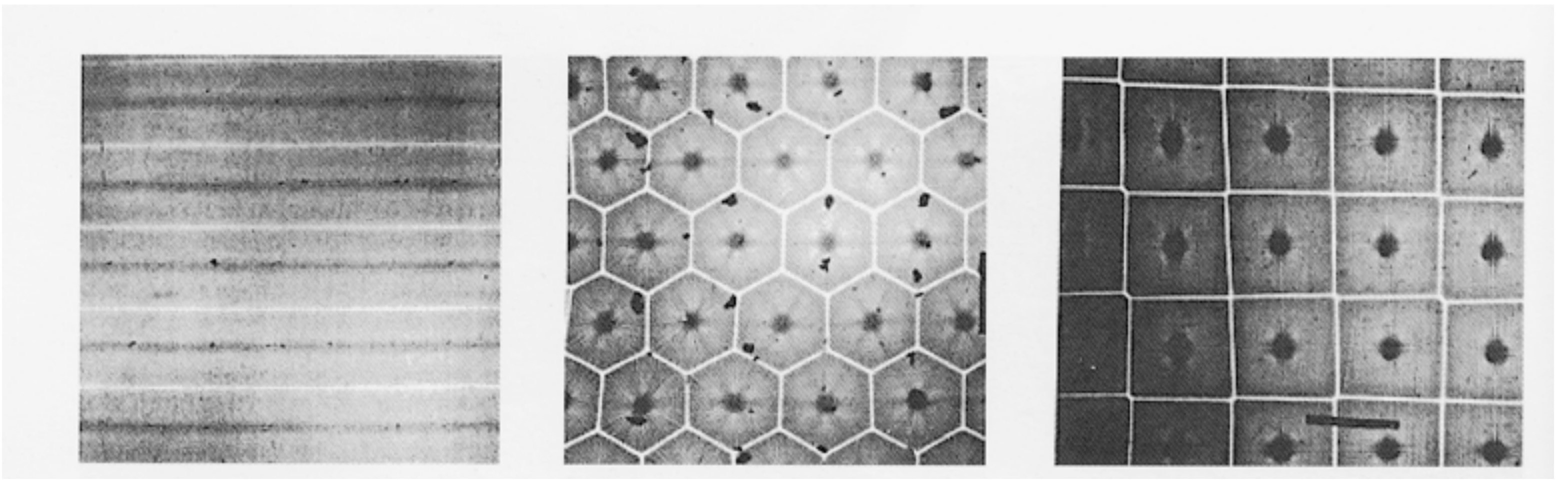
Rayleigh's number

$$Ra = \frac{g\alpha\Gamma d^4}{\kappa\nu}$$

α Thermal expansion coefficient
κ Diffusivity
ν Viscosity
Γ Temperature gradient

Rayleigh-Benard instability





Rayleigh-Benard instability

Rayleigh number: ratio between buoyancy forces (temperature gradient) and viscous forces the governing parameter

- **Linear stability theory**: above a critical Rayleigh number of 1708 the conductive state becomes unstable to infinitesimal perturbations

- **Energy stability theory**: below a critical Rayleigh number of 1708 finite-amplitude perturbations superimposed on the conductive state decay monotonically in energy

Rayleigh-Benard instability

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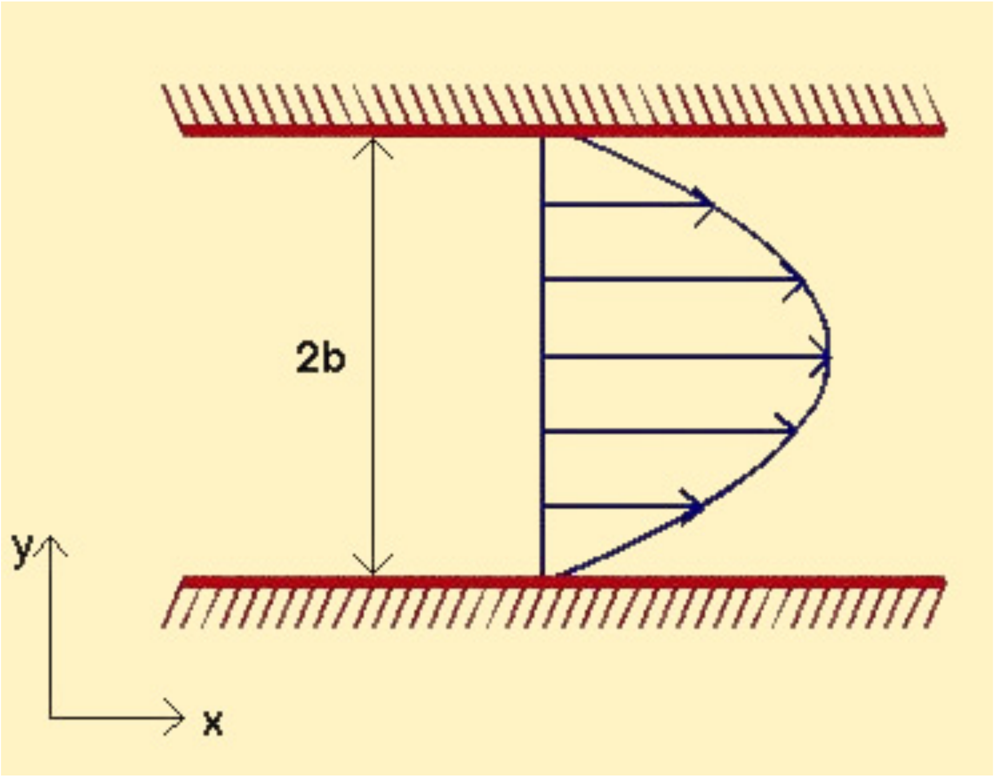
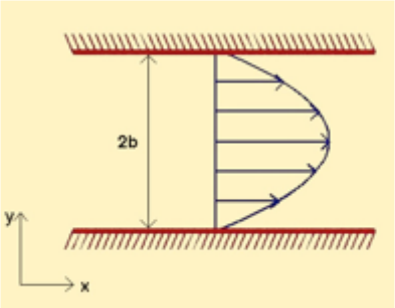
Experiments show the onset of convective instabilities at a critical Rayleigh number of about 1710 !!!

Plane Poiseuille flow

Reynolds number: ratio between inertial forces and viscous forces the governing parameter



Plane Poiseuille flow

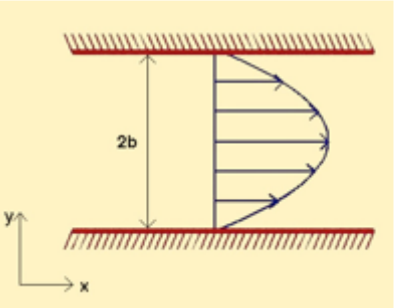


Reynolds number: ratio between inertial forces and viscous forces the governing parameter

- **Linear stability theory**: above a critical Reynolds number of 5772 the parabolic profile becomes unstable to infinitesimal perturbations

- **Energy stability theory**: below a critical Reynolds number of 49.6 finite-amplitude perturbations superimposed on the parabolic profile decay monotonically in energy

Plane Poiseuille flow



Reynolds number: ratio between inertial forces and viscous forces the governing parameter

- **Linear stability theory**: above a critical Reynolds number of 5772 the parabolic profile becomes unstable to infinitesimal perturbations

- **Energy stability theory**: below a critical Reynolds number of 49.6 finite-amplitude perturbations superimposed on the parabolic profile decay monotonically in energy

Experiments show turbulent patches at a critical Reynolds number of about 1000 !!!

Two opposite behaviors

- Linear stability theory, energy stability theory and experiments are in excellent agreement for Rayleigh-Bénard problem
- Linear stability theory, energy stability theory and experiments show significant discrepancies for plane Poiseuille flow

Questions

- Can we explain the success and failure of stability theory for the two examples?
- Is there a better way to investigate the stability of plane Poiseuille flow (and many other wall-bounded shear flows)?

Energy equation

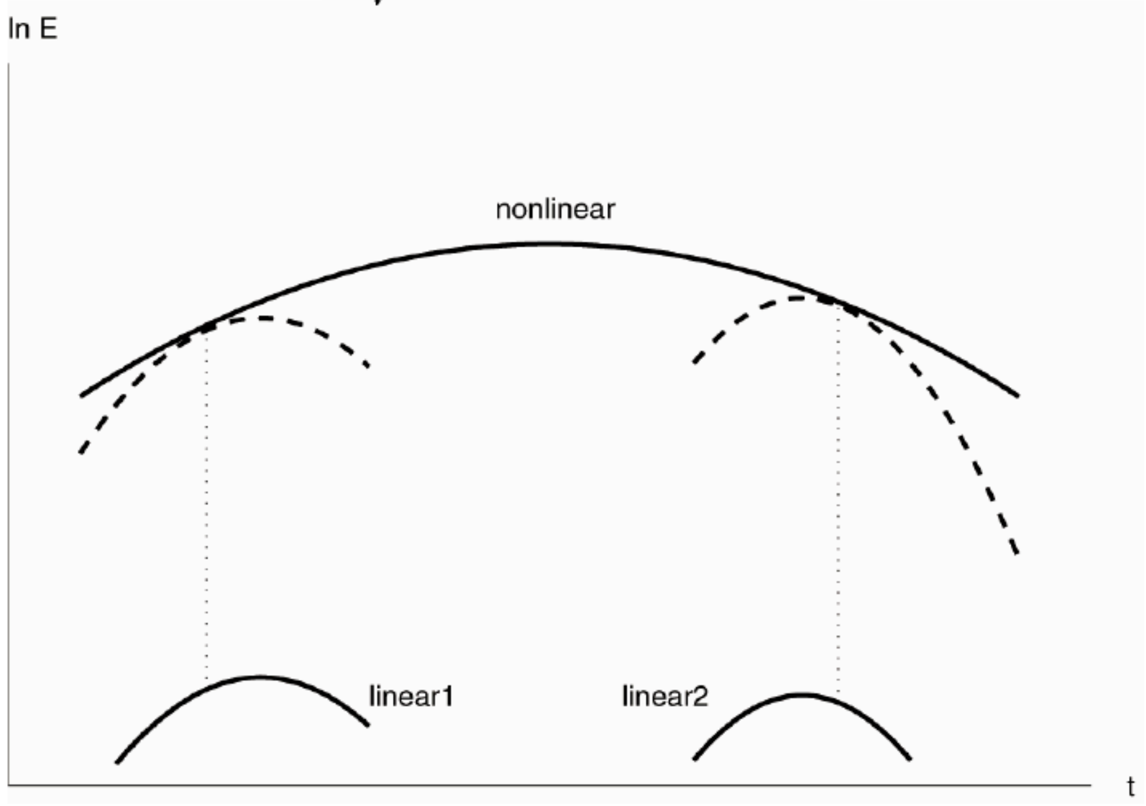
$$
\begin{aligned}
u_i\frac{\partial u_i}{\partial t} &= -u_iu_j\frac{\partial U_i}{\partial x_j} - \frac{1}{\mathrm{Re}}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j}\left[-\frac{1}{2}u_iu_iU_j - \frac{1}{2}u_iu_iu_j - u_ip\delta_{ij} + \frac{1}{\mathrm{Re}}u_i\frac{\partial u_i}{\partial x_j}\right] \\
&\Rightarrow \\
\frac{dE_V}{dt} &= -\int_V u_iu_j\frac{\partial U_i}{\partial x_j}dV - \frac{1}{\mathrm{Re}}\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}dV
\end{aligned}
$$

Theorem: Linear mechanisms required for energy growth

Proof: $\frac{1}{E_V}\frac{dE_V}{dt}$ independent of disturbance amplitude

Linear growth mechanisms

$$\frac{1}{E_V}\frac{dE_V}{dt} = \frac{d}{dt}\ln E_V$$



Subcritical transition

- The nonlinear terms of the Navier-Stokes equations conserve energy
- During transition to turbulence we observe a substantial increase in kinetic perturbation energy, even for Reynolds numbers below the critical one.

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Subcritical transition

- The nonlinear terms of the Navier-Stokes equations conserve energy
- During transition to turbulence we observe a substantial increase in kinetic perturbation energy, even for Reynolds numbers below the critical one.

➢ **The increase in energy for subcritical Reynolds numbers is related to a linear process, without relying on an exponential instability;**

linear instability without an unstable eigenvalue!

Non-modal approach

Linear stability problem

- Start from linearised Navier-Stokes about base flow U written as initial value problem

$$\frac{d}{dt}q = Lq$$

With solution in the form of matrix exponential

$$q = \exp(tL)q_0; \quad q(t=0) = q_0$$

Norm of matrix exponential

$$q = \exp(tL)q_0$$

- Input output analysis:
  maximum possible amplification at time *t* over all initial conditions

$$G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2} = \max_{q_0} \frac{||\exp(tL)q_0||^2}{||q_0||^2}$$

Norm of matrix exponential

$$q = \exp(tL)q_0$$

- Input output analysis:
  maximum possible amplification at time *t* over all initial conditions

$$G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2} = ||\exp(tL)||^2$$

Definition of the norm of a matrix

Matrix norm $\|G\| = \max_{\|w\|=1} \|Gw\|$

- Euclidean scalar product $\|w\|_2^2 = w_j^H w_j$ $\|w\|^2 = \langle w, w \rangle$

- Matrix as transformation with associated amplification

$$\frac{\|Gw\|_2}{\|w\|_2} = \left[\frac{w^H G^H G w}{w^H w}\right]^{1/2}$$

- Eigenvalues of $G^H G$ or singular values of G: $G^H G v_i = \lambda_i v_i$

- Largest amplification for the largest singular value: $\lambda_1, \; v_1$

$$\sqrt{\lambda_1} = \max_{\|w\|_2=1} \|Gw\|_2 = \frac{\|Gv_1\|_2}{\|v_1\|_2}, \quad Gv_1 = u_1$$

- Input and output basis $v_i, \; u_i$: $G = U \Lambda V^H$

Eigenvalues vs Propagator Norm

$$q = \exp(tL)q_0$$

- Matrix exponential difficult to compute
  System eigenvalues used

$$L = S\Lambda S^{-1}$$

Eigenvalue decomposition

S: Column eigenvector

Λ: Diagonal eigenvalues

$$||\exp(tL)||^2 = ||\exp(tS\Lambda S^{-1})||^2 = ||S\exp(t\Lambda)S^{-1}||^2$$

Traditional stability analysis:
Behavior deduced by system eigenvalues

Eigenvalues vs Propagator Norm

- Upper and lower bounds of $G(t)$

Lower bound $$e^{2t\lambda_{max}} \leq ||\exp(tL)||^2$$

The energy cannot decay at a faster rate than that given by the least stable eigenvalue λ_{max}

Upper bound $$||\exp(tL)||^2 = ||\exp(tS\Lambda S^{-1})||^2$$
$$\leq ||S||^2||S^{-1}||^2 e^{2t\lambda_{max}}$$

Bounds of the matrix exponential

$$e^{2t\lambda_{max}} \leq ||\exp(tL)||^2 \leq ||S||^2||S^{-1}||^2 e^{2t\lambda_{max}}$$

Condition number:

$$\kappa(S) = ||S||^2||S^{-1}||^2$$

Two distinct cases:

$$\kappa(S) = 1$$

upper and lower bound coincide:
the energy amplification is governed by the least stable eigenvalue

Bounds of the matrix exponential

$$e^{2t\lambda_{max}} \leq ||\exp(tL)||^2 \leq ||S||^2||S^{-1}||^2 e^{2t\lambda_{max}}$$

Condition number:

$$\kappa(S) = ||S||^2||S^{-1}||^2$$

Two distinct cases:

$$\kappa(S) \gg 1$$

upper and lower bound can differ significantly:
the energy amplification is governed by the least stable eigenvalue only at large times

Non-normality

$$\kappa(S) = ||S||^2||S^{-1}||^2 = 1$$

Normal stability problem:

orthogonal eigenvectors

Eigenvalues capture the dynamics

$$\kappa(S) = ||S||^2||S^{-1}||^2 \gg 1$$

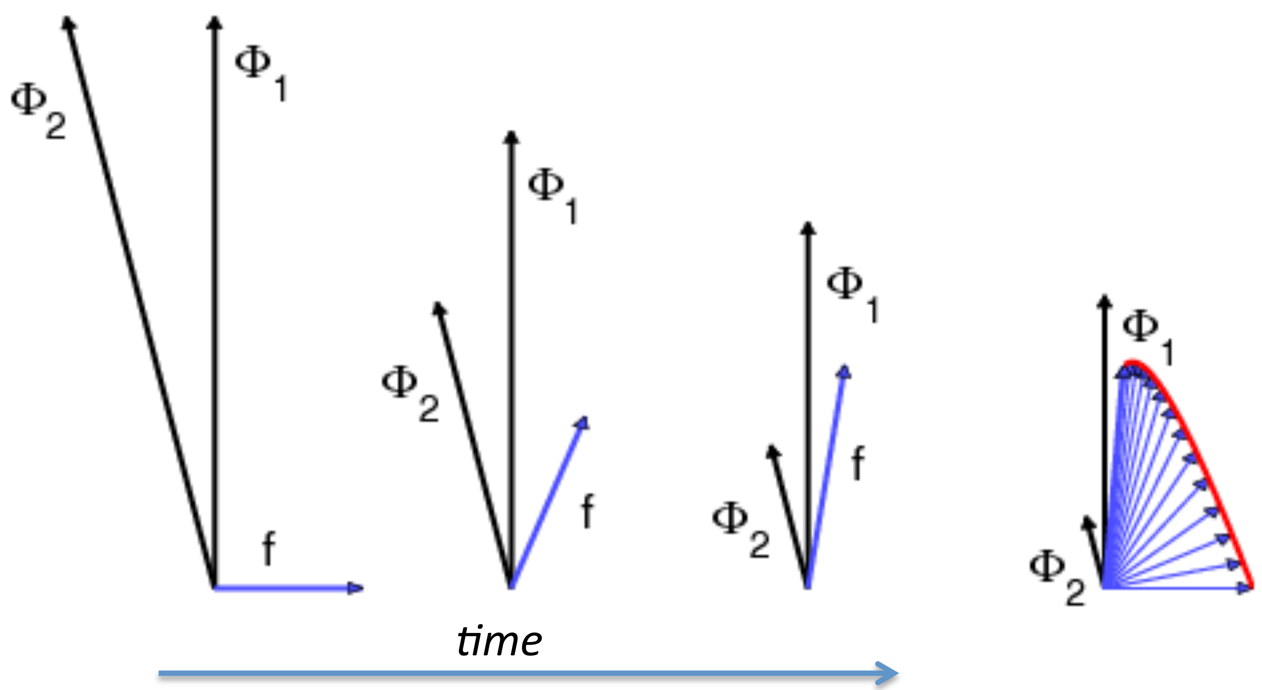
Non-Normal stability problem:

non-orthogonal eigenvectors

Eigenvalues capture the asymptotic dynamics, not the transient behavior

Non-modal transient growth

- The non-normality of the system can give rise to transient energy amplification



Although we observe exponential decay for large times, the non-orthogonal superposition of eigenvectors can lead to short-time growth of energy.

Short time dynamics

- Taylor expansion of matrix exponential at t=0

$$
\begin{aligned}
E(t) = ||S||^2 = \langle q, q \rangle = \\
\langle \exp(tL) q_0, \exp(tL) q_0 \rangle \\
\approx \langle (I + tL) q_0, (I + tL) q_0 \rangle \\
\approx \langle q_0, q_0 \rangle + t \langle q_0, (L + L^H) q_0 \rangle
\end{aligned}
$$

L^H Adjoint matrix defined by the norm used (Energy)

Short time dynamics

- Initial energy growth rate

$$\frac{1}{E}\frac{dE}{dt}|_{0^+} = \frac{\langle q_0, (L + L^H) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$$\frac{1}{E}\frac{dE}{dt}|_{0^+} = \lambda_{max}(L + L^H)$$

$(L + L^H)$ **Hermitian matrix: numerical abscissa of L**

The numerical range

- Generalization of the numerical abscissa

$$\frac{d}{dt}||q||^2 = \langle \frac{d}{dt}q, q\rangle + \langle q, \frac{d}{dt}q\rangle =$$
$$\langle Lq, q\rangle + \langle q, Lq\rangle =$$
$$2\Re\{\langle Lq, q\rangle\}$$

Definition of numerical range:
Rayleigh quotient of L

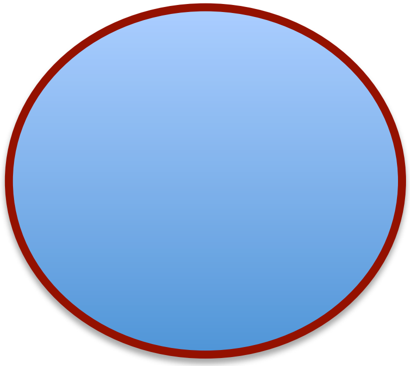
$$\mathcal{F}(L) = \left\{ z | z = \frac{\langle Lq, q\rangle}{\langle q, q\rangle} \right\}$$

The numerical range

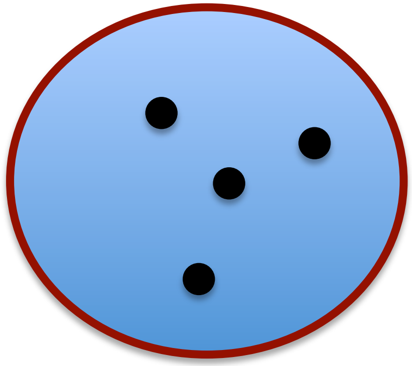
Properties of the numerical range

$$\mathcal{F}(L) = \left\{ z \middle| z = \frac{\langle Lq, q \rangle}{\langle q, q \rangle} \right\}$$

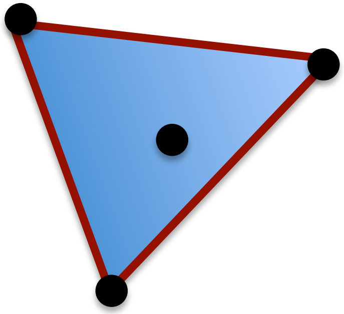
1. The numerical range is convex
2. The numerical range contains the spectrum of L
3. If L is normal, the numerical range is the convex hull of the spectrum



convex



Non-normal system

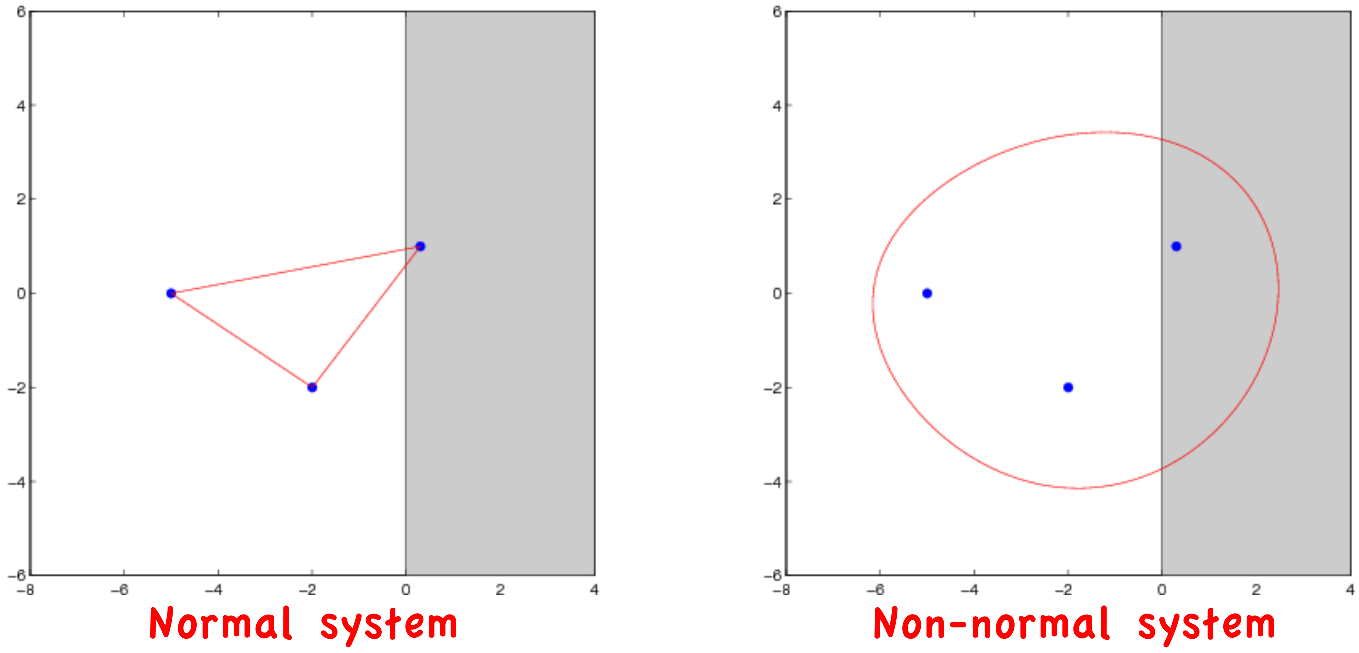


Normal system

The numerical range

- It can substantially larger than the convex hull of the spectrum: positive energy growth even if stable eigenvalues!

