NORDITA/FLOW Spring School on
Turbulent Boundary Layers

Lecture VII

Issues in DNS

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Pioneers

- Orszag & Patterson (1972) – *Homogeneous isotropic turbulence* [Fourier^3 spectral]
- Kim, Moin & Moser (1987) – R_τ=180 *turbulent plane channel/Poiseuille flow* [Fourier^2/Chebyshev-τ spectral]
- Spalart (1988) – R_θ=1410 *ZPG boundary layer* [Fourier^2/Jacobi spectral]

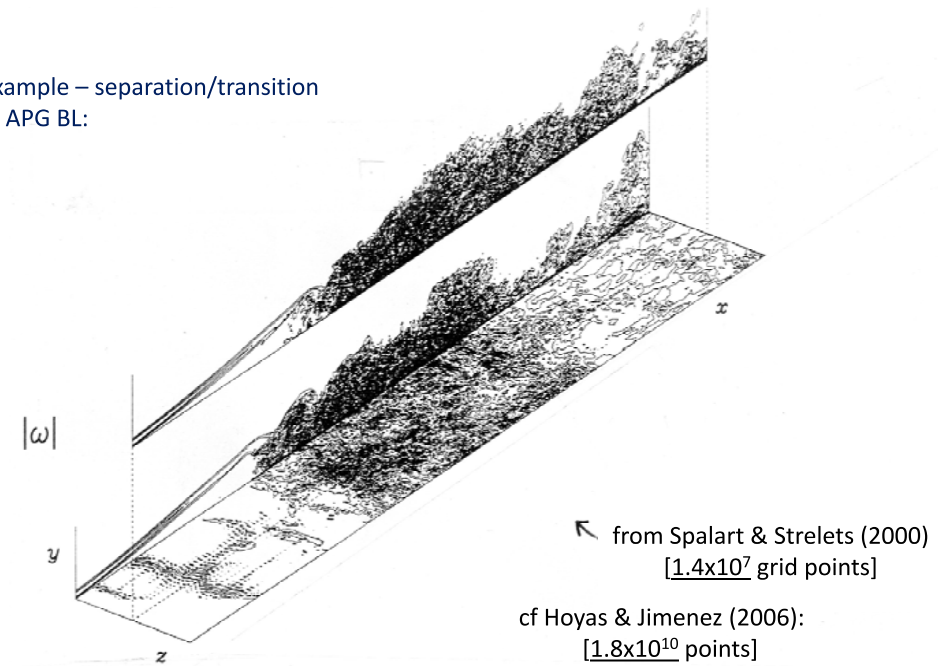
Overview

(following *"Primer on DNS of turbulence")*

- **Introduction***
  - **Context, seminal studies***
- **The Reynolds-number constraint***
- **Numerical considerations***
  - **Spatial discretisation***
    - **Spectral methods***
    - **Finite-difference methods***
  - (Temporal discretisation)
  - (Boundary/initial conditions)
  - (Code validation & resolution guidelines)
- (Outlook)
- **Case study: Near-wall similarity and DNS* (*Lecture VIIb*)**

*** Covered here**

Example – separation/transition in APG BL:



↖ from Spalart & Strelets (2000) [1.4x10^7 grid points]

cf Hoyas & Jimenez (2006): [1.8x10^{10} points]

Direct numerical simulation (DNS)

Conceptually straightforward:

- Discretise 3D Navier-Stokes equations on fine grid
- Time step resolving the fine-scale motion (*unsteady* questions must be solved)
- Collect statistics, etc just as for a short-duration experimental realisation of flow:

  DNS = `*numerical experiment*'

- Resolve all spatial & temporal scale of motion

Or LES…

DNS is not RANS…

DNS = 'Exact' high-fidelity CFD

Benefits:

- Capable of providing complete knowledge of turbulent flow
- Complete control of IC/BCs and **parameters & equations** (can "play with nature"!)