$$\frac{d}{dt}||q||^2 = 2\Re\{\langle Lq, q\rangle\}$$

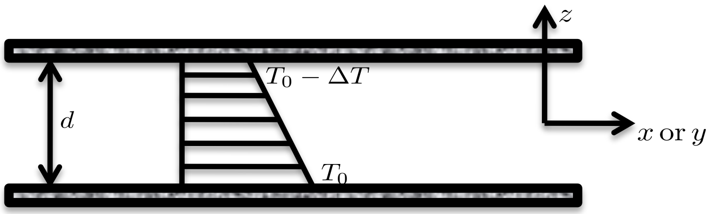


Non-normal stability problems

The numerical abscissa (numerical range) governs the short-time behavior. The sign determines the initial energy growth or decay

The least stable eigenvalue governs the long-time behavior. The sign of the real part of λ_{max} determines the initial energy growth or decay

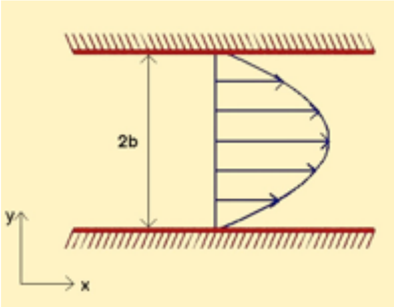
Rayleigh-Benard convection



- Normal stability problem
- The numerical range is the convex hull of the spectrum
- The numerical range and the spectrum cross into the unstable half-plane at the same Rayleigh number
- Energy growth and instability occur at the Rayleigh number
- The spectrum governs the flow behavior at all times

$$Ra_{lin} = Ra_{en} = 1708$$

Plane Poiseuille flow



- Non-Normal stability problem
- The numerical range is larger than the convex hull of the spectrum
- The numerical range crosses into the unstable half-plane before the spectrum
- Initial Energy growth occur before asymptotic instability
- The spectrum governs the flow behavior only at long times

$$Re_{en} = 49.6 \ll Re_{lin} = 5772$$

Non-modal analysis

$$\frac{1}{E}\frac{dE}{dt}|_{0^+} = \lambda_{max}(L + L^H)$$

Short-time: numerical abscissa

$$||\exp(tL)||^2$$

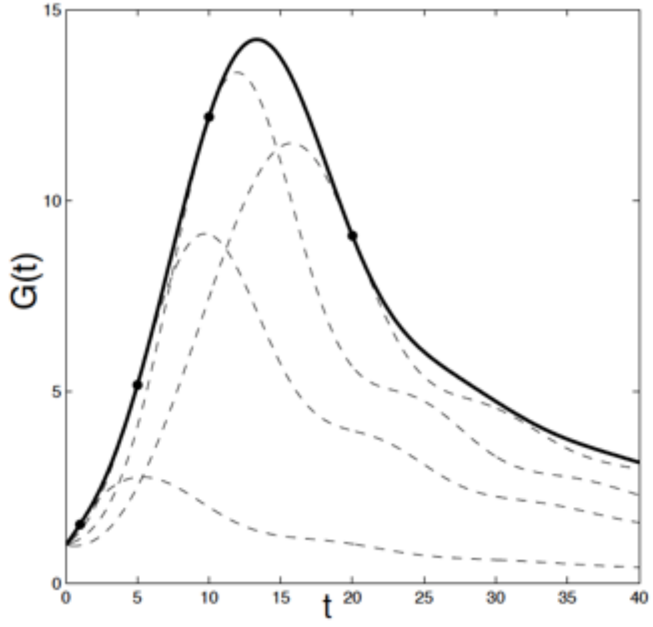
Any time: matrix exponential

$$G(t \to \infty) = \lim_{t\to\infty} ||\exp(tL)||^2 = e^{t\lambda_{max}}$$

Long time: eigenvalues

Results for Poiseuille flow

- *G(t)* envelope over many individual growth curves
- For each point, a specific initial condition reaches its maximum energy amplification at this point in time



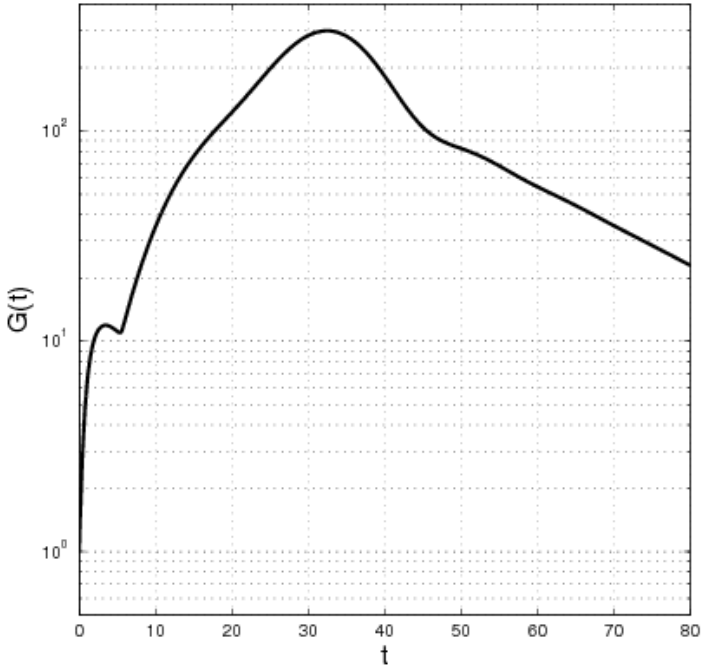
$$Re = 1000, \alpha = 1$$

Solid line: envelope G(t)

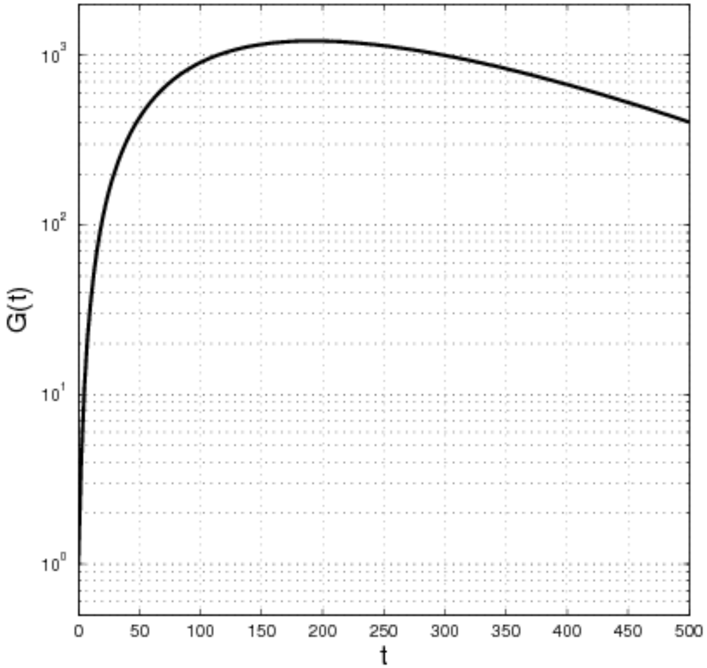
Dashed lines: evolution of selected initial conditions

Results for Poiseuille flow

- *G(t)* envelope over many individual growth curves
- Potential for strong amplification of spanwise periodic disturbances



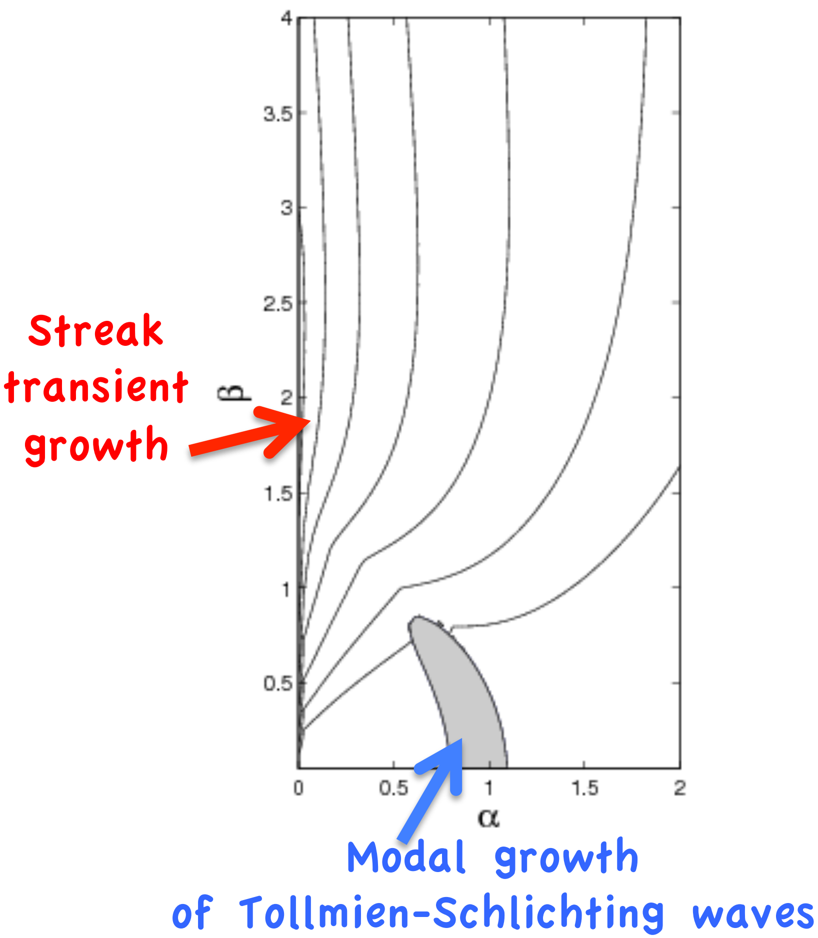
$$Re = 2500, \alpha = 1, \beta = 1$$



$$Re = 2500, \alpha = 0, \beta = 2$$

Bypass transition

- Supercritical Poiseuille flow, $Re = 10000$

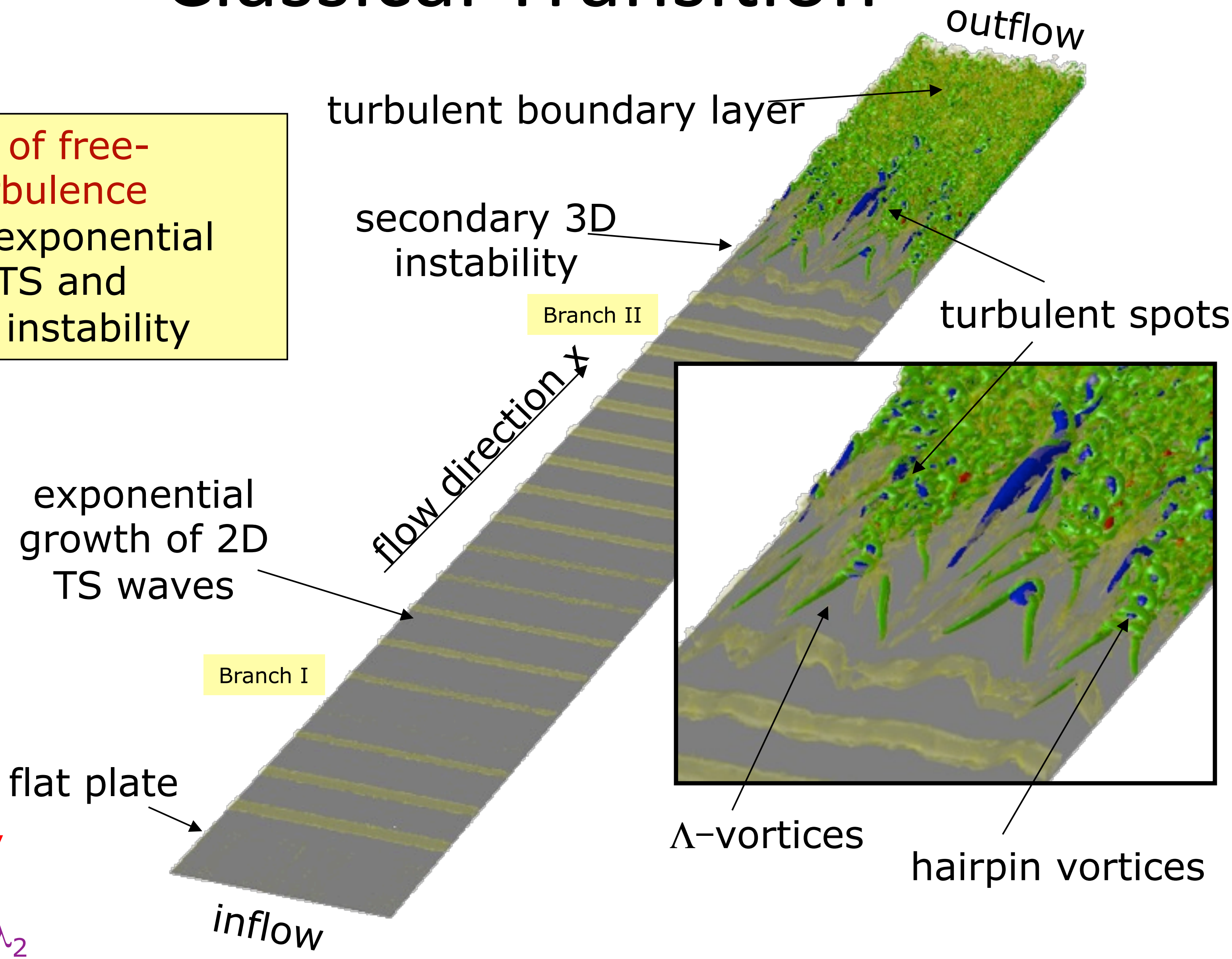


Over **short time** horizon, streamwise independent ***streaks*** are favored

Over **long time** horizon, waves with weak or no streamwise dependence are favored

Classical Transition

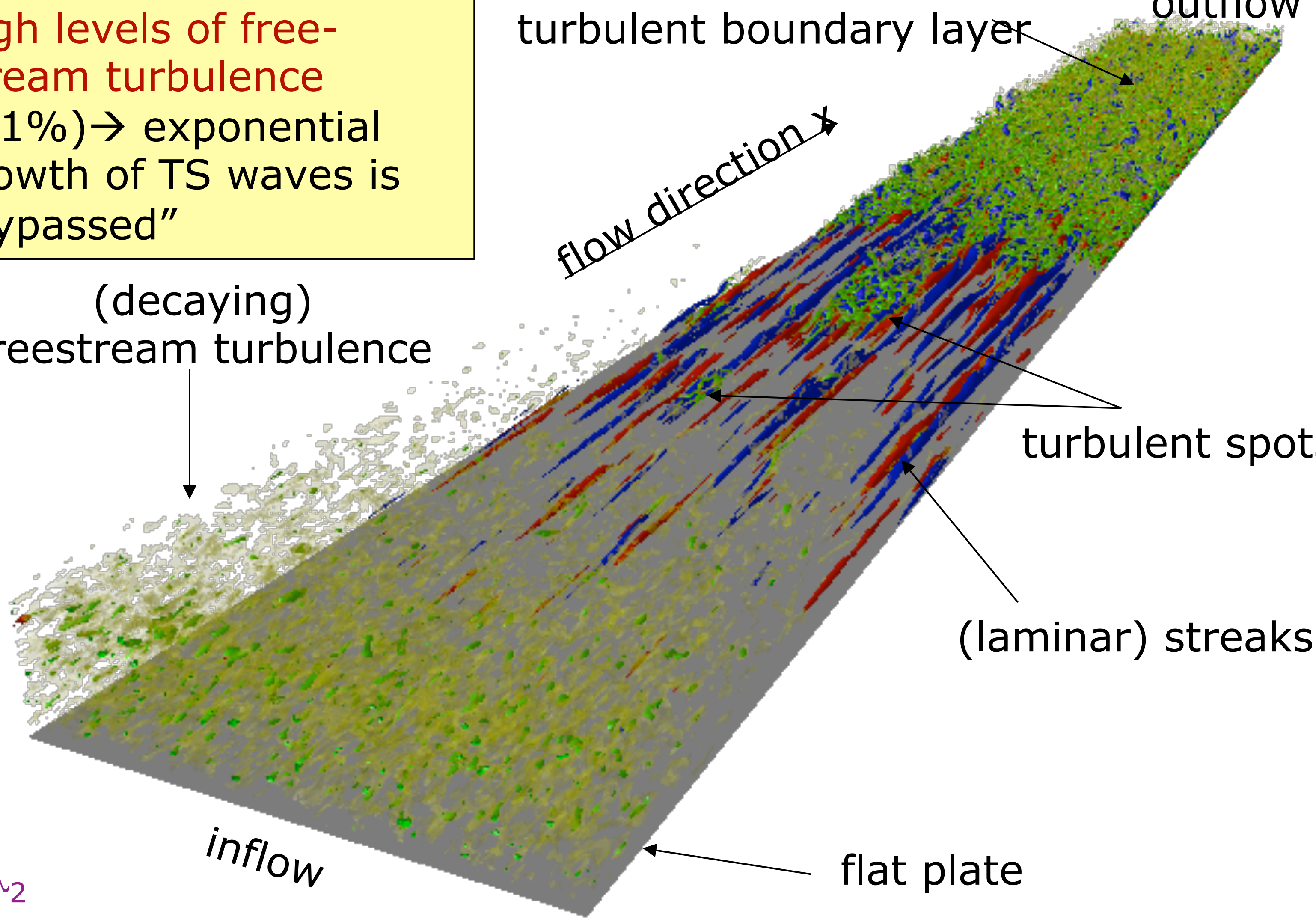
Low levels of free-stream turbulence (<1%) → exponential growth of TS and secondary instability



high velocity
low velocity
contours of λ_2

Bypass Transition

High levels of free-stream turbulence (>1%)→ exponential growth of TS waves is "bypassed"



high velocity
low velocity
contours of λ_2

Optimal initial condition

Initial condition that results in the maximum energy amplification at a given time

$$\bar{q}(t^*) = \exp(t^* L) q_0$$

$$||q(t^*)||^2 = ||q_0||^2 = 1$$

$$\exp(t^* L)\ q_0 = \ ||\exp(t^* L)||\ q(t^*)$$

propagator input amplification output

Optimal initial condition

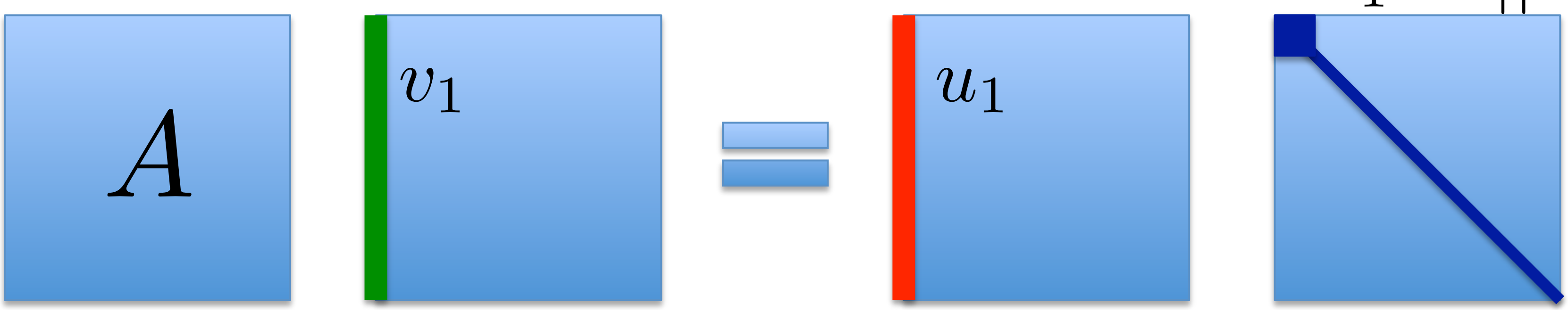
$$\exp(t^* L)\ q_0 = \ ||\exp(t^* L)||\ q(t^*)$$

propagator input amplification output

Singular value decomposition of a matrix

$$A = U \Sigma V^H \qquad AV = U\Sigma$$

$$\sigma_1 = ||A||^2$$



Optimal initial condition

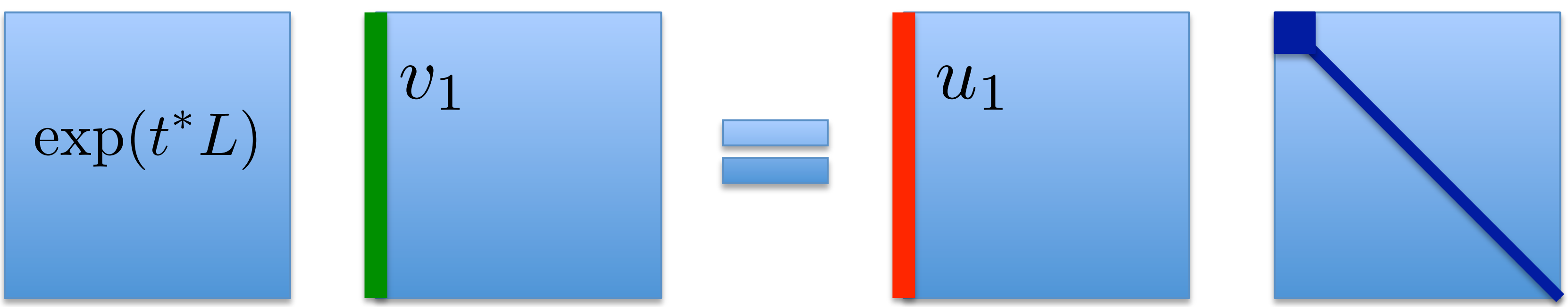
$$\exp(t^* L)\ q_0 = \ ||\exp(t^* L)||\ q(t^*)$$

propagator input amplification output

Singular value decomposition of a matrix

$$\mathrm{svd}(\exp(t^* L)) = U \Sigma V^H$$

$G(t^*) = ||\exp(t^* L)||$



Optimal initial condition

$$\exp(t^* L)\ q_0 = \ ||\exp(t^* L)||\ q(t^*)$$

propagator input amplification output

Singular value decomposition of a matrix

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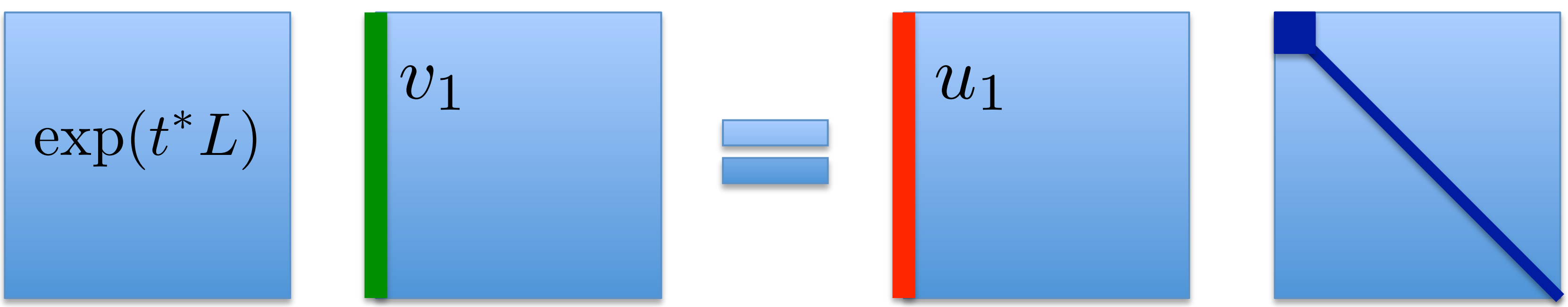


Optimal initial condition
left principal
singular vector

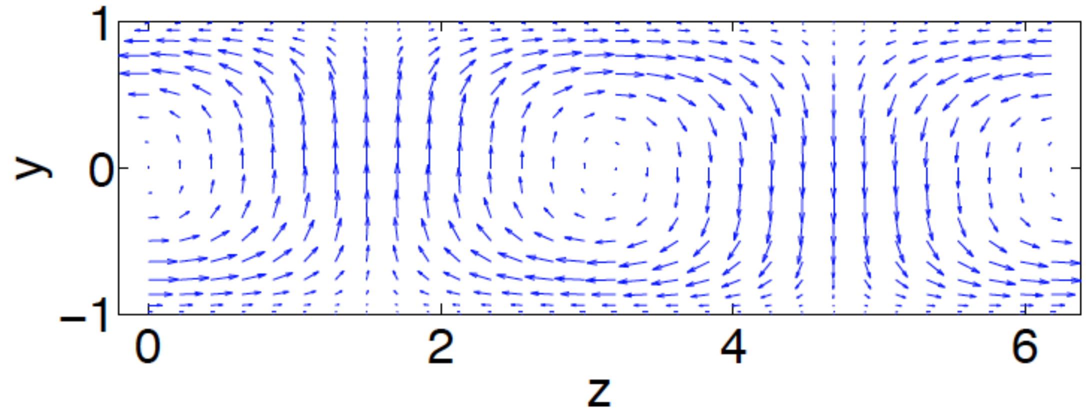
Optimal final condition
right principal
singular vector

Optimal disturbance: Poiseuille flow

$Re = 5000; \alpha = 0, \beta = 2$

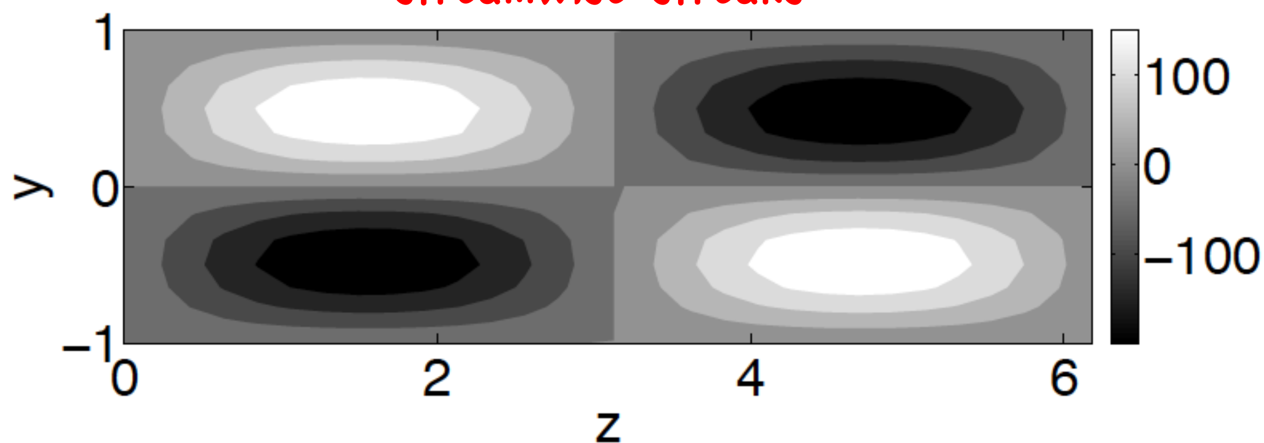
Optimal initial condition:
Counter-rotating streamwise vortices

Cross-stream perturbation velocity



Optimal final condition:
streamwise streaks

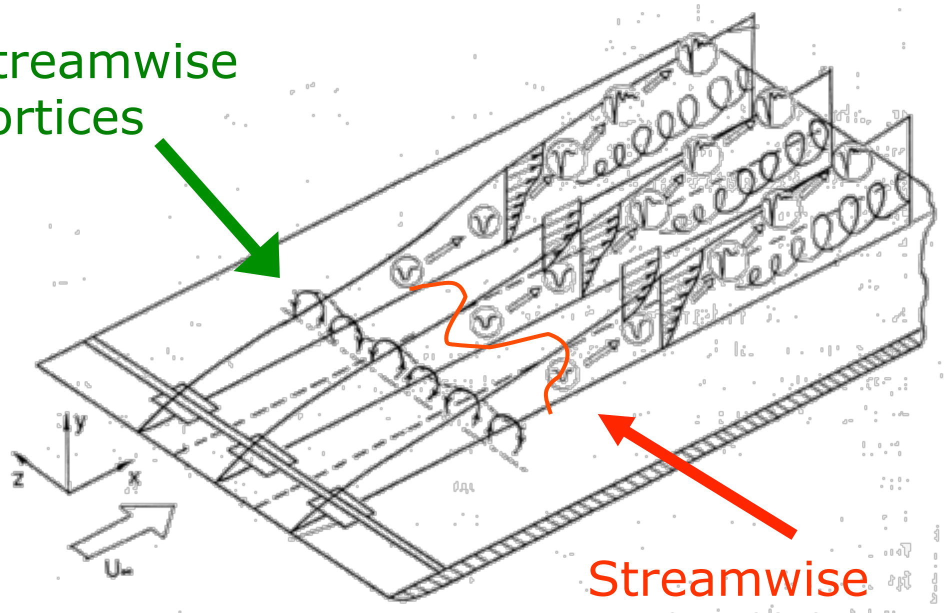
Streamwise perturbation velocity



Non-Modal Growth

Lift-up **mechanism in shear layers**
(Ellingsen & Palm 1975, Landahl 1980)

Streamwise vortices



Streamwise streaks

Particle displaced in the wall-normal direction retain their horizontal momentum

⬇

Streamwise vortices induce streamwise streaks

In boundary layers: wall-normal shear is *O(Re)*

⇨ streak growth *O(Re)*

Outline

- Stability of fluid systems
  - modal limit
  - short time dynamics, matrix exponential
- Receptivity
  - resolvent norm
  - Resonance limit
  - Adjoint modes
- Sensitivity
  - Structural sensitivity
  - Base-flow sensitivity

Receptivity

- Interested in the response of a fluid system to external excitations (free-stream turbulence, roughness, acoustic waves…)

$$\frac{d}{dt}q = Lq + f$$

External forces

General solution for zero initial condition: convolution integral

$$q_p = \int_0^t \exp((\tau - t)L) f(\tau) d\tau$$

Optimal response

- Linear problem: consider the case of harmonic forcing $f = \hat{f} e^{i\omega t}$

Regime solution given by the resolvent $\hat{q}_p = (i\omega - L)^{-1} \hat{f}$

Written as an input-output problem: optimal response

$$\mathcal{R}(\omega) = \max_{\hat{f}} \frac{||\hat{q}_p||}{||\hat{f}||} = \max_{\hat{f}} \frac{||(i\omega - L)^{-1} \hat{f}||}{||\hat{f}||} = ||(i\omega - L)^{-1}||$$

Bounds of resolvent norm

- Diagonalize the system matrix *L*

$$L = S\Lambda S^{-1}$$

Eigenvalue decomposition

S: Column eigenvector

Λ: Diagonal eigenvalues

$$\frac{1}{dist\{i\omega, \Lambda\}} \leq ||(i\omega - L)^{-1}|| = ||S(i\omega - \Lambda)^{-1}S^{-1}|| \leq \kappa(S)\frac{1}{dist\{i\omega, \Lambda\}}$$

Bounds of resolvent norm

- Diagonalize the system matrix *L*

$$L = S\Lambda S^{-1}$$

Eigenvalue decomposition

S: Column eigenvector

Λ: Diagonal eigenvalues

$$\frac{1}{dist\{i\omega, \Lambda\}} \leq ||(i\omega - L)^{-1}|| = ||S(i\omega - \Lambda)^{-1} S^{-1}|| \leq \kappa(S) \frac{1}{dist\{i\omega, \Lambda\}}$$

$$\kappa(S) = 1$$

Normal system: upper and lower bound coincide:
the classical resonance conditions holds
minimum distance from the spectrum

Bounds of resolvent norm

- Diagonalize the system matrix *L*

$$L = S\Lambda S^{-1}$$

Eigenvalue decomposition

S: Column eigenvector

Λ: Diagonal eigenvalues

$$\frac{1}{dist\{i\omega, \Lambda\}} \leq ||(i\omega - L)^{-1}|| = ||S(i\omega - \Lambda)^{-1}S^{-1}|| \leq \kappa(S)\frac{1}{dist\{i\omega, \Lambda\}}$$

$$\kappa(S) \gg 1$$

Non-Normal system: upper and lower bound differ

we can have a pseudo-resonance

Strong amplification also far from system eigenfrequency

Optimal forcing

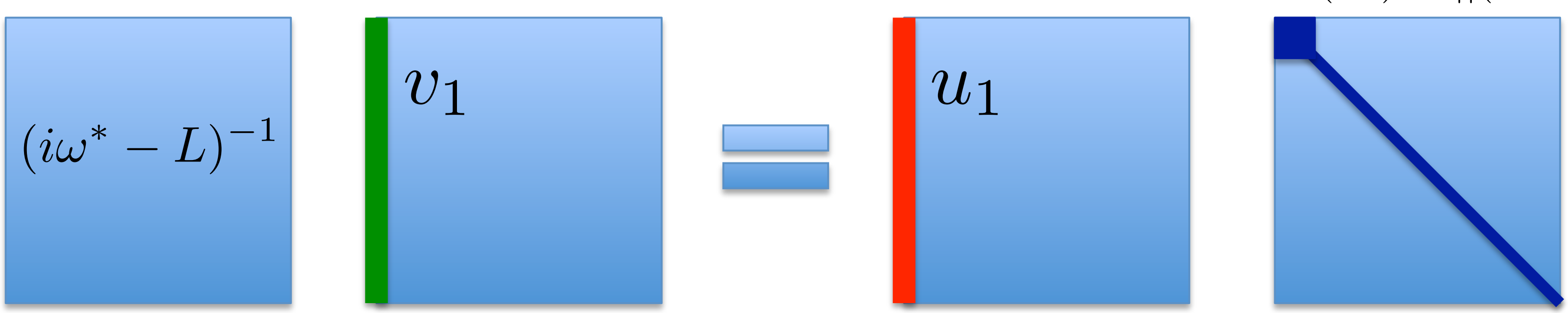
- Singular value decomposition svd of the resolvent norm

$$(i\omega^* - L)^{-1}\bar{f} = ||(i\omega^* - L)^{-1}||\,\bar{q}_p$$

Transfer function Input Amplification Output

$$||\bar{q}_p|| = ||\bar{f}|| = 1$$

$$\mathcal{R}(\omega^*) = ||(i\omega^* - L)^{-1}||$$



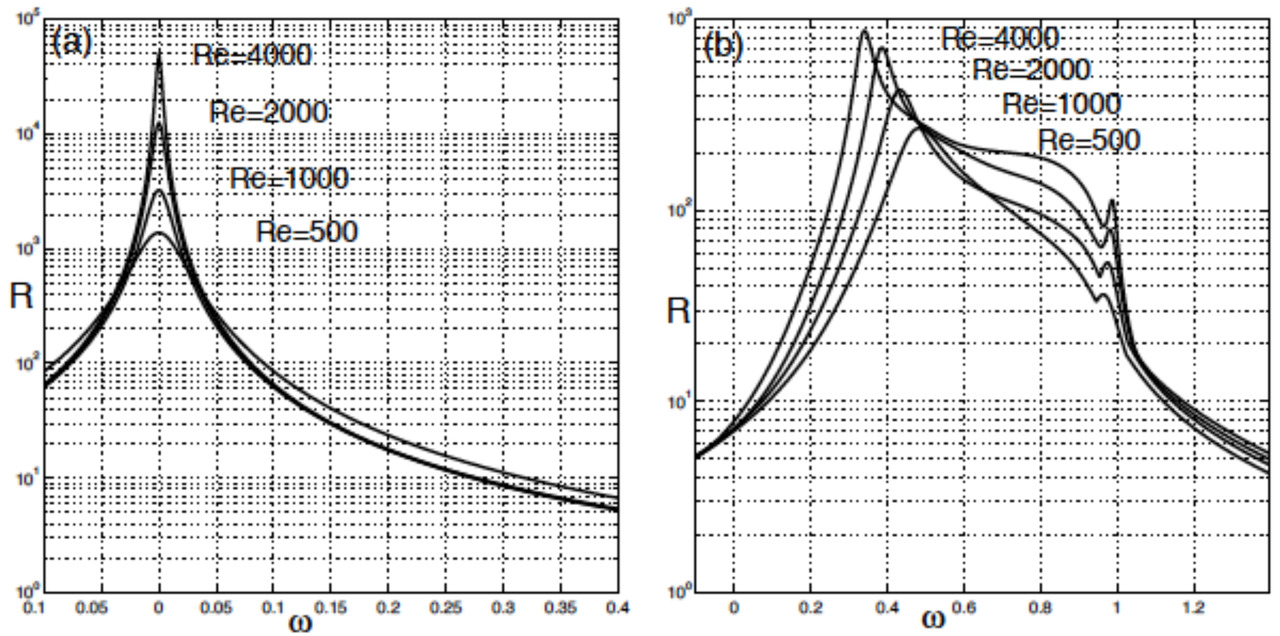
Optimal harmonic forcing
left principal
singular vector

Optimal harmonic response
right principal
singular vector

Results for Poiseuille flow

Largest possible amplification versus forcing frequency

$\mathcal{R}(\omega)$

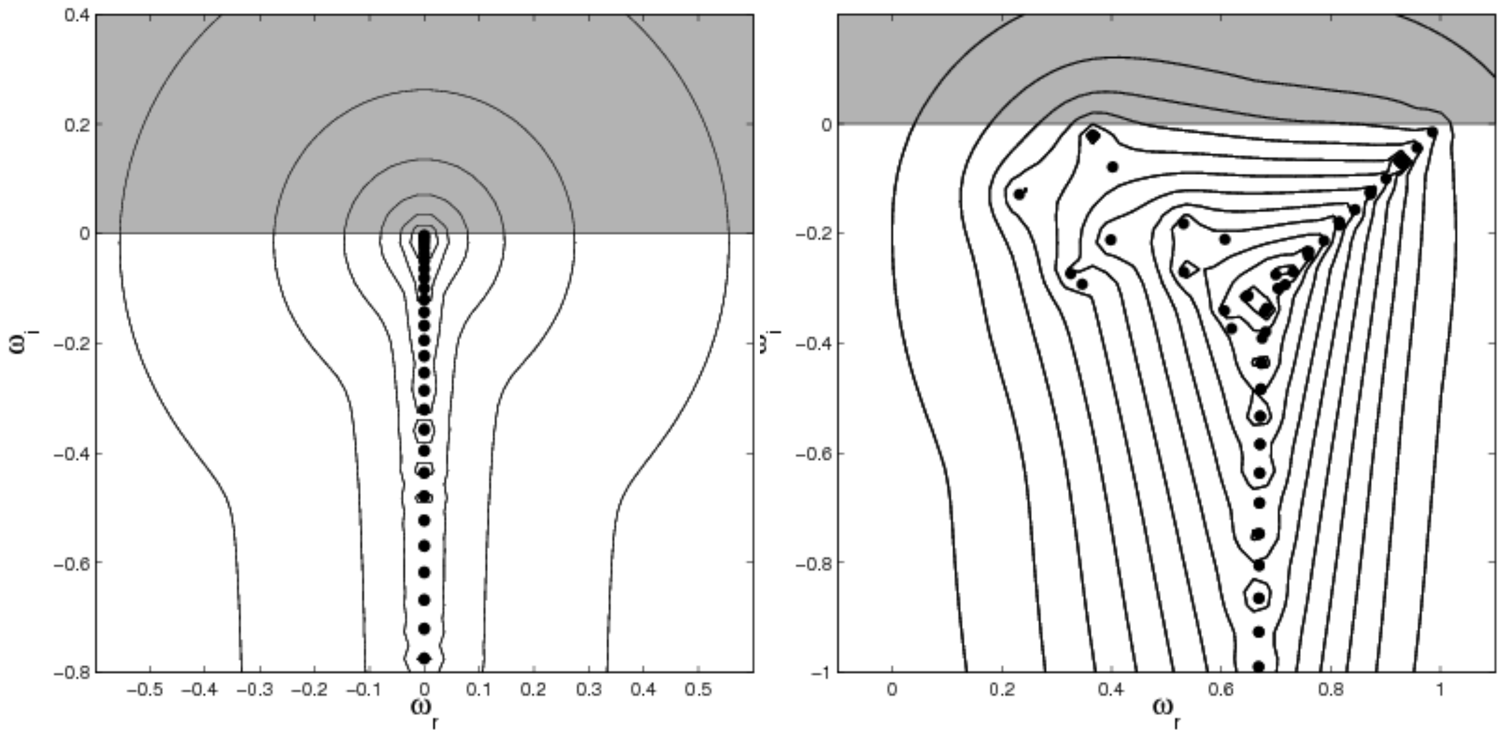


$\alpha = 0, \beta = 2$ $\qquad$ $\alpha = 1, \beta = 1$

Results for Poiseuille flow

Resolvent norm in the complex plane

$\mathcal{R}(\omega)$



$Re = 2500, \alpha = 0, \beta = 2$ $Re = 2500, \alpha = 1, \beta = 1$

Component-wise energy transfer

- Define transfer function with input/output matrices

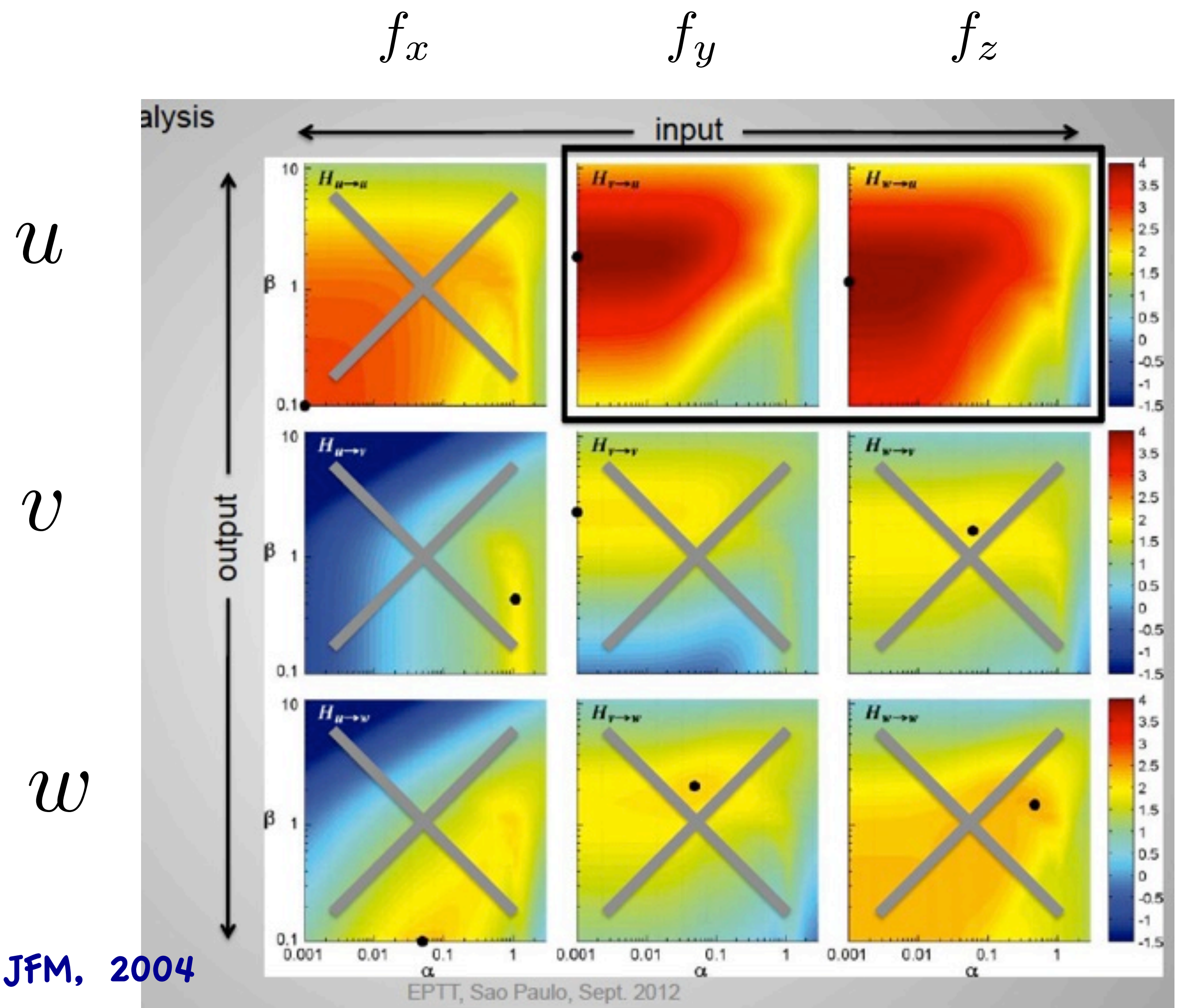
$$\hat{q}_p = (i\omega - L)^{-1}\hat{f}$$

$$\mathcal{H}(\omega) = C(i\omega - L)^{-1}B$$

Take the worst case amplification over all frequencies

$$||\mathcal{H}||_\infty = \max_{-\infty<\omega<\infty} \sigma_{max}(\mathcal{H})$$

Component-wise input-output analysis



Jovanovic & Bamieh, JFM, 2004

Adjoint system

Inner product $$\langle p, q \rangle = q^H p$$

Adjoint system $$\langle p, Lq \rangle = q^H L p = (L^H q)^H p = \langle L^H p, q \rangle$$

Eigenmodes/values $$\lambda_i q_i = L q_i; \quad \lambda_j^* q_j^+ = L^H q_j^+$$

Bi-orthogonality $$\langle q_m^+, q_n \rangle = \delta_{mn}$$

Adjoint system

Eigenmodes/values

$$\lambda_i q_i = L q_i; \quad \lambda_j^* q_j^+ = L^H q_j^+$$

Bi-orthogonality

$$\langle (L^H - \lambda_m^*) q_m^+, \quad q_n \rangle = 0$$

$$\langle q_m^+, (L - \lambda_m) q_n \rangle = \langle q_m^+, (L - \lambda_m - L + \lambda_n) q_n \rangle = 0$$

$$(\lambda_n - \lambda_m) \langle q_m^+, q_n \rangle = 0$$

$$\langle q_m^+, q_n \rangle = \delta_{mn}$$

Expansion using eigenmodes

Bi-orthogonality $$\langle q_m, q_n \rangle = \delta_{mn}$$

Expand solution $$q(t) = \sum_i \kappa_i \, q_i \, e^{\lambda_m t}$$

Given initial condition q_0

compute coefficients κ_n
using bi-orthogonality

$$\langle q_n^+, q_0 \rangle = \sum_i \kappa_m \langle q_n^+, q_m \rangle = \kappa_n$$

Use of adjoint field for receptivity

Linearized Navier-Stokes

(u, p)

$$\frac{\partial u}{\partial t} + L(U, Re)u + \nabla p = 0$$
$$\nabla \cdot u = 0$$

Differentiable fields

(f^+, m^+)

Sum and multiply

$$\left[\left(\frac{\partial u}{\partial t} + L(U, Re)u + \nabla p\right) \cdot f^+ + (\nabla \cdot u)m^+\right]$$

Integrate by parts over time and space

Use of adjoint field for receptivity

Integrate by parts over time and space

$$\int_0^t \int_{\mathcal{D}} \left[\left(\frac{\partial u}{\partial t} + L(U,Re)u + \nabla p \right) \cdot f^+ + (\nabla \cdot u) m^+ \right]$$

$$= - \int_0^t \int_{\mathcal{D}} \left[u \cdot \left(\frac{\partial f^+}{\partial t} + L^+(U,Re) f^+ + \nabla m^+ \right) + p(\nabla \cdot f^+) \right] + \int_0^t \frac{\partial u \cdot f^+}{\partial t} + \int_{\mathcal{D}} \nabla \cdot J$$

where

$$L(U,Re)u = U \cdot \nabla u + u \cdot \nabla U - \frac{1}{Re} \nabla^2 u$$

$$L^+(U,Re) f^+ = U \cdot \nabla f^+ - \nabla U \cdot f^+ + \frac{1}{Re} \nabla^2 f^+$$

$$J = U(u \cdot f^+) + \frac{1}{Re}(\nabla f^+ \cdot u - \nabla u \cdot f^+) + m^+ u + p f^+$$

Use of adjoint field for receptivity

Integrate by parts over time and space

$$\int_0^t \int_{\mathcal{D}} \left[\left(\frac{\partial u}{\partial t} + L(U, Re)u + \nabla p \right) \cdot f^+ + (\nabla \cdot u) m^+ \right]$$

$$= - \int_0^t \int_{\mathcal{D}} \left[u \cdot \left(\frac{\partial f^+}{\partial t} + L^+(U, Re) f^+ + \nabla m^+ \right) + p(\nabla \cdot f^+) \right] + \int_0^t \frac{\partial u \cdot f^+}{\partial t} + \int_{\mathcal{D}} \nabla \cdot J$$

Definition of adjoint problem

Assume volume forcing, mass source and integrate in time

$$\frac{\partial u}{\partial t} + L(U, Re)u + \nabla p = f \qquad \nabla \cdot u = Q$$

$$u(t) \cdot f^+(t) - u(0) \cdot f^+(0) = \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Q m^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Use of adjoint field for receptivity

$$u(t) \cdot f^+(t) - u(0) \cdot f^+(0) = \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Assume initial condition for adjoint system $\quad f^+(t) = u(t)$

$$u(t) \cdot u(t) = u(0) \cdot f^+(0) + \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Use of adjoint field for receptivity

$$u(t) \cdot f^+(t) - u(0) \cdot f^+(0) = \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Assume initial condition for adjoint system $f^+(t) = u(t)$

$$u(t) \cdot u(t) = u(0) \cdot f^+(0) + \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Adjoint velocity gives sensitivity to initial condition and forcing

$$\frac{\delta u^2(t)}{\delta u(0)} = f^+(0) \qquad \frac{\delta u^2(t)}{\delta f} = f^+$$