$\therefore$ *ideal research tool*

Drawbacks:

- Computationally *very* expensive, due to need to resolve fine-scale motion: **HPC required**
- Computational costs scale horribly with Reynolds number…

Assume LS & SS in equilibrium (eg that all scales have adjusted to present state of mean (ends))

$\Rightarrow$ ϵ = Rate at which TKE 'cascades from LS' ← i.e. **Flux** of TKE…

to be dissipated by ν at SS, and

that $l \sim q^3/\epsilon$ where $q \sim TKE^{1/2}$ is velocity scale of large eddies.

thus $N \sim \frac{l}{\eta} \sim \left(\frac{q^3}{\epsilon}\right)\left(\frac{\epsilon}{\nu^3}\right)^{1/4} = \frac{q^3}{(\epsilon\nu)^{3/4}}$

for turbulent flows we define $Re_T \equiv \frac{q\,l}{\nu} \sim \frac{q^4}{\epsilon\nu} \Rightarrow \frac{q^3}{(\epsilon\nu)^{3/4}} \sim \boxed{Re_T^{3/4} \sim N}$

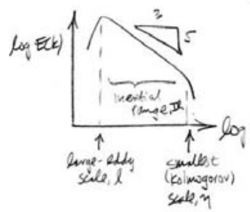
($l \sim q^3/\epsilon$, for LS/SS equil case)

Turbulence Reynolds no.

NB. In general $Re_T = Re_T(\underline{x},t)$ (ie is a time-dep 3D field variable), but since here $\frac{\partial(\text{anything})}{\partial x_j} = 0$ (due to 3D homogeneity) a single No. at each time characterises entire flow.

The Reynolds-number constraint:

Range of Scales in turbulent flow



$E(k)$ = energy spectrum

k = wavenumber, $2\pi/\lambda$

λ = wavelength of interest

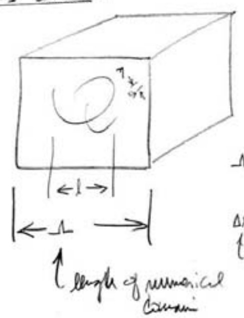
NB. as $Re \Rightarrow \infty$ IR widens

Recall $\eta = (\nu^3/\epsilon)^{1/4}$

ϵ = rate of TKE dissipation

Consider case with 3 homogeneous directions, (x_1, x_2, x_3):

DNS domain:



$\Lambda \sim l$ (size of largest eddies)

$\Delta x = \frac{\Lambda}{N} \sim \eta = (\nu^3/\epsilon)^{1/4}$ (grid spacing; No. of grid pts)

$\therefore N = \frac{\Lambda}{\Delta x} \sim \frac{l}{\eta}$

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since $N \sim \frac{q^3}{(\epsilon\nu)^{3/4}} \sim Re_T^{3/4}$

For complete DNS: 'cost' $\sim N^3 \cdot N_t$

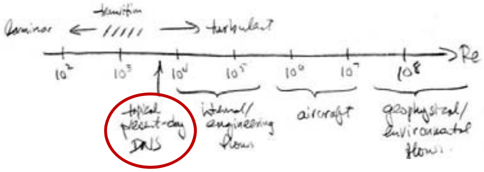
(Mem size, CPU-time) (x_1, x_2, x_3) (No. of time steps/'Numerical Exp')

where $N_t \sim \frac{T_{\text{Exp}}}{\Delta t} \sim \frac{1}{\Delta x} \sim N$

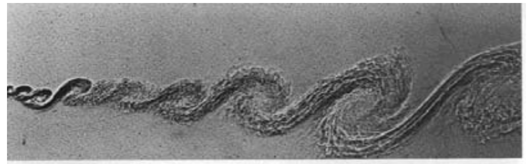
... and $\boxed{\text{cost} \sim \mathrm{Re}^3}$ ← yikes!

⚠ Thus to increase Re by 10,
cost goes up by 1000! (1day → 3 years!)