Use of adjoint field for receptivity

$$u(t) \cdot f^+(t) - u(0) \cdot f^+(0) = \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Assume initial condition for adjoint system $\quad f^+(t) = u(t)$

$$u(t) \cdot u(t) = u(0) \cdot f^+(0) + \int_0^t \int_{\mathcal{D}} (f \cdot f^+ + Qm^+) + \int_{\Gamma_{\mathcal{D}}} J \cdot n$$

Adjoint pressure gives sensitivity to mass source

$$\frac{\delta u^2(t)}{\delta Q} = m^+$$

Use of adjoint field for receptivity

$$u(t)\cdot f^{+}(t) - u(0)\cdot f^{+}(0) = \int_0^t \int_{\mathcal{D}} (f\cdot f^{+} + Qm^{+}) + \int_{\Gamma_{\mathcal{D}}} J\cdot n$$

Assume initial condition for adjoint system $f^{+}(t) = u(t)$

$$u(t)\cdot u(t) = u(0)\cdot f^{+}(0) + \int_0^t \int_{\mathcal{D}} (f\cdot f^{+} + Qm^{+}) + \int_{\Gamma_{\mathcal{D}}} J\cdot n$$

Gradient of adjoint field gives sensitivity to boundary conditions

$$\frac{\delta u^2(t)}{\delta u_{wall}} = \frac{1}{Re}\nabla f^{+} + m^{+}\cdot n$$

Outline

- Stability of fluid systems
  - modal limit
  - short time dynamics, matrix exponential
- Receptivity
  - resolvent norm
  - Resonance limit
  - Adjoint modes
- Sensitivity
  - Structural sensitivity
  - Base-flow sensitivity

Structural sensitivity

- Sensitivity to internal changes

$$A(p)q = \lambda Bq$$

p Governing parameter: Reynolds number, base flow, wavenumber

Perturbation expansion

$$(A + \delta A)(q + \delta q) = (\lambda + \delta\lambda)B(q + \delta q)$$

Structural sensitivity

- Sensitivity to internal changes

$$A(p)q = \lambda B q$$

p Governing parameter: Reynolds number, base flow, wavenumber

Perturbation expansion

$$(A + \delta A)(q + \delta q) = (\lambda + \delta\lambda)B(q + \delta q)$$

$$\cancel{(A - \lambda B)q} + (A - \lambda B)\delta q + (\delta A - \delta\lambda\, B)q + \cancel{(\delta A - \delta\lambda\, B)\delta q} = 0$$

Higher order

Structural sensitivity

- Sensitivity to internal changes

$$A(p)q = \lambda B q$$

p Governing parameter: Reynolds number, base flow, wavenumber

Perturbation expansion

$$(A + \delta A)(q + \delta q) = (\lambda + \delta\lambda) B (q + \delta q)$$

$$(A - \lambda B)\delta q + (\delta A - \delta\lambda\, B) q \approx 0$$

Structural sensitivity

- Sensitivity to internal changes

$$(A - \lambda B)\delta q + (\delta A - \delta\lambda\, B)q \approx 0$$

Use adjoint, left eigenvector

$$q^+(A - \lambda B) = 0 \quad \longleftrightarrow \quad (A^+ - \lambda^* B^+)q^+ = 0$$

$$q^+(A - \lambda B)\delta q + q^+(\delta A - \delta\lambda\, B)q \approx 0$$

Structural sensitivity

- Sensitivity to internal changes

$$A(p)q = \lambda B q$$

Perturbation expansion, linearize and use adjoint

$$\delta\lambda = \frac{q^{+}\delta A q}{q^{+} B q}$$

$$\nabla_p \lambda = \frac{q^{+}\nabla_p A\, q}{q^{+} B q}$$

Gradient: constraint optimization (C. Cossu)

Sensitivity to a scalar parameter

Complex Ginzburg-Landau $u_t = (-\nu\partial_x + \gamma\partial_{xx} + \mu(x))u$

$$\nu = U + ic_u$$

Eigenvalue sensitivity

$$\nabla_p \lambda = \frac{q^+ \nabla_p A\, q}{q^+ B q}$$

$$\nabla_\nu A = -\partial_x$$

$$A\hat{u} = \lambda\hat{u}$$

$$A^+\hat{u}^+ = \lambda^*\hat{u}^+$$

$$\lambda = \sigma + i\omega$$

$$\nabla_\nu \lambda = \hat{u}^+ \nabla_\nu A\, \hat{u} = -\hat{u}^+ \partial_x\, \hat{u}$$

$$\nabla_\nu \sigma = \Re\{\nabla_\nu \lambda\}$$

Sensitivity of growth rate

$$\nabla_\nu \omega = \Im\{\nabla_\nu \lambda\}$$

Sensitivity of frequency

Sensitivity to base flow modifications

Linearized Navier-Stokes

$$u_t + U\nabla u + u\nabla U = \frac{1}{Re}\nabla^2 u$$

Eigenvalue sensitivity

$$\delta\lambda = \frac{q^+ \delta A q}{q^+ B q}$$

$$\delta A = -(\delta U)\nabla \hat{u} - \hat{u}\nabla(\delta U)$$

$$A\hat{u} = \lambda\hat{u}$$

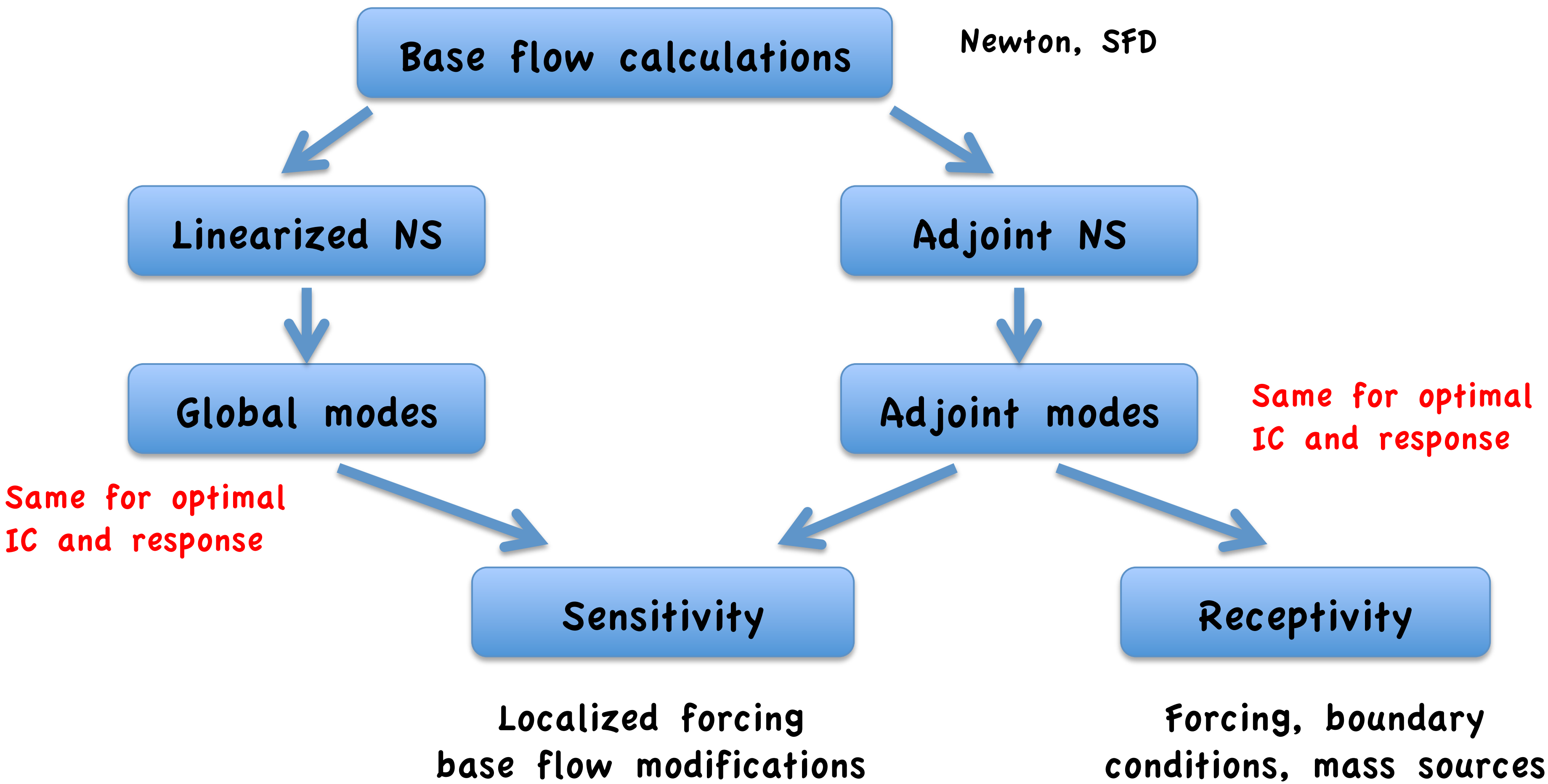
$$A^+\hat{u}^+ = \lambda^*\hat{u}^+$$

$$\lambda = \sigma + i\omega$$

$$\nabla_U \lambda = \nabla\hat{u}^+ \cdot \hat{u}^* - (\nabla\hat{u})^H \cdot \hat{u}^+$$

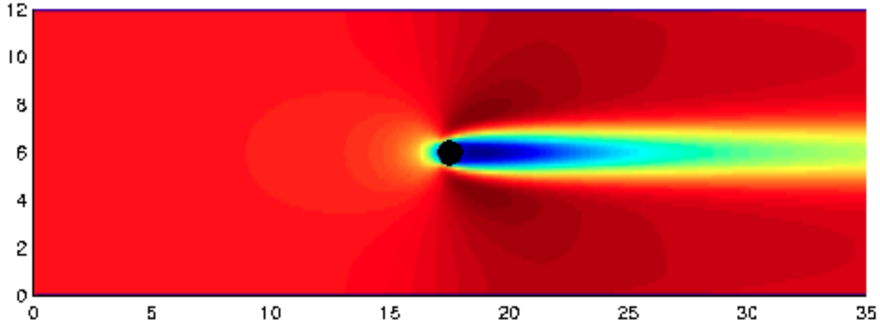
Relate mean flow modification to passive control: small control forcing

Flow chart for sensitivity/receptivity

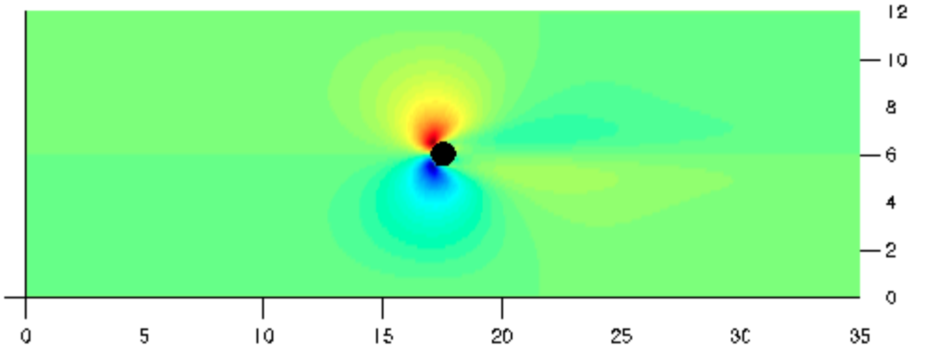


Flow around a cylinder

- 1. base flow



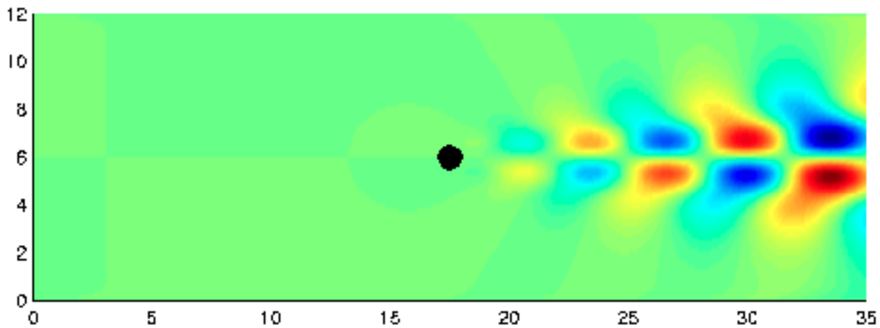
U



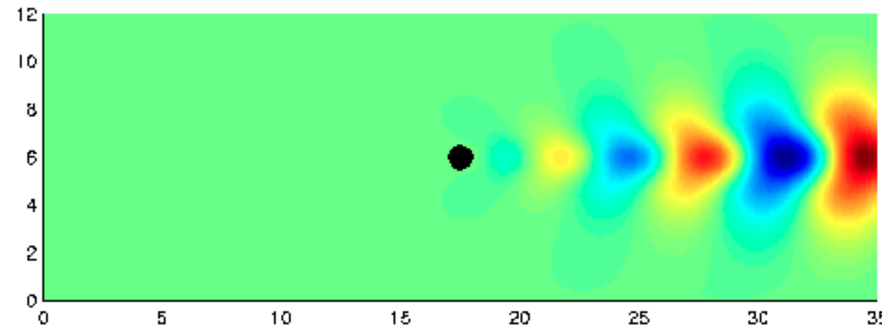
V

Flow around a cylinder

- 1. base flow
- 2. global modes



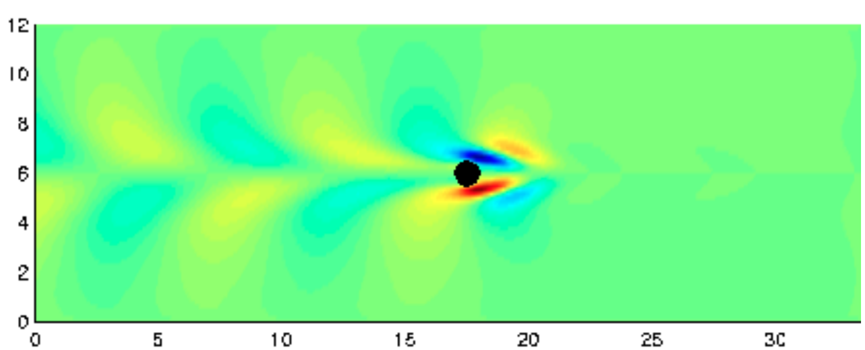
$\hat{u}$



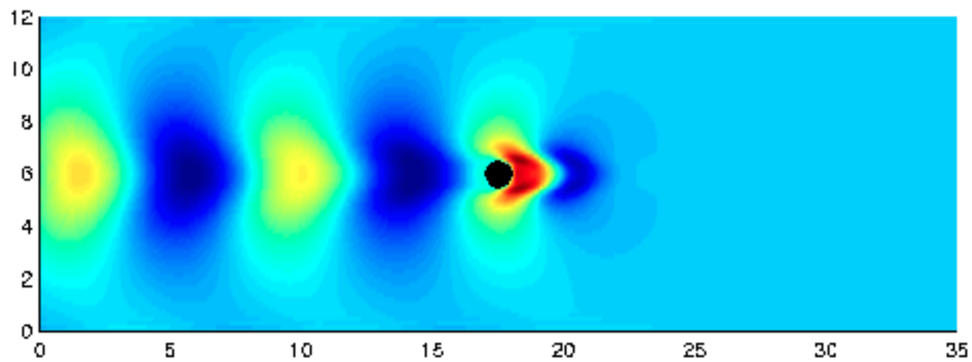
$\hat{v}$

Flow around a cylinder

- 1. base flow
- 2. global modes
- 3. adjoint modes and receptivity



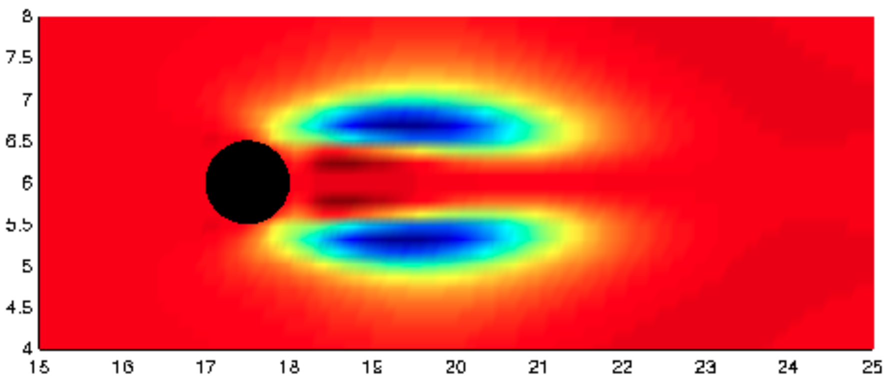
$\hat{u}^+$



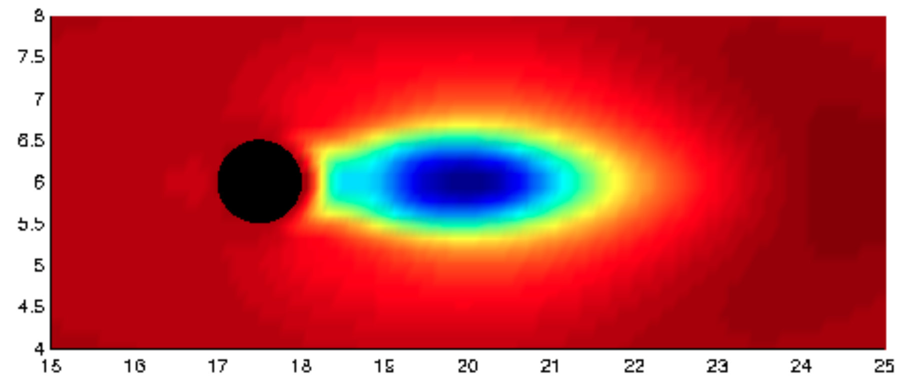
$\hat{v}^+$

Flow around a cylinder

- 1. base flow
- 2. global modes
- 3. adjoint modes and receptivity
- 4. sensitivity, wavemaker



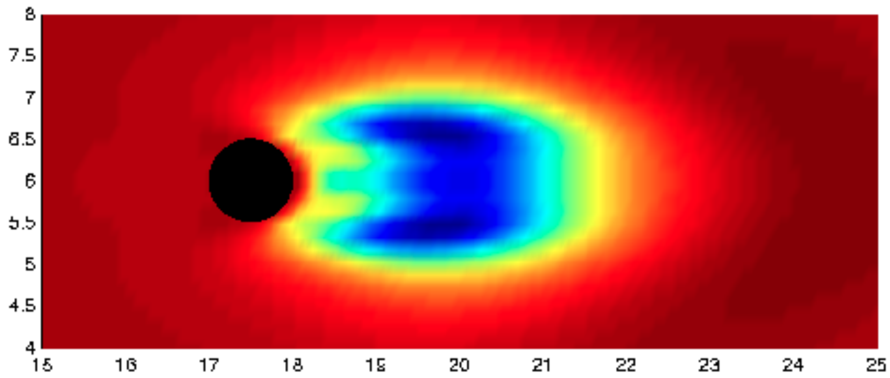
$u_{wavemaker}$



$v_{wavemaker}$

Flow around a cylinder

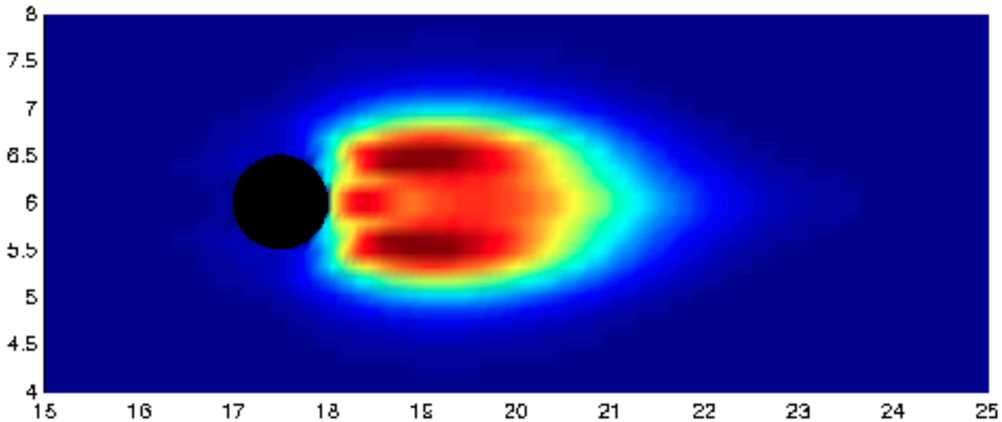
- 1. base flow
- 2. global modes
- 3. adjoint modes and receptivity
- 4. sensitivity, wavemaker



Spatial feedback

Flow around a cylinder

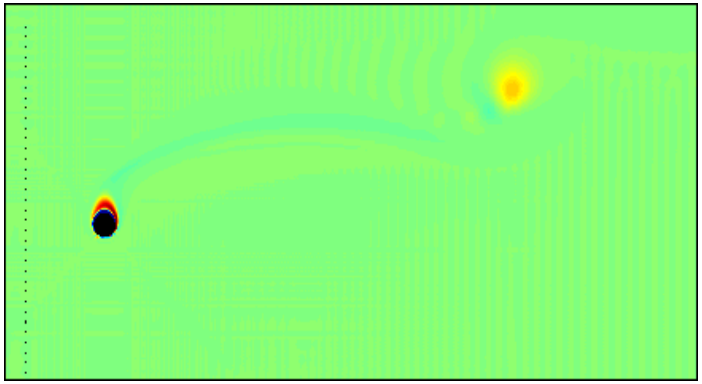
- 1. base flow
- 2. global modes
- 3. adjoint modes and receptivity
- 4. sensitivity, wavemaker
- 5. sensitivity to base flow modifications



Oscillators vs. noise amplifiers

Open flows: global instability and transient growth

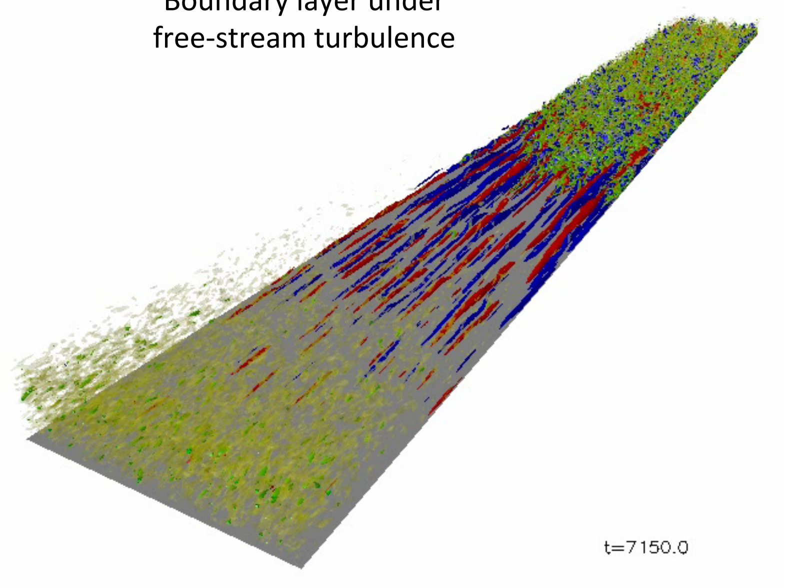
Rotating cylinder



Hydrodynamic oscillators:

Global instability
Intrinsic frequency
Local absolute instability (WKB)

Boundary layer under
free-stream turbulence



Noise amplifiers:

Globally stable
Broad-band frequency spectrum
Local convective instability (WKB)

Globally transient growth of perturbations!

Stability analysis

- Oscillators → Modal analysis

  - Largest Eigenvalue gives the asymptotic behavior

- Noise amplifiers → Non-modal analysis

  Optimal initial condition

  – Initial condition that gives the maximum energy growth at a fixed final time.

  Optimal forcing (pseudo-spectra)

  – Forcing function the gives the maximum energy of the regime response when the forcing is applied with a fixed frequency

Instability mechanisms: Globally unstable flows

- The flow fields behave like an oscillator

  Huerre & Monkewitz, Annu. Rev. Fluid Mech., 1990

- In weakly parallel flows the WKBJ approach identifies a specific spatial position in the absolutely unstable region which acts as a wavemaker.

  Chomaz, Annu. Rev. Fluid Mech., 2005

- A concept similar to that of wavemaker can be introduced by investigating where in space a modification in the structure of the problem produces the largest drift of the eigenvalue:

  determine the region where feedback from velocity to force is most effective

  Giannetti & Luchini, Journal of Fluid Mech., 2007, Pralits et al, 2010

Lagrange identity and adjoint equations

- Using differentiation by parts

$$\left[\left(\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{L}\{U_b, Re\}\boldsymbol{u} + \nabla p\right)\cdot \boldsymbol{f}^+ + \nabla\cdot\boldsymbol{u}\hat{m}^+\right]$$
$$+\left[\boldsymbol{u}\cdot\left(\frac{\partial \boldsymbol{f}^+}{\partial t} + \mathbf{L}^+\{U_b, Re\}\boldsymbol{f}^+ + \nabla m^+\right) + p\,\nabla\cdot\boldsymbol{f}^+\right] = \frac{\partial \boldsymbol{u}\cdot\boldsymbol{f}^+}{\partial t} + \nabla\cdot\boldsymbol{J}(\boldsymbol{q}, \boldsymbol{g}^+).$$

Introduce the adjoint fields f^+ and m^+, and the adjoint linearised Navier-Stokes L^+

Structural sensitivity w.r.t. perturbations

- Perturbed eigenvalue problem

$$\sigma \hat{\mathbf{u}}' + L\{\mathbf{U}_b, Re\}\hat{\mathbf{u}}' + \boldsymbol{\nabla}\hat{p}' = \delta H(\hat{\mathbf{u}}', \hat{p}')$$

- Structural perturbation: local force related to local velocity

$$\delta H(\hat{\mathbf{u}}', \hat{p}') = \delta M(x,y) \cdot \hat{\mathbf{u}}' = \delta(x - x_0, y - y_0)\, \delta M_0 \cdot \hat{\mathbf{u}}'$$

- Linear variation of eigenvalue-eigenfunction expressed as

$$\sigma \delta \hat{\mathbf{u}} + L\{\mathbf{U}_b, Re\}\delta \hat{\mathbf{u}} + \boldsymbol{\nabla}\delta \hat{p} = -\delta\sigma \hat{\mathbf{u}} + \delta M \cdot \hat{\mathbf{u}}$$

- Using Lagrange identity for perturbation field

$$\delta\sigma(x_0, y_0) = \frac{\hat{\mathbf{f}}^{+} \cdot \delta M_0 \cdot \hat{\mathbf{u}}}{\int_{\mathcal{D}} \hat{\mathbf{f}}^{+} \cdot \hat{\mathbf{u}} \, \mathrm{d}S} = S : \delta M_0$$

✓ The sensitivity is a spatial map given by the product above

✓ Different norms can be displayed (Spectral norm=worst case here)

Production of perturbation kinetic energy

- Classic approach in stability analysis

$$\frac{d}{dt}\int_{\mathcal{D}}\left(\frac{1}{2}\overline{u_i u_i}\right)dx_i = \int_{\mathcal{D}}\frac{\partial U_i}{\partial x_j}\overline{\tau_{ij}}dx_i - \frac{1}{Re}\int_{\mathcal{D}}\overline{\omega_i\omega_i}dx_i$$

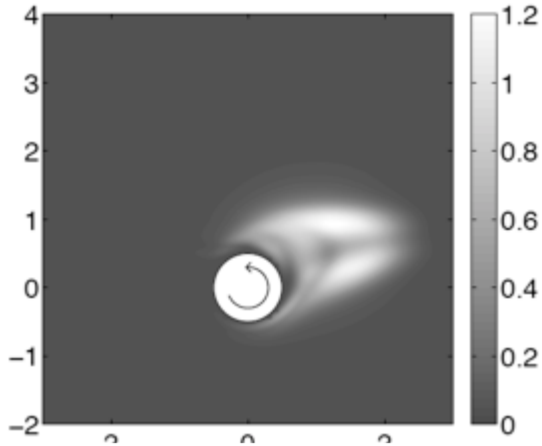
Work of Reynolds stresses against the base flow shear

Wavemaker: where in space a modification in the structure of the problem produces the largest drift of the eigenvalue:

determine the region where feedback from velocity to force is most effective

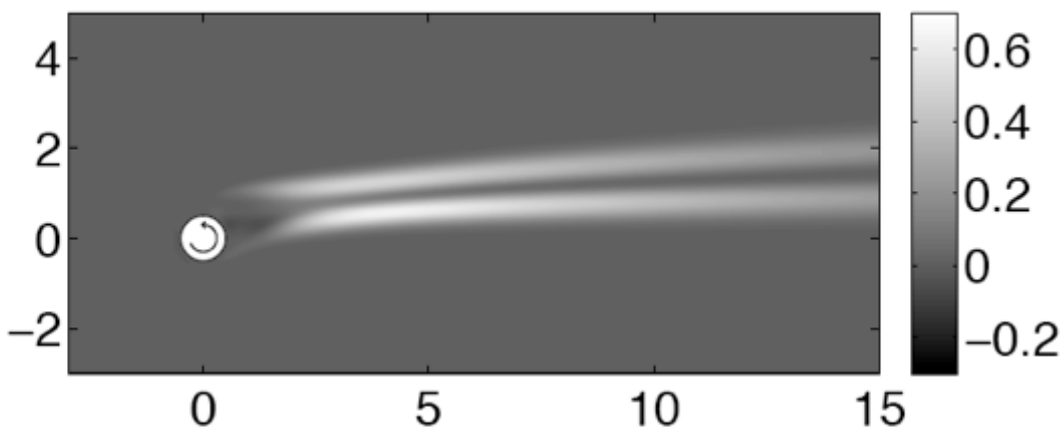
Comparison with kinetic energy production

- Shedding mode I: wavemaker vs. energy growth



Structural sensitivity

Production

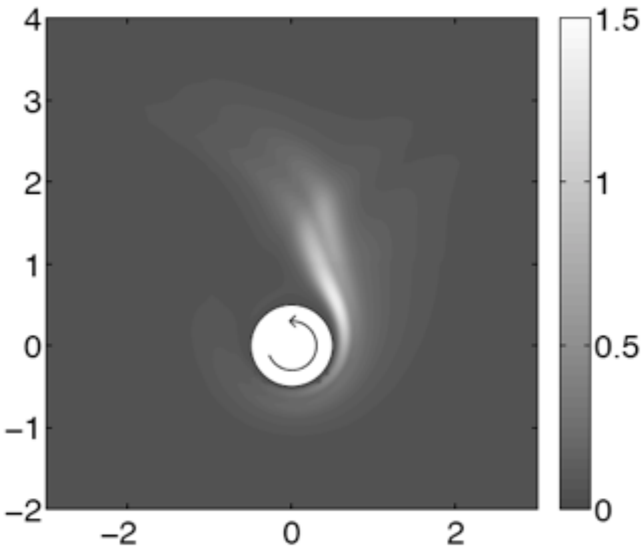


Mode generated in region of absolute instability

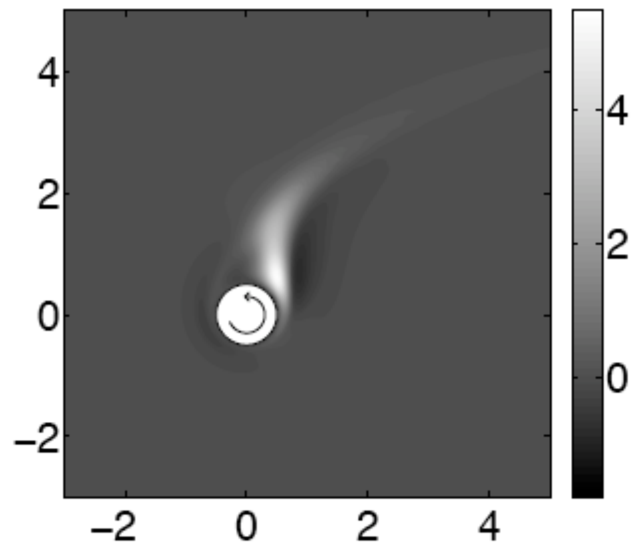
Pralits et al, JFM, 2010

Comparison with kinetic energy production

- Shedding mode I: wavemaker vs. energy growth



Structural sensitivity



Production

✓ **Same region identified**

Pralits et al, JFM, 2010

Structural sensitivity w.r.t. base flow variations

- Consider variations of the eigenvalue due to structural variations of the base flow.

- Assume arbitrary variations of the base flow

$$\sigma\delta\hat{\mathbf{u}} + L\{\mathbf{U}_b, Re\}\delta\hat{\mathbf{u}} + \boldsymbol{\nabla}\delta\hat{p} = -[\delta\sigma\hat{\mathbf{u}} + \delta C(\delta\mathbf{U}_b, \hat{\mathbf{u}})]$$

$$\delta C(\delta\mathbf{U}_b, \hat{\mathbf{u}}) = \delta\mathbf{U}_b \cdot \boldsymbol{\nabla}\hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \boldsymbol{\nabla}\delta\mathbf{U}_b$$

- δC bilinear operator expressing variations of L w.r.t. $\delta\mathbf{U}_b$

- Eigenvalue drift

$$\delta\sigma = \frac{\int_{\mathcal{D}} \hat{\mathbf{f}}^{+} \cdot \delta C(\delta\mathbf{U}_b, \hat{\mathbf{u}})\, \mathrm{d}S}{\int_{\mathcal{D}} \hat{\mathbf{f}}^{+} \cdot \hat{\mathbf{u}}\, \mathrm{d}S}$$

Structural sensitivity w.r.t. base flow variations

- Consider variations of the eigenvalue due to structural variations of the base flow.

- Assume arbitrary variations of the base flow

$$\sigma\delta\hat{\mathbf{u}} + L\{\mathbf{U}_b, Re\}\delta\hat{\mathbf{u}} + \nabla\delta\hat{p} = -[\delta\sigma\hat{\mathbf{u}} + \delta C(\delta\mathbf{U}_b, \hat{\mathbf{u}})]$$

$$\delta C(\delta\mathbf{U}_b, \hat{\mathbf{u}}) = \delta\mathbf{U}_b \cdot \nabla\hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla\delta\mathbf{U}_b$$

- δC bilinear operator expressing variations of L w.r.t. $\delta\mathbf{U}_b$

- Eigenvalue drift

$$\delta\sigma = \frac{\int_{\mathcal{D}} \delta\mathbf{U}_b \cdot \delta C(\hat{\mathbf{f}}^+, \hat{\mathbf{u}})\,\mathrm{d}S + \oint_{\Gamma_c} (\delta\mathbf{U}_b \cdot \hat{\mathbf{f}}^+)\hat{\mathbf{u}} \cdot \mathbf{n}\,\mathrm{d}l}{\int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}}\,\mathrm{d}S}$$

Structural sensitivity w.r.t. base flow variations

- Structural variations of the base flow: linearized steady Navier-Stokes equations

$$L\{\mathbf{U}_b, Re\}\delta\mathbf{U}_b + \nabla\delta P_b = \delta M \cdot \mathbf{U}_b$$

- Using Lagrange identity

$$\int_{\mathcal{D}} \delta\mathbf{U}_b \cdot \delta C(\hat{\mathbf{f}}^+, \hat{\mathbf{u}})\, \mathrm{d}S = \int_{\mathcal{D}} \mathbf{f}_b^+ \cdot \delta M \cdot \mathbf{U}_b \,\mathrm{d}S = \mathbf{f}_b^+(x_0, y_0) \cdot \delta M_0 \cdot \mathbf{U}_b(x_0, y_0)$$

with $$L^+\{\mathbf{U}_b, Re\}\mathbf{f}_b^+ + \nabla m_b^+ = \delta C(\hat{\mathbf{f}}^+, \hat{\mathbf{u}})$$

- Structural sensitivity

$$\delta\sigma = \frac{\displaystyle\int_{\mathcal{D}} \mathbf{f}_b^+ \cdot \delta M \cdot \mathbf{U}_b \,\mathrm{d}S}{\displaystyle\int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}} \,\mathrm{d}S} = \frac{\mathbf{f}_b^+ \cdot \delta M_0 \cdot \mathbf{U}_b}{\displaystyle\int_{\mathcal{D}} \hat{\mathbf{f}}^+ \cdot \hat{\mathbf{u}} \,\mathrm{d}S} = S_b : \delta M_0$$

Passive control

- Wake control by means of small obstacles in the flow

- Small control cylinder as localized structural perturbation: First term of Lamb-Oseen expansion for drag at low *Re*

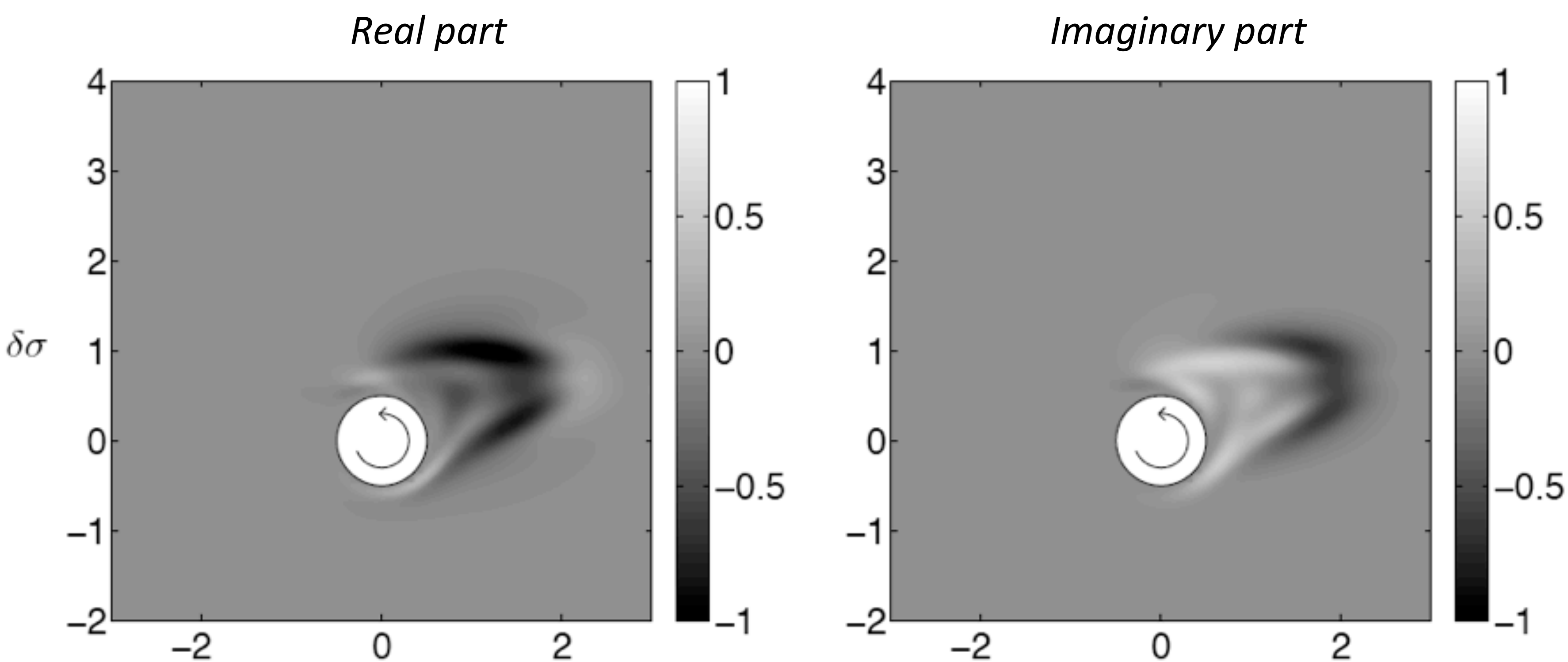
$$\delta A = \frac{4\pi}{Re \ \ln\left(\frac{7.4}{Re_c}\right)}$$

- Structural perturbation is reacting force, aligned with local velocity and acting at perturbation and base-flow level

$$\delta\sigma_t = \delta\sigma_b + \delta\sigma = S_b : \delta M_0 + S : \delta M_0 = S_t : \delta M_0$$

- DNS of passive control

Structural sensitivity for Shedding Mode I

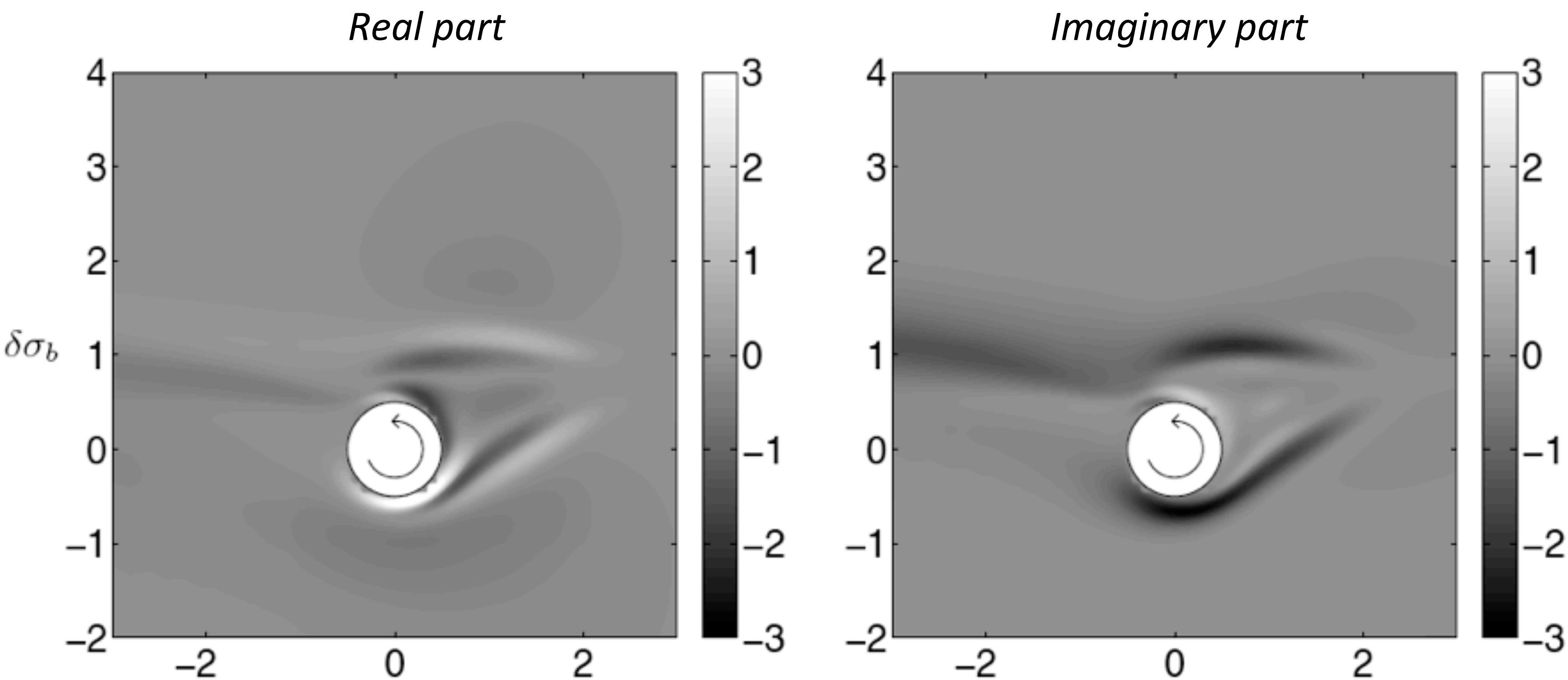


$$\alpha = 1.8$$

Sensitivity w.r.t. perturbations (spectral norm)

Pralits et al, JFM, 2010

Structural sensitivity for Shedding Mode I

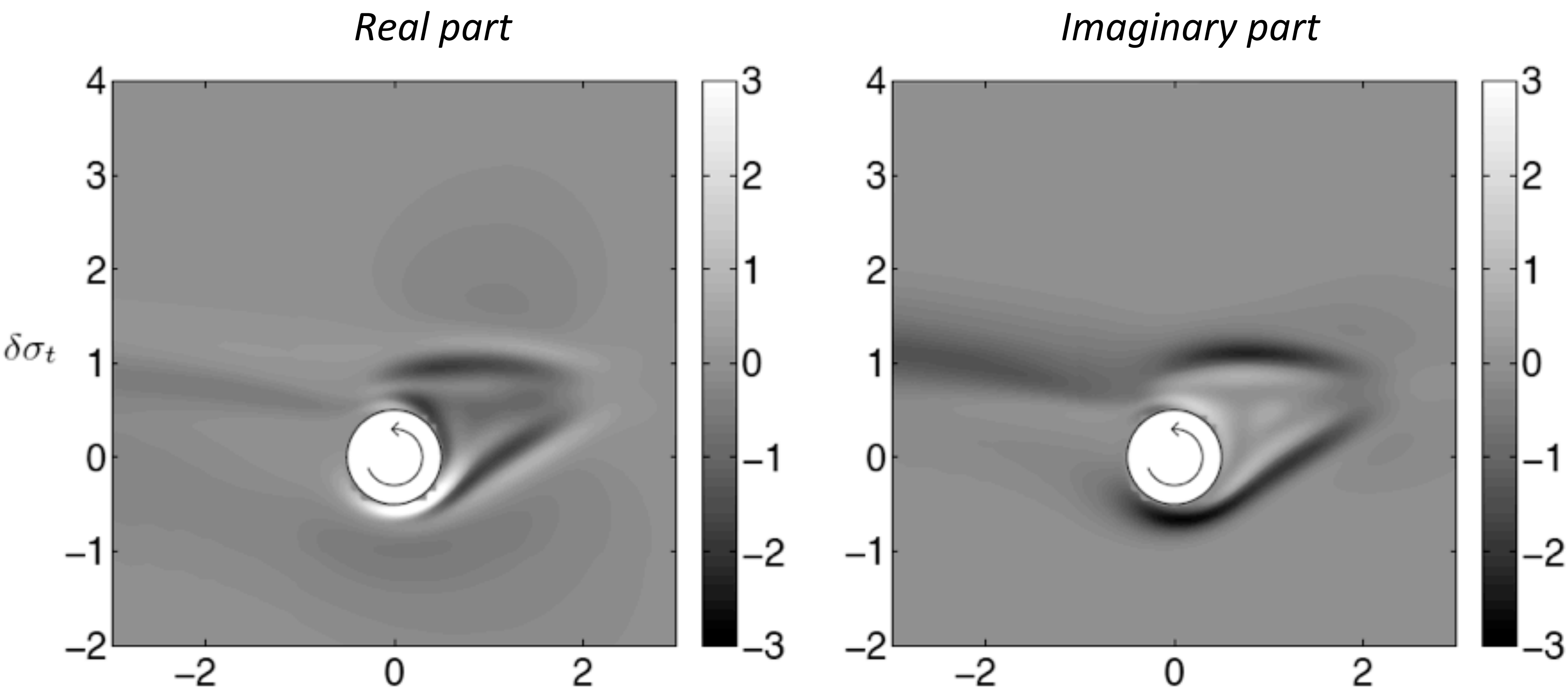


$\alpha = 1.8$

Sensitivity w.r.t. base flow (spectral norm)

Pralits et al, JFM, 2010

Structural sensitivity for Shedding Mode I



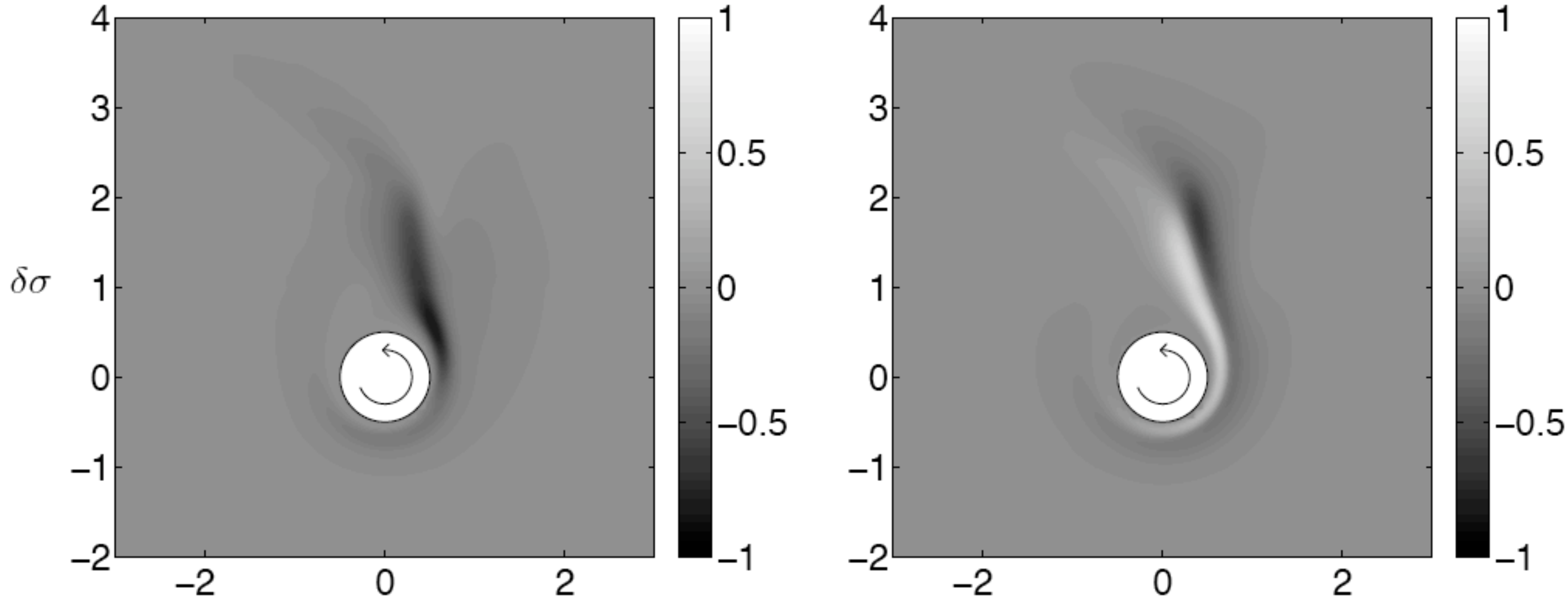
$$\alpha = 1.8$$

Total Sensitivity (spectral norm)