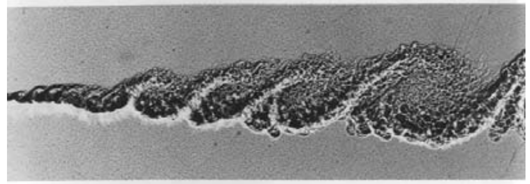
Different N=N(Re) relationships for other types of flows, but in general cost ~ Re^n where n~2-3. Consequently…



Re 'similarity'…



176. Large-scale structure in a turbulent mixing layer. Nitrogen above flowing at 1000 cm/s mixes with a helium-argon mixture below at the same density flowing at 380 cm/s under a pressure of 4 atmospheres. Spark shadow photography shows simultaneous edge and plan views, demonstrating the spanwise organization of the large eddies. The streamwise streaks in the plan view (of which half the span is shown) correspond to a system of secondary vortex pairs oriented in the streamwise direction. Their spacing at the downstream side of the layer is larger than near the beginning. *Photograph by J. H. Konrad, Ph.D. thesis, Calif. Inst. of Tech., 1976.*



177. Coherent structure at higher Reynolds number. This flow is as above but at twice the pressure. Doubling the Reynolds number has produced more small-scale structure without significantly altering the large-scale structure. *M. R. Rebollo, Ph.D. thesis, Calif. Inst. of Tech., 1976; Brown & Roshko 1974*

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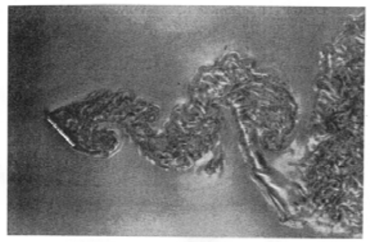
(from *An Album of Fluid Motion*, van Dyke 1982)

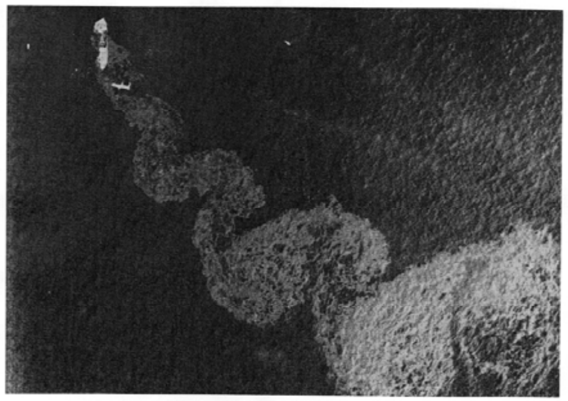
►Most engineering flows are beyond the reach of DNS – by a large factor (it is *not* a design tool)

BUT:

- DNS can give complete and accurate knowledge of turbulence at moderate Reynolds numbers, and is therefore *ideally suited for studying fundamental physics*, testing RANS/SGS models
- And results at lower Re can often be safely extrapolated to higher Re via *Reynolds-number similarity* (i.e. Re-independence of Large Scales)…



172. Wake of an inclined flat plate. The wake behind a plate at 45° angle of attack is turbulent at a Reynolds number of 4300. Aluminum flakes suspended in water show its characteristic sinuous form. Cantwell 1981. *Reproduced, with permission, from the* Annual Review of Fluid Mechanics, *Volume 13. © 1981 by Annual Reviews Inc.*



173. Wake of a grounded tankship. The tanker *Argo Merchant* went aground on the Nantucket shoals in 1976. Leaking crude oil shows that she happened to be inclined at about 45° to the current. Although the Reynolds number is approximately 10^7, the wake pattern is remarkably similar to that in the photograph at the top of the page. *NASA photograph, courtesy of O. M. Griffin, Naval Research Laboratory.*

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(from van Dyke 1982)

…Real-world relevance of DNS

- For phenomena and statistics associated with *largest scales* (mixing, transport), DNS at "high enough Re" captures typical high-Re behaviour
- And for *wall-bounded turbulence* (for which effective Re→0 as $y^+ \