Pralits et al, JFM, 2010

Structural sensitivity for Shedding Mode II

Real part *Imaginary part*

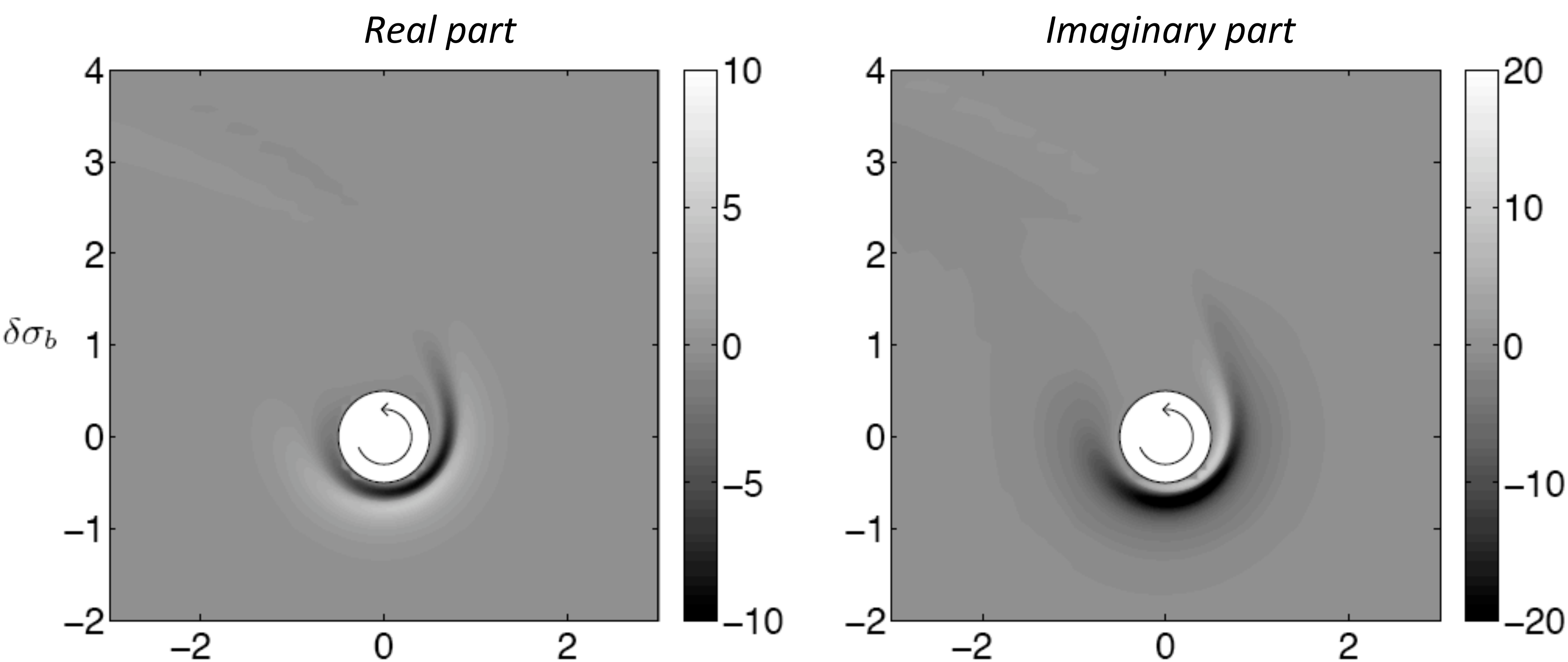


$$\alpha = 4.85$$

Sensitivity w.r.t. perturbations (spectral norm)

Pralits et al, JFM, 2010

Structural sensitivity for Shedding Mode II

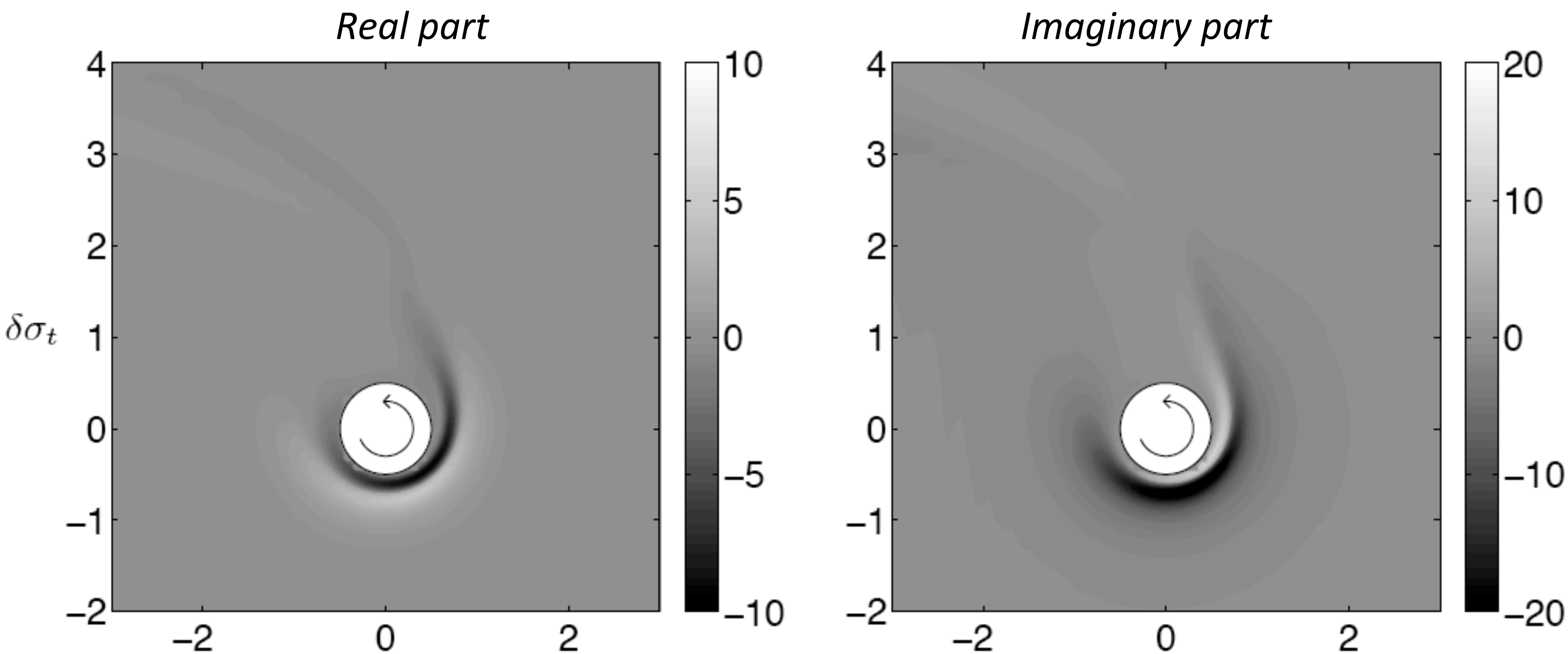


$$\alpha = 4.85$$

Sensitivity w.r.t. base flow (spectral norm)

Pralits et al, JFM, 2010

Structural sensitivity for Shedding Mode II



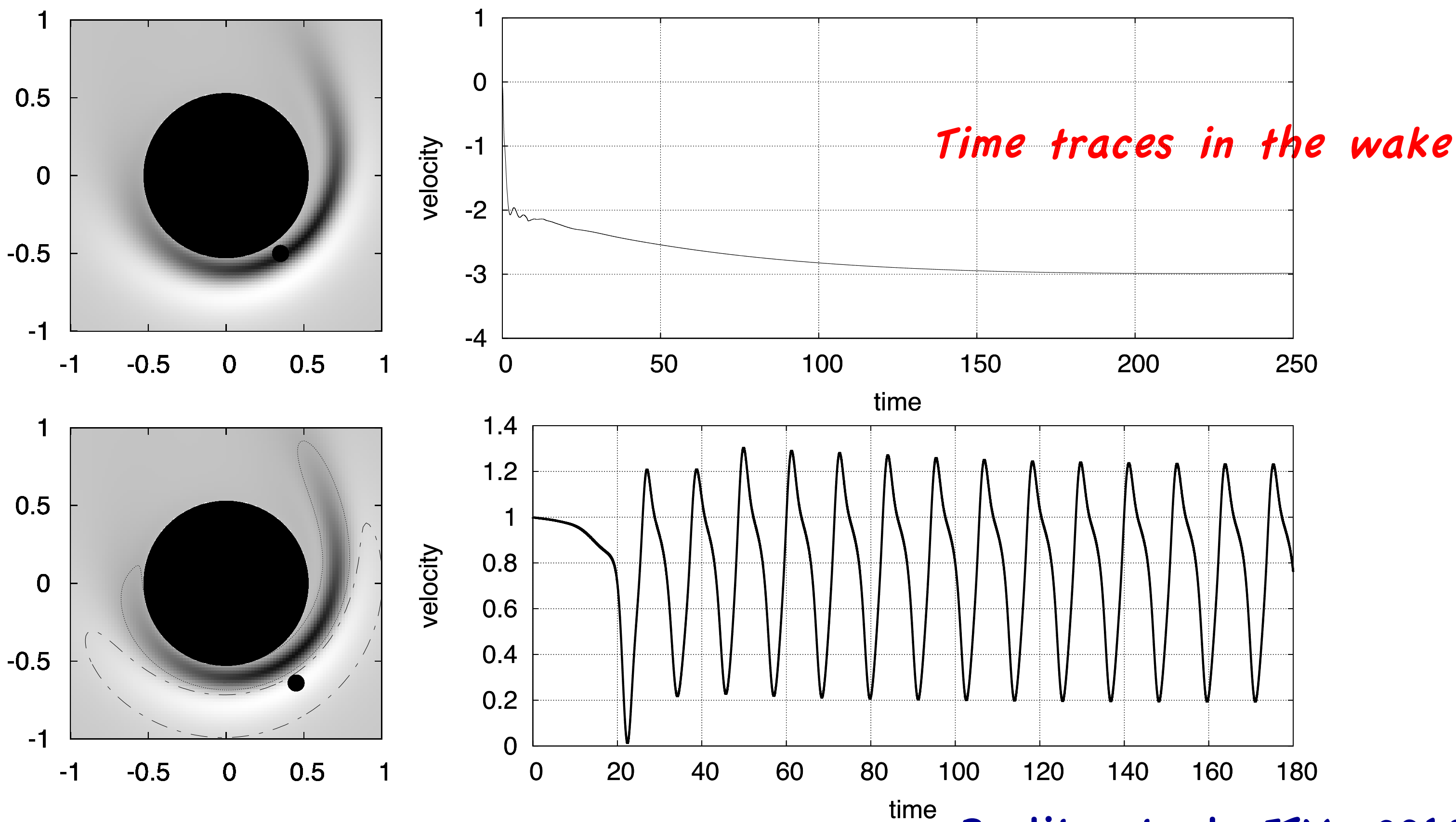
$\alpha = 4.85$

Total Sensitivity (spectral norm)

Pralits et al, JFM, 2010

Passive control based on sensitivity analysis

- Simulations of a secondary small cylinder positioned as indicated by sensitivity maps



Pralits et al, JFM, 2010

Instability mechanisms: Noise amplifiers

- The aim of the present work is to extend the sensitivity analysis to flow behaving as noise amplifiers

  Huerre & Monkewitz, Annu. Rev. Fluid Mech., 1990

- The target of the analysis is the largest singular value of the system. We consider the resolvent norm in frequency domain.

  Schmid & Henningson, Stability and Transition of shear flows, 2001

- A concept similar to that of wavemaker can be introduced by investigating where in space a modification of the base flow produces the largest drift of the optimal response: wave-amplifier

  Brandt et al., Journal of Fluid Mech., 2011

Gradient of the resolvent norm

We assume

$$\mathbf{u} = \mathcal{R}(\omega, \mathbf{U})\, \mathbf{f}\ , \quad \mathcal{R} = \ \mathcal{S}(\omega, \mathbf{U})^{-1} \ , \quad \mathcal{S} = -i\,\omega\, \mathcal{M} + \mathcal{L}(\mathbf{U})$$

Optimal forcing and response

$$G \quad = \quad \frac{(\mathbf{u}, \mathbf{u})}{(\mathbf{f}, \mathbf{f})} = \frac{(\mathcal{R}\,\mathbf{f}\ ,\ \mathcal{R}\,\mathbf{f})}{(\mathbf{f}\ ,\ \mathbf{f})} = \frac{\left(\mathcal{R}^{\dagger}\mathcal{R}\,\mathbf{f}\ ,\ \mathbf{f}\right)}{(\mathbf{f}\ ,\ \mathbf{f})}$$

Sensitivity to base-flow modifications:
Overlap of optimal forcing and response

$$\nabla_{\mathbf{U}}\ \sigma^2 = 2\sigma^2 \Re\{(\nabla \mathbf{f})\, \mathbf{u}^* - \left(\nabla \mathbf{u}^H\right) \mathbf{f}\}$$

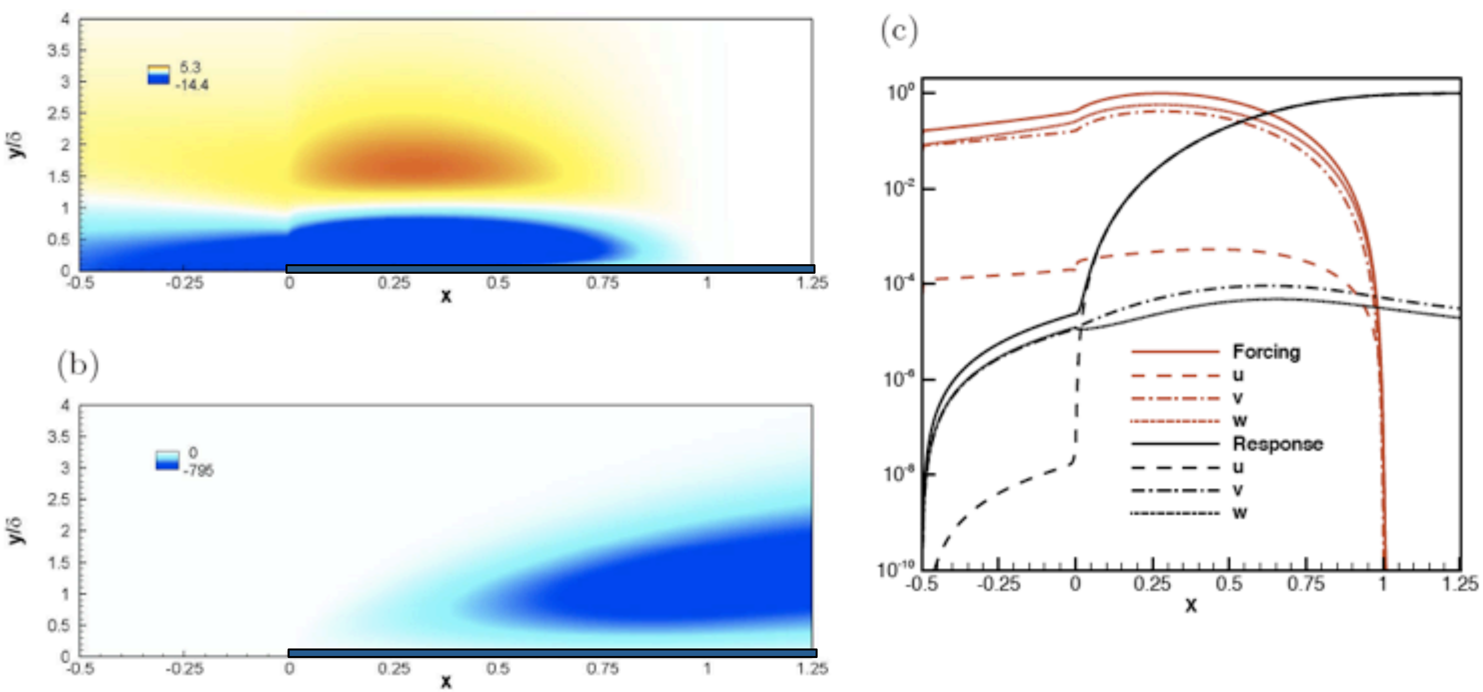
Constrained optimization

$$\mathcal{K} = \sigma^2 \quad - \quad (\mathbf{u}^{\dagger}\ ,\ \mathcal{S}(\mathbf{U})\,\mathbf{u} - \mathbf{f}\ ) - (\mathbf{a}^{\dagger}\ ,\ \mathcal{S}^{\dagger}(\mathbf{U})\,\mathbf{a} - \mathbf{u}\ ) - (\mathbf{f}^{\dagger}\ ,\ \sigma^2\mathbf{f} - \mathbf{a}\ )$$

Brandt et al, JFM, 2011

Boundary layer over flat plate

Optimal forcing: Lift-Up effect

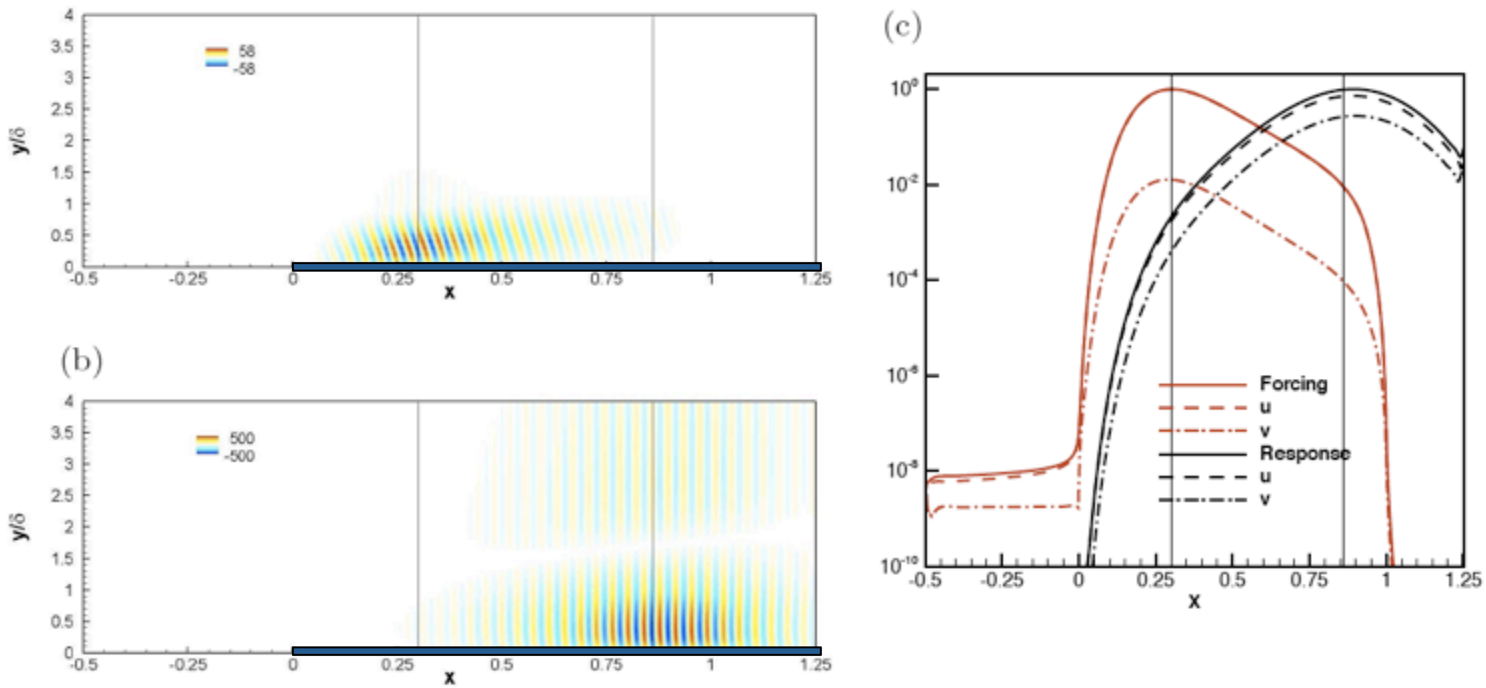


✓ Streamwise vortices induce streamwise streaks: component-wise non-normality, zero frequency and β=1

✓ Forcing active upstream (x<0) and above the flat plate

Brandt et al, JFM, 2011

Boundary layer over flat plate
Optimal forcing: TS-waves



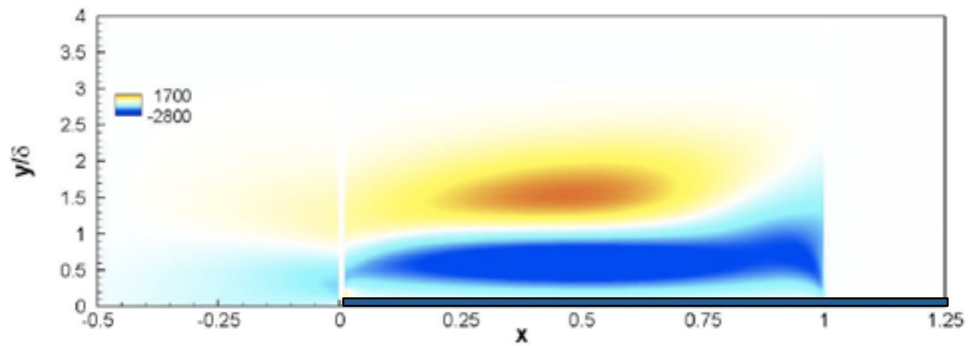
✓ Convective non-normality, 2D forcing at frequency F=100.

✓ Forcing active between branch I and II

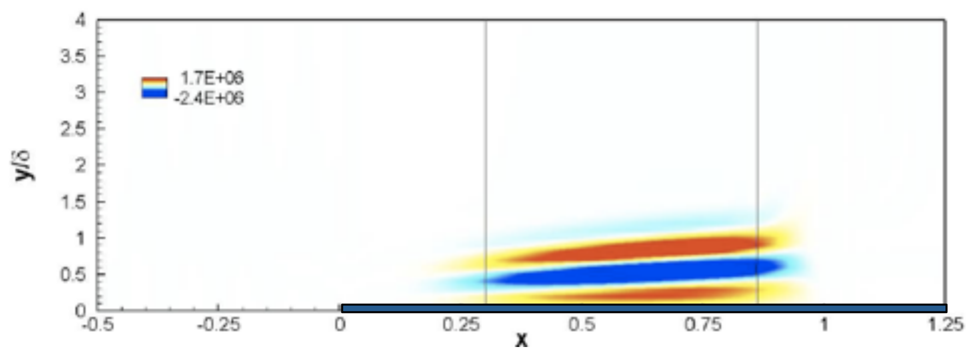
Brandt et al, JFM, 2011

Sensitivity to base-flow modifications streamwise velocity component

Lift-up



TS-waves



✓ Largest sensitivity for TS-instability

✓ Streak amplification: very robust mechanism

Brandt et al, JFM, 2011

Outline

- Stability of fluid systems
  - modal limit
  - short time dynamics, matrix exponential
- Receptivity
  - resolvent norm
  - Resonance limit
  - Adjoint modes
- Sensitivity
  - Structural sensitivity
  - Base-flow sensitivity

SOME MORE ...

Time-dependent flows

- Generalize to time-periodic and time-dependent flows
- Relate non-modal analysis to optimization problems

$$\frac{d}{dt}q = L(t)q$$

With solution

$$q(t) = A(t)q_0$$

Propagator

Initial condition

Input-output analysis

- Seek optimal energy amplification

$$G(t) = \max_{q_0} \frac{\langle q, q \rangle}{\langle q_0, q_0 \rangle}$$

Input-output analysis

- Seek optimal energy amplification

$$G(t) = \max_{q_0} \frac{\langle q, q \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A(t) q_0, A(t) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

Input-output analysis

- Seek optimal energy amplification

$$G(t) = \max_{q_0} \frac{\langle q, q \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A(t) q_0, A(t) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A^H(t) A(t) q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

Input-output analysis for time-dependent system

$$G(t) = \max_{q_0} \frac{\langle A^H(t) A(t) q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$A^H A$ **is a normal matrix**

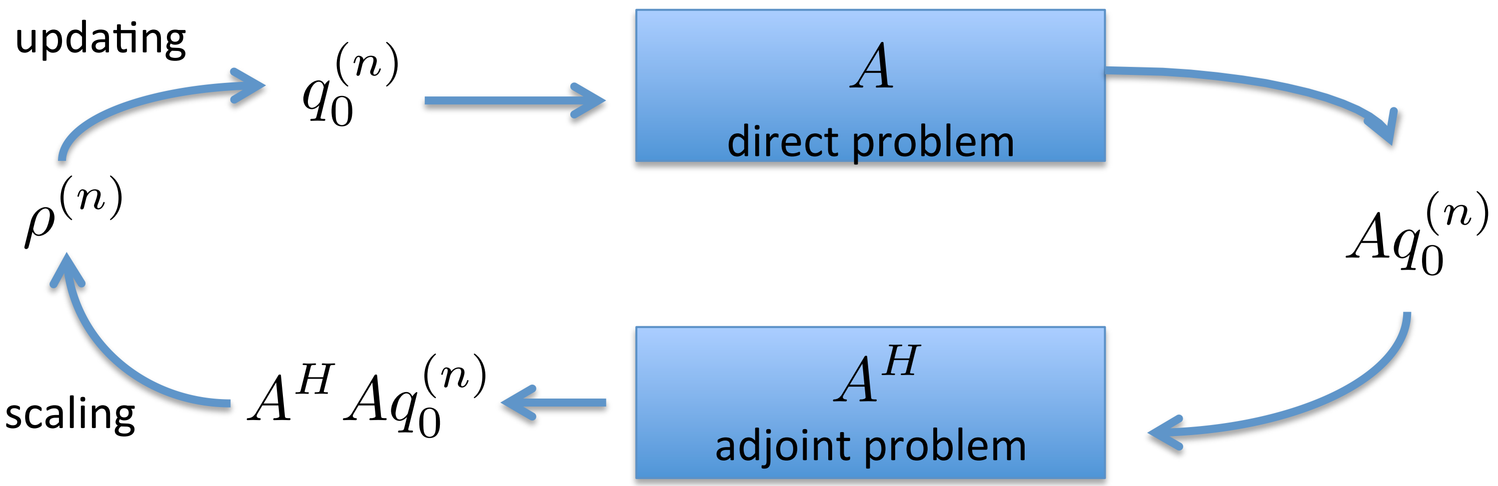
Maximum amplification for largest eigenvector of $A^H A$

Principal eingenvector and eigenvalue can be found by power iterations

$$q_0^{(n+1)} = \rho^{(n)} A^H A \, q_0^{(n)}$$

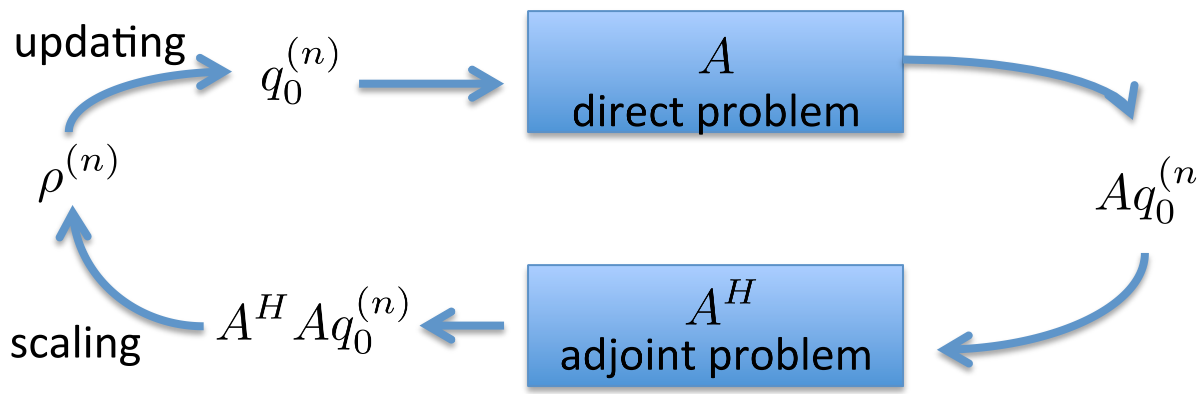
Input-output analysis for time-dependent system

$$q_0^{(n+1)} = \rho^{(n)} \; A^H A \, q_0^{(n)}$$



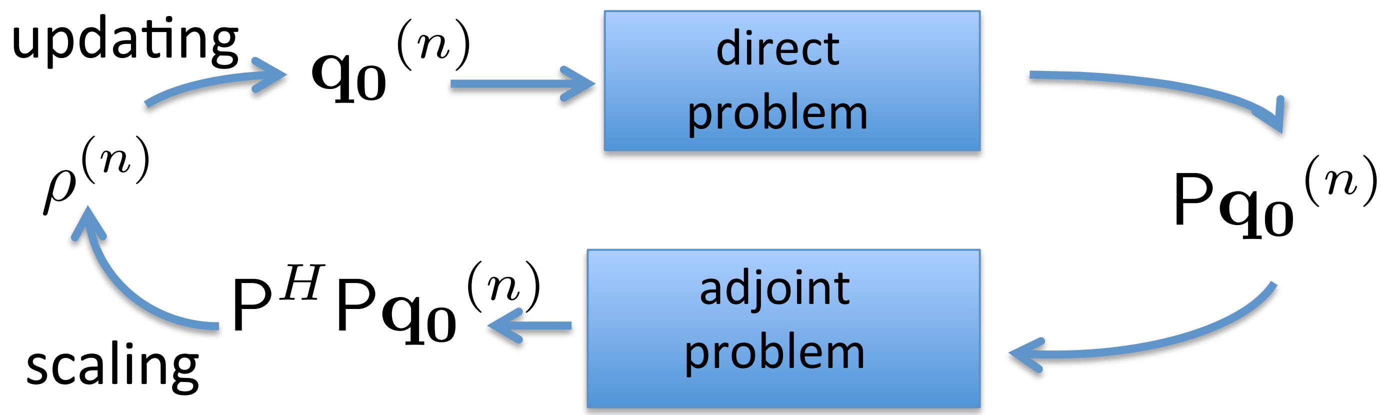
Input-output analysis for time-dependent system

$$q_0^{(n+1)} = \rho^{(n)}\, A^H A\, q_0^{(n)}$$



- This technique (adjoint looping) can be applied to any general time-dependent stability problem
- Propagation of initial condition forward in time and of adjoint initial condition backward in time

$$\mathbf{q_0}^{(n+1)} = \rho^{(n)}\, \mathsf{P}^H \mathsf{P} \mathbf{q_0}^{(n)}$$



Variational formulation of the problem

- Variational formulation of the optimal growth problem

more general

We wish to maximize

$$J = \frac{||q||^2}{||q_0||^2} \to \max$$

with the constraint

$$\frac{d}{dt} q = L(t) q$$

Listen Carlo Cossu on Friday!

Global modes: how to

- Most problems with inhomogeneous directions, complex geometry
- Cannot use Fourier transform in two directions
- Eigenfunctions and optimals depend on more than one direction $\hat{u}^{+}(x, y)$

Computational issues

- **One vs two inhomogeneous directions:**
  state vector, matrix, operation count

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ q_N \end{pmatrix} \qquad \textbf{State vector} \qquad q = \begin{pmatrix} q_{1,1} \\ q_{1,2} \\ \cdot \\ \cdot \\ q_{N,N} \end{pmatrix}$$

$$L \in \mathbb{C}^{N \times N} \sim \mathcal{O}(N^2) \qquad \textbf{Matrix size} \qquad L \in \mathbb{C}^{N^2 \times N^2} \sim \mathcal{O}(N^4)$$

$$\sim \mathcal{O}(N^3) \qquad \textbf{Operation count} \qquad \boxed{\sim \mathcal{O}(N^6)}$$

Computational issues

- One vs two inhomogeneous directions:
  state vector, matrix, operation count

$$L \in \mathbb{C}^{N^2 \times N^2} \sim \mathcal{O}(N^4)$$

storage

$$\sim \mathcal{O}(N^6)$$

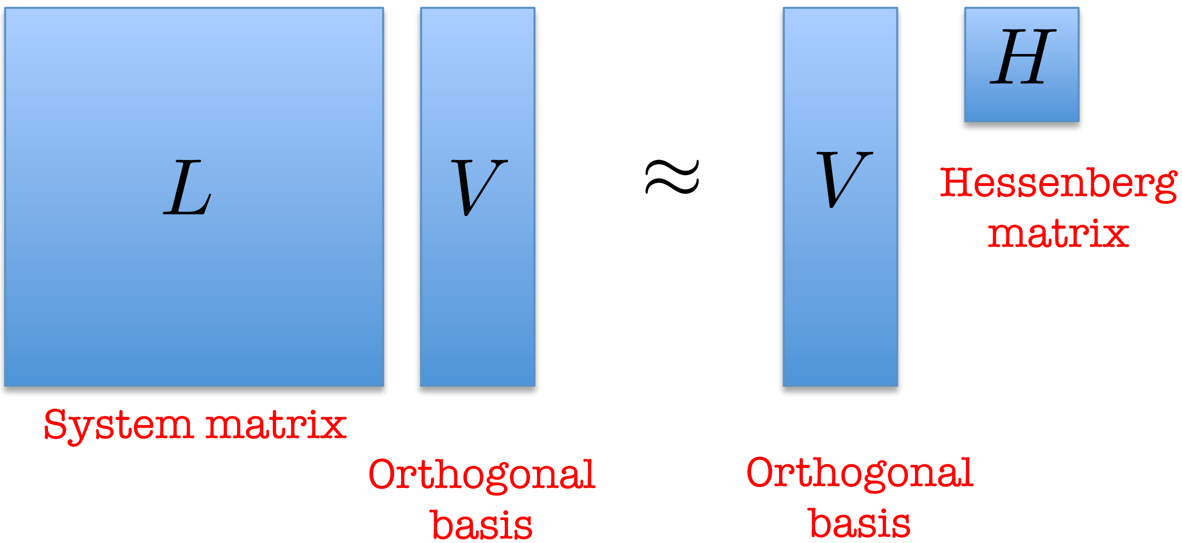
CPU time

➢ Direct eigenvalue algorithms become too expensive

➢ Iterative algorithms, Arnoldi technique

Arnoldi algorithm

- Action of the linear operator within an orthonormal basis V

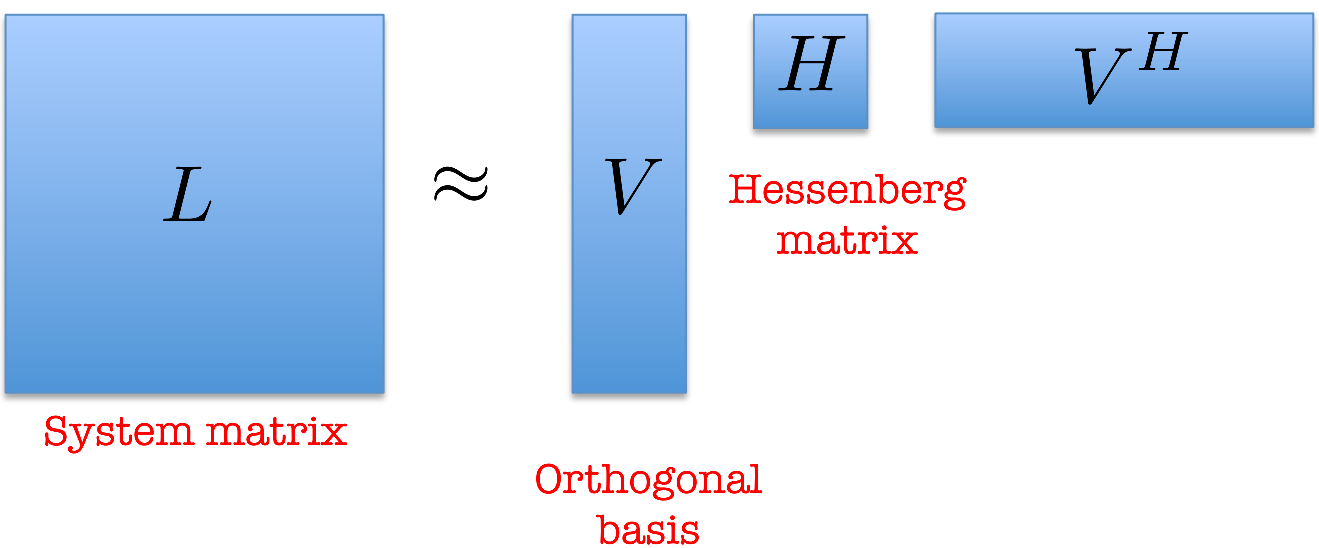


System matrix: $L, \exp(tL), LL^H, \exp(tL)\exp(tL^H)$

USE THIS APPROACH TO COMPUTE MODAL AND NON-MODAL STABILITY

Arnoldi algorithm

- Action of the linear operator within an orthonormal basis V
- Represent stability matrix by a low-order approximation based on V



Hessenberg matrix

- **Only multiplication by L are necessary**

$$q_k = L\ q_{k-1}$$

$$\text{for}\quad j = 1 : k-1$$

$$\quad H_{j,k-1} = \langle q_j, q_k \rangle$$

$$\quad q_k = q_k - H_{j,k-1}\ q_j$$

$$\text{end}$$

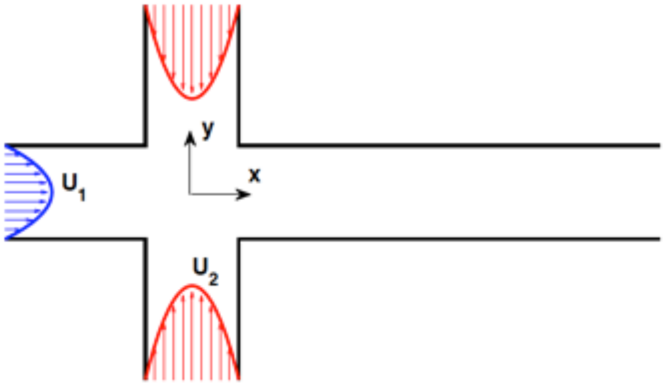
$$H_{k,k-1} = \|q_k\|$$

$$q_k = q_k / H_{k,k-1}$$

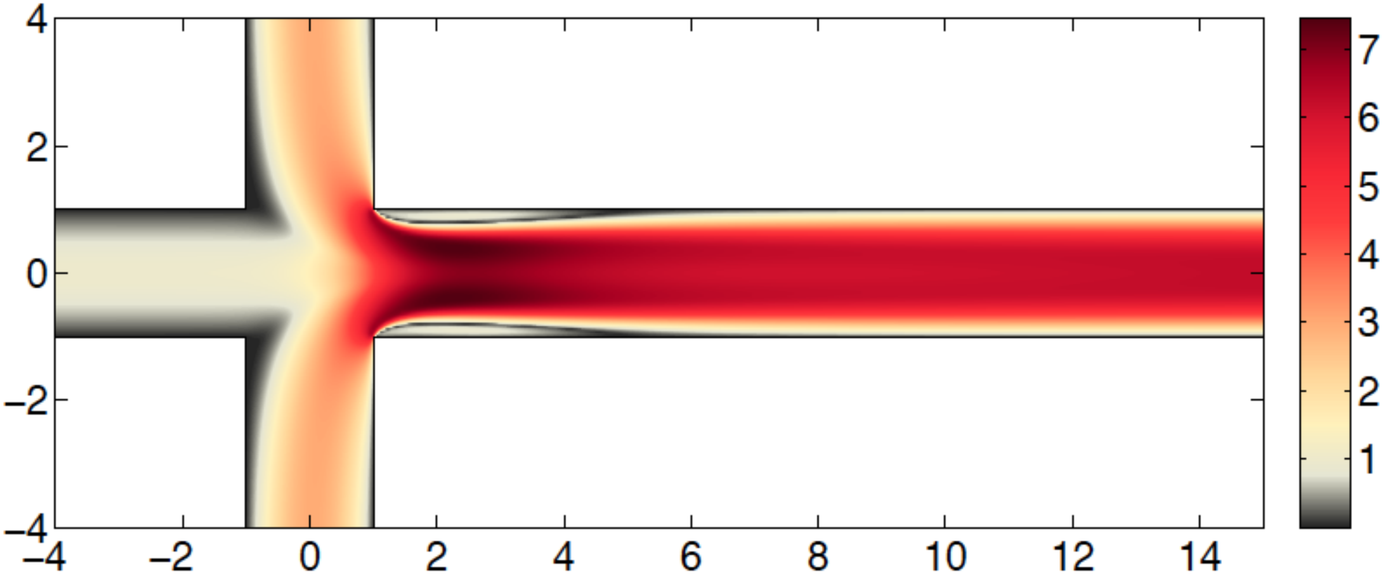
System eigenvalues approximated by eigenvalues of H

$$Eig(L) \approx Eig(H)$$

Example in 2D: flow in X-junction

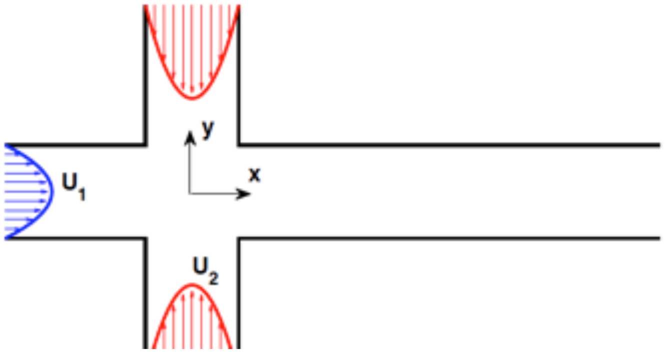


Base flow

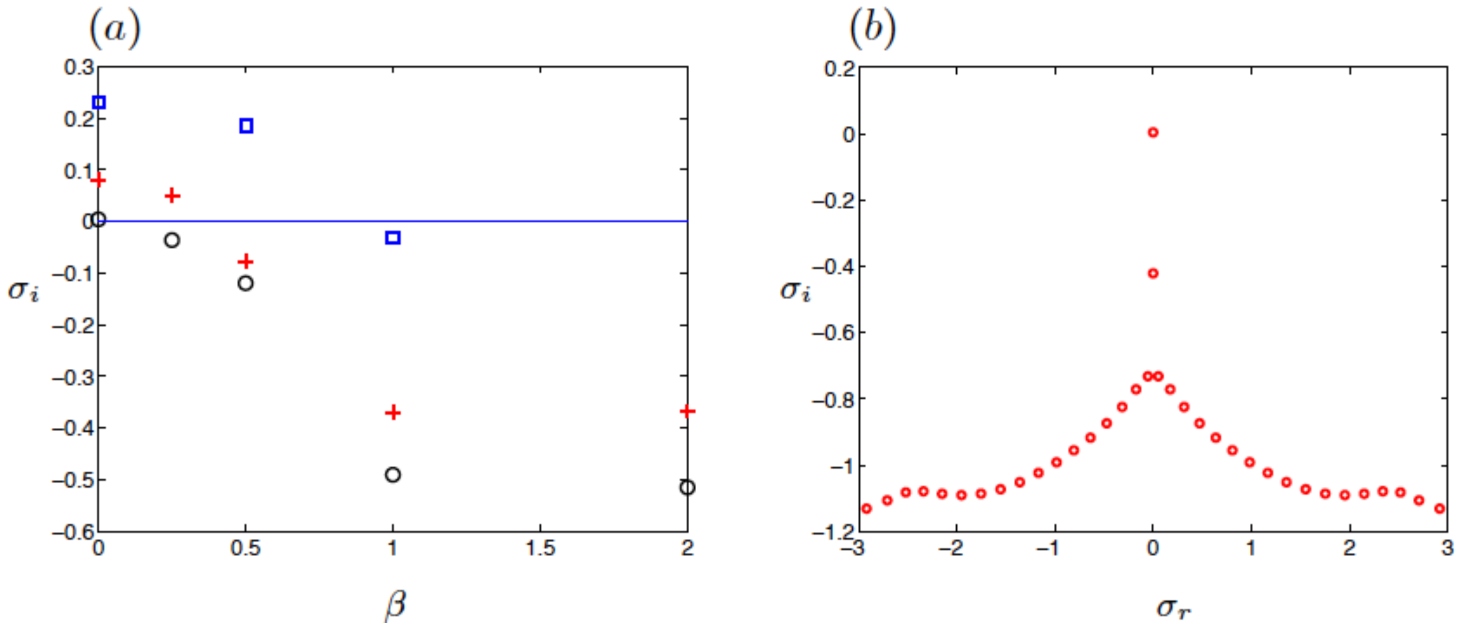


Lashgari et al, in preparation

Example in 2D: flow in X-junction



Spectrum

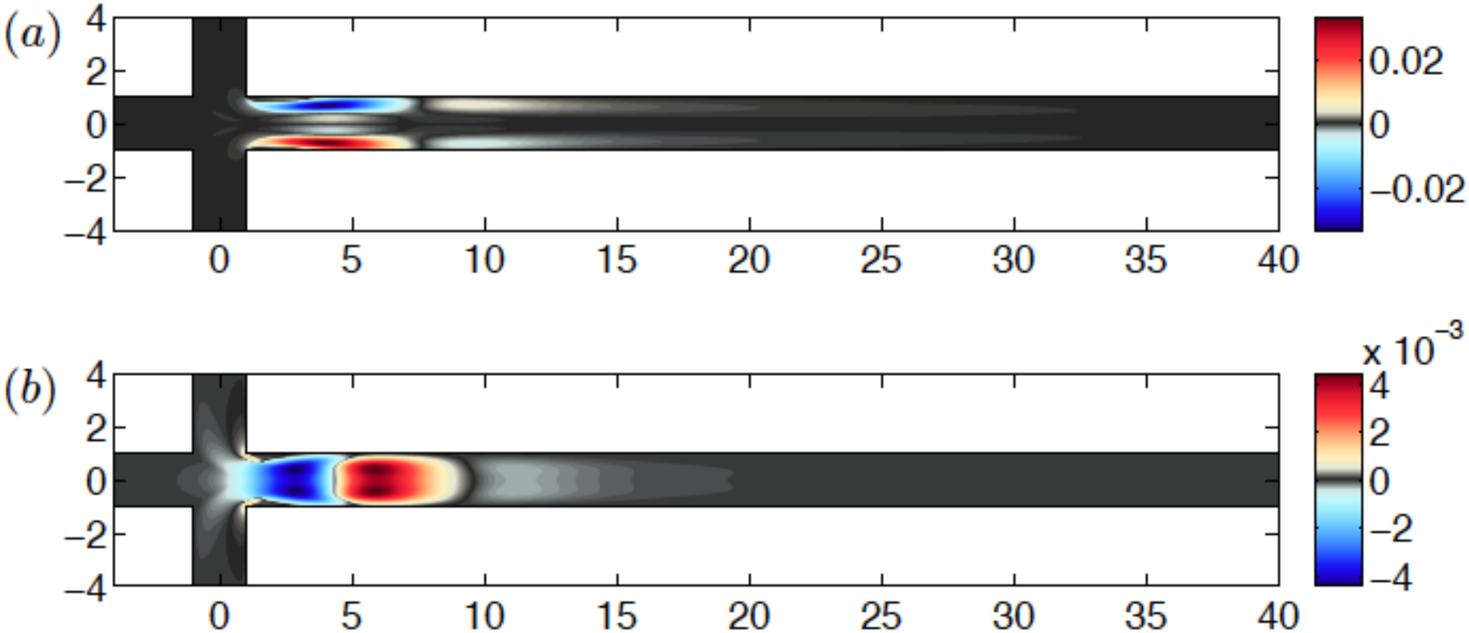


- Steady two-dimensional bifurcation

Lashgari et al, in preparation

Example in 2D: flow in X-junction

Eigenfunction

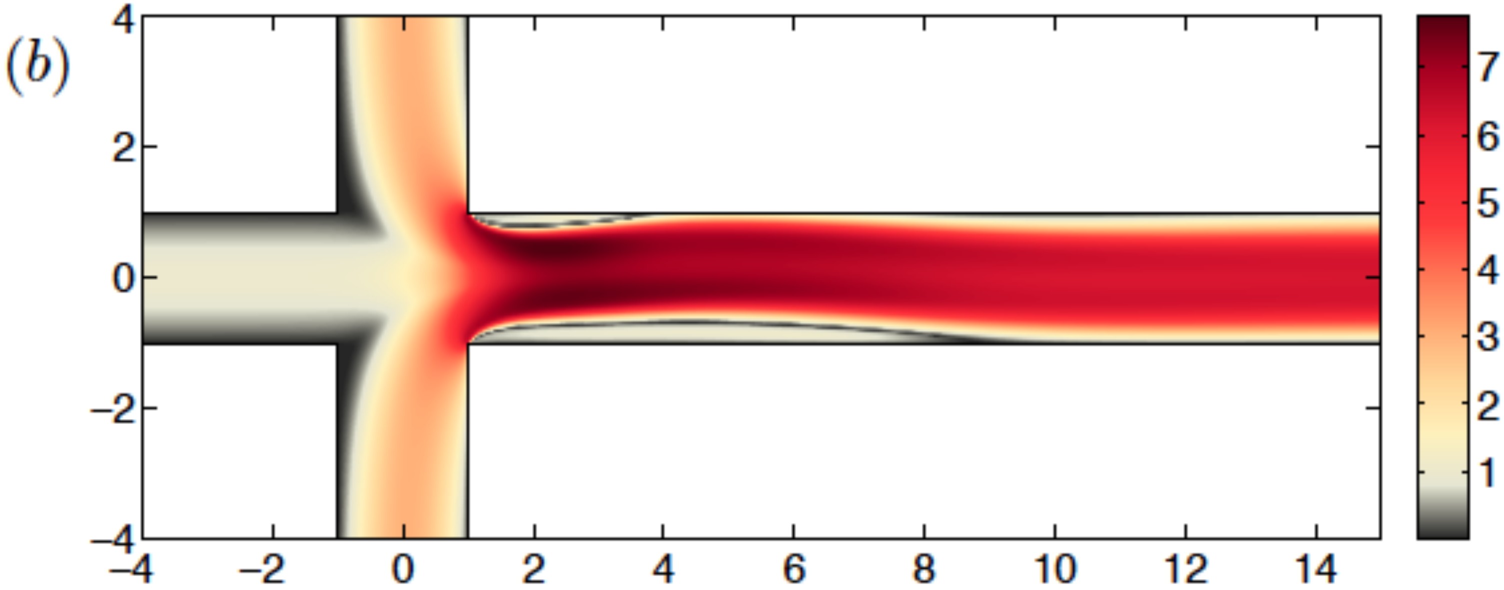


- Steady two-dimensional bifurcation

Lashgari et al, in preparation

Example in 2D: flow in X-junction

New asymmetric state

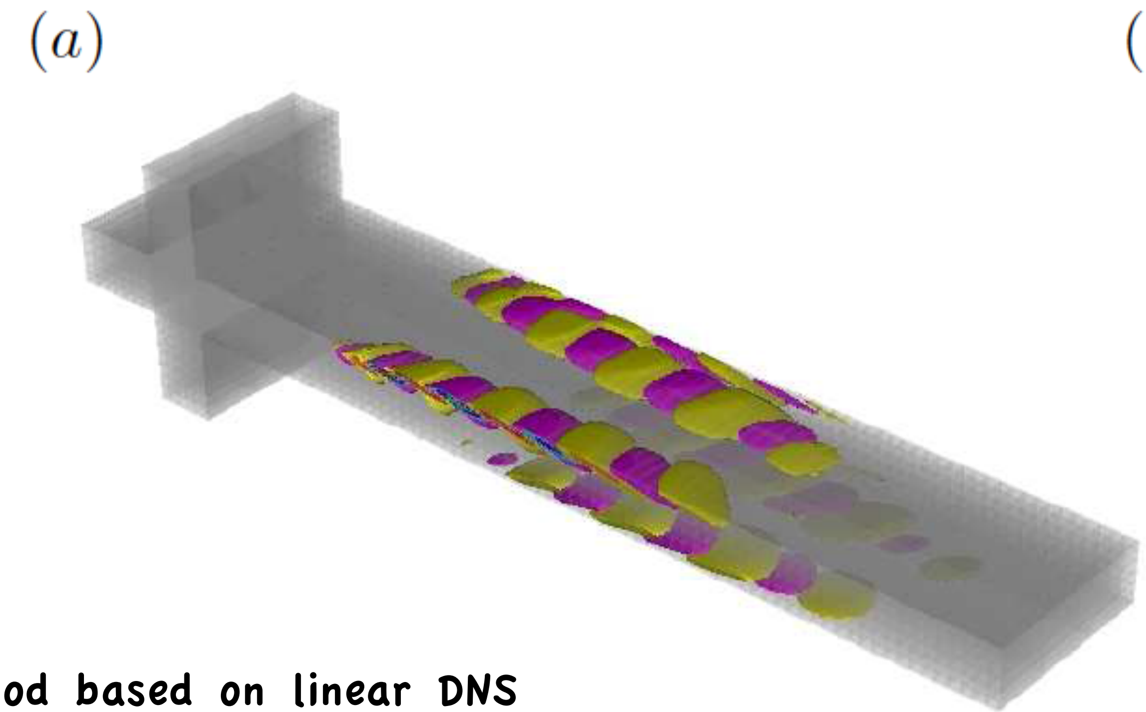


➢ Steady two-dimensional bifurcation

Lashgari et al, in preparation

X-junction: Example in 2,5D...

Asymmetric state unstable to 3D periodic disturbances

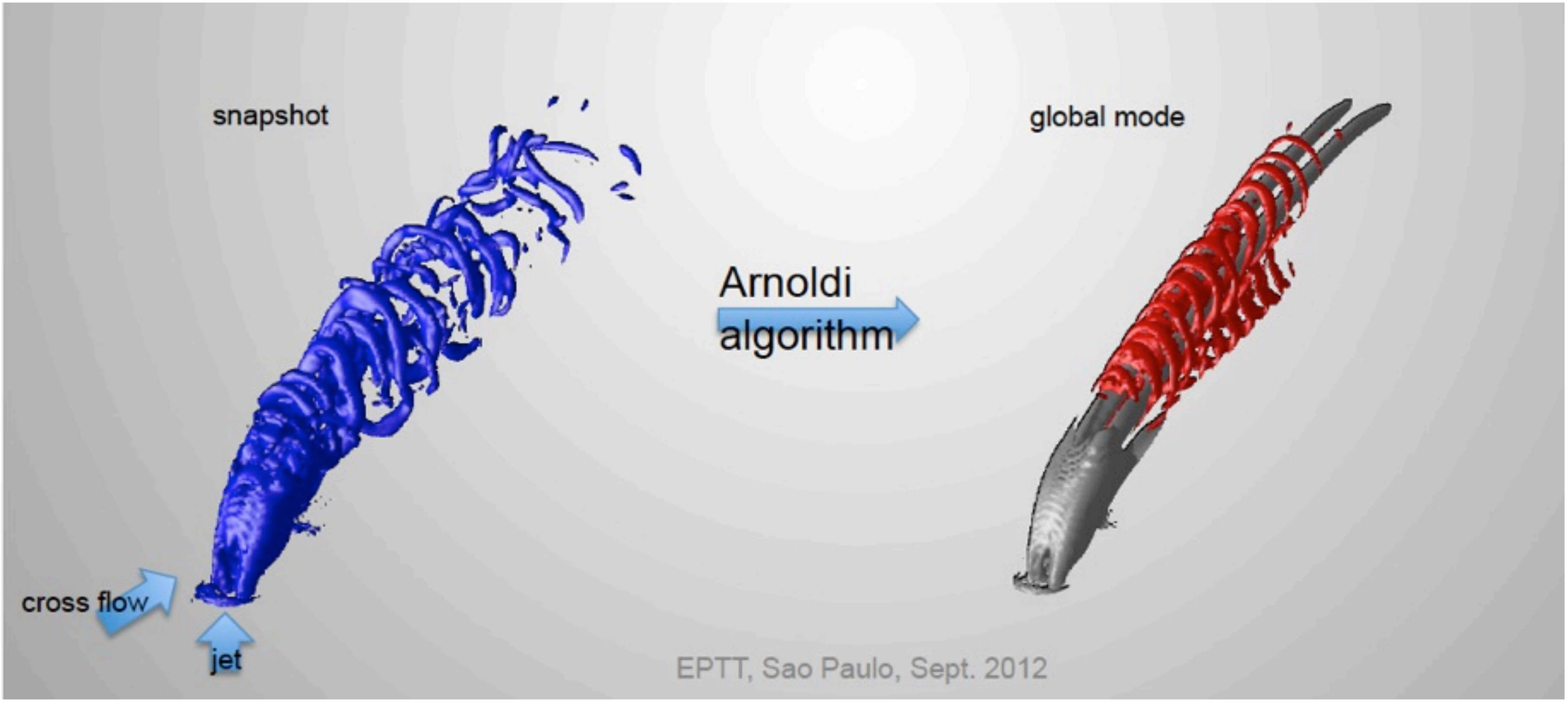


Snapshot method based on linear DNS

Lashgari et al, in preparation

Example in 3D: Jet in cross flow

Use DNS and compute spectrum of matrix exponential $q = \exp(tL)q_0$



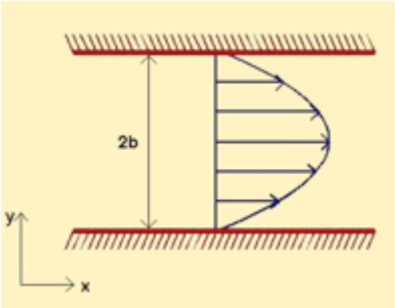
Bagheri et al, JFM, 2011, Schlatter et al.

Summary

- Stability of fluid systems
  - non-normal operators are ubiquitous in fluid flow problems
  - non-modal (multi-modal) effects therefore dominant
  - non-modal analysis computationally more costly, many extensions possible though

- Receptivity and sensitivity
  - Use of adjoint modes
  - Structural sensitivity, wavemaker
  - Base-flow sensitivity, passive control

Tutorials

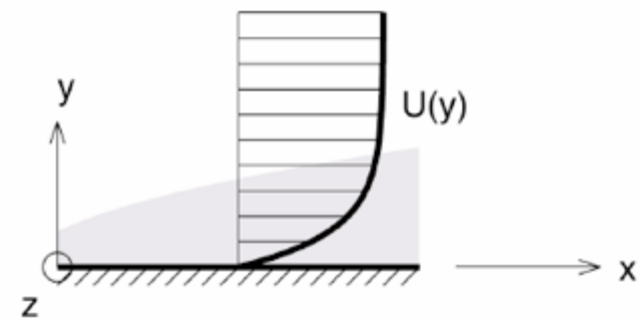


- **Consider Poiseuille or Couette flow**

- **Orr-Sommerfeld and Squire system for 3D disturbances**

Parallel shear flows $U_i = U(y)\delta_{1i}$

$$\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}$$



divergence of the momentum equations gives $\nabla^2 p = -2U'\frac{\partial v}{\partial x}$

elliminate pressure in v-equation $\Rightarrow$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4\right] v = 0$$

Parallel shear flows, cont

normal vorticity describes horizontal flow

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta = -U'\frac{\partial v}{\partial z}$$

with the boundary conditions

$v = v' = \eta = 0$ at a solid wall and in the far field

Orr-Sommerfeld and Squire equations

Assume wavelike solutions: $v(x,y,z,t) = \tilde{v}(y)\ e^{i(\alpha x + \beta z - \omega t)} \quad \Rightarrow$

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - k^2)^2\right]\tilde{v} \;=\; 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(D^2 - k^2)\right]\tilde{\eta} \;=\; -i\beta U'\tilde{v}$$

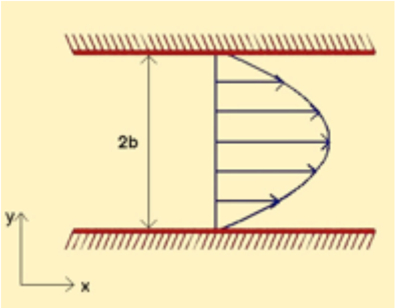
Orr-Sommerfeld modes: $\left\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\right\}_{n=1}^{N}$

Squire modes: $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^{M}$

$$\tilde{u} = \frac{i}{k^2}(\alpha D\tilde{v} - \beta\eta)$$
$$\tilde{w} = \frac{i}{k^2}(\beta D\tilde{v} + \alpha\eta)$$

Tutorials

- Consider Poiseuille or Couette flow

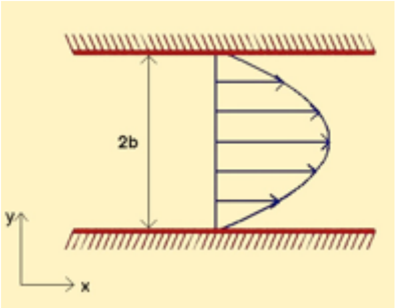


- **TransientGrowth.m**
  G(t) for selected α,β; Re and given max time

- **Resolvent.m**
  R(omega) for selected α,β; Re

- **NumRange.m**
  Numerical range for selected α,β; Re

Tutorials



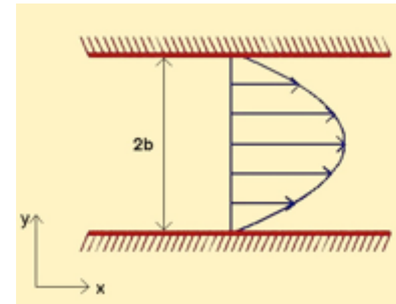
- Consider Poiseuille or Couette flow

- **Neutral_a_Re.m**
  Eigenvalues and G(t) for selected β and given max time, for a range of α and Re

- **Neutral_alpha_beta.m**
  Eigenvalues and G(t) for given Re and max time, for a range of α,β

- **OptimalDisturbance.m**
  Optimal disturbance and response for selected α,β; Re

Tutorials



- Consider Poiseuille or Couette flow

- **Sens_OptDist.m**
  Structural sensitivity of least stable mode and

$$\delta\lambda = \frac{q^+ \delta A q}{q^+ B q}$$

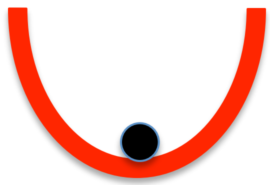
of optimal disturbance to base flow modification
for selected α,β; Re

$$\nabla_U \lambda^2 = 2\lambda^2 \Re\{(\nabla \mathbf{u}_{in})\, \mathbf{u}^*_{out} - (\nabla \mathbf{u}_{out})^H \mathbf{u}_{in}\}$$

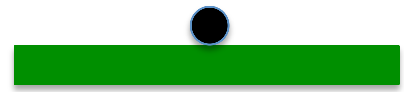
Brandt et al, JFM, 2011

Extra slides

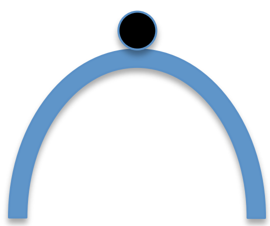
Stability



Stable



Neutral



Unstable

Nonlinearly
unstable

Stability definitions

$$E = \frac{1}{2}\int_V u_i u_i dV$$

Stable: $\lim_{t\to\infty} \frac{E(t)}{E(0)} \to 0$

Conditionally stable: $\exists\delta : E(0) < \delta \Rightarrow$ Stable

Globally stable: Conditionally Stable with $\delta \to \infty$

Linearly unstable: $\delta \to 0$

Monotonically stable: $\frac{dE}{dt} < 0 \quad \forall \quad t > 0$

Critical Reynolds numbers

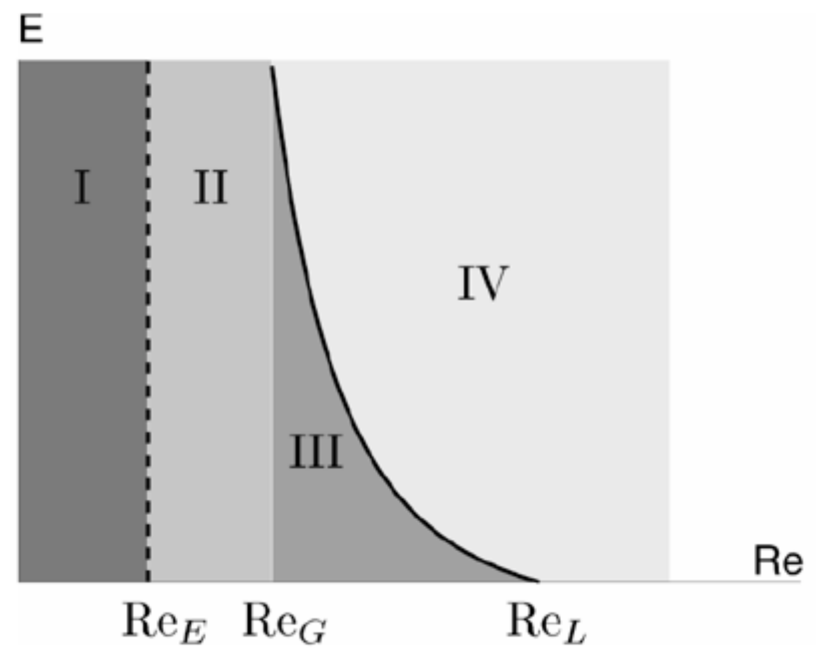
Flow monotonically stable $Re < Re_E$

Flow globally stable $Re < Re_G$

Flow linearly unstable $Re > Re_L$

Critical values for shear flows

Flow	Re_E	Re_G	Re_T	Re_L
Hagen-Poiseuille	81.5	—	2000	∞
Plane Poiseuille	49.6	—	1000	5772
Plane Couette	20.7	125	360	∞



Stability analysis

- Search for $Re_{E:}$ monotonically stable flows

- Linear analysis: Re_L

- Amplitude δ ? Re_G ?

Disturbance equations

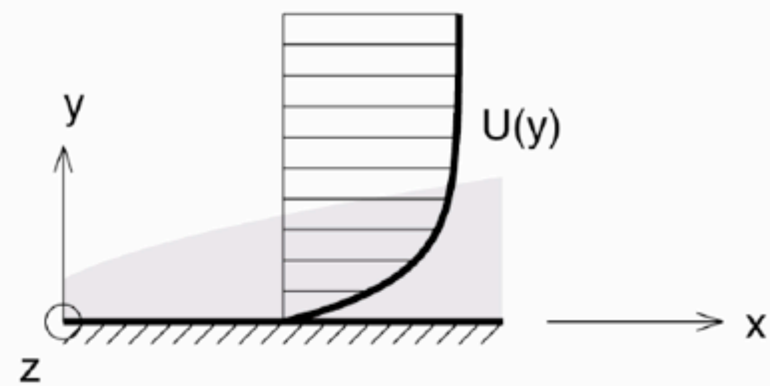
$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\nabla^2 u_i \\
\frac{\partial u_i}{\partial x_i} &= 0
\end{aligned}
$$

$$
\begin{aligned}
u_i(x_i, 0) &= u_i^0(x_i) \\
u_i(x_i, t) &= 0 \quad \text{on solid boundaries}
\end{aligned}
$$

$$
\mathrm{Re} = U_\infty \delta_* / \nu
$$

$$
\begin{aligned}
u_i &= U_i + u_i' \\
p &= P + p' \quad \text{drop primes}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -U_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\nabla^2 u_i - u_j\frac{\partial u_i}{\partial x_j} \\
\frac{\partial u_i}{\partial x_i} &= 0
\end{aligned}
$$



Energy equation

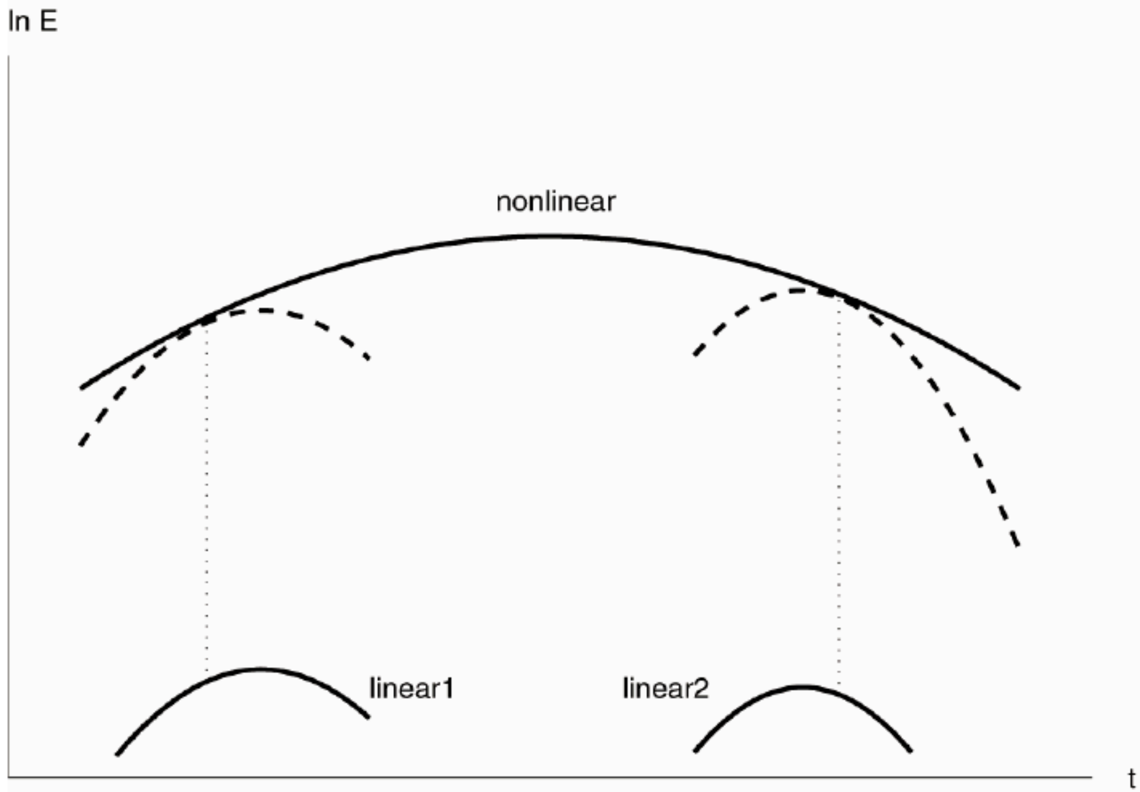
$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{\mathrm{Re}} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j} \left[-\frac{1}{2} u_i u_i U_j - \frac{1}{2} u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{\mathrm{Re}} u_i \frac{\partial u_i}{\partial x_j} \right] \\
&\Rightarrow \\
\frac{dE_V}{dt} &= -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{\mathrm{Re}} \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV
\end{aligned}
$$

Theorem: Linear mechanisms required for energy growth

Proof: $\frac{1}{E_V} \frac{dE_V}{dt}$ independent of disturbance amplitude

Linear growth mechanisms

$$\frac{1}{E_V}\frac{dE_V}{dt} = \frac{d}{dt}\ln E_V$$



Energy theory: Re_E

$$\frac{1}{Re_E} = \max_{u_i} \frac{-\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV}{\int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV}$$

Variational problem: very conservative estimate, need for constraints!

Stability analysis

- Search for $Re_{E:}$ monotonically stable flows

- Linear analysis: Re_L

- Amplitude δ ? Re_G ?

Linear analysis

$$\begin{aligned}\frac{\partial u_i}{\partial t} &= -U_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\nabla^2 u_i - u_j\frac{\partial u_i}{\partial x_j}\\ \frac{\partial u_i}{\partial x_i} &= 0\end{aligned}$$

$$\begin{aligned}u_i(x_i,0) &= u_i^0(x_i)\\ u_i(x_i,t) &= 0 \quad \text{on solid boundaries}\end{aligned}$$

Departure from equilibrium: slope

Identify relevant mechanisms

Examine receptivity and sensitivity

Linear analysis

$$\frac{\partial u}{\partial t} = \mathcal{L}(U, t; Re)\, u$$

• Time-independent problem: eigenvalue problem in time. Ex?

$$u(x_i, t) = \Re(\hat{u}(x_i) e^{\sigma t}), \quad \sigma u = \mathcal{L}(U; Re)\, u$$

• Classic 1d problem:
Orr-Sommerfeld, Squire system for parallel flows

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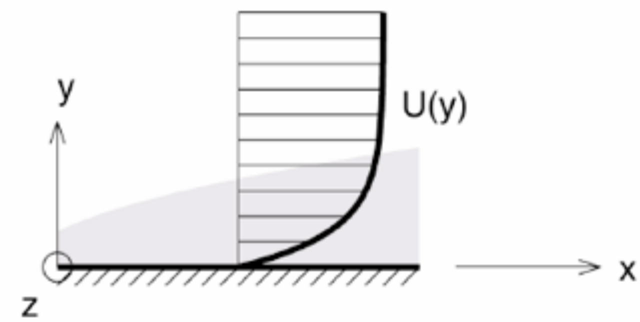

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Parallel shear flows, cont

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Squire modes: $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^{M}$

Nonmodal stability analysis

- Input-output approach
- Nonnormal operators
- Non-orthogonal eigenvectors

Matrix norm $\|G\| = \max_{\|w\|=1} \|Gw\|$

- Euclidean scalar product $\|w\|_2^2 = w_j^H w_j$
- Matrix as transformation with associated amplification

$$\frac{\|Gw\|_2}{\|w\|_2} = \left[\frac{w^H G^H G w}{w^H w}\right]^{1/2}$$

- Eigenvalues of $G^H G$ or singular values of G: $G^H G v_i = \lambda_i v_i$
- Largest amplification for the largest singular value: $\lambda_1,\ v_1$

$$\sqrt{\lambda_1} = \max_{\|w\|_2=1} \|Gw\|_2 = \frac{\|Gv_1\|_2}{\|v_1\|_2}, \quad Gv_1 = u_1$$

- Input and output basis $v_i,\ u_i$: $G = U \Lambda V^H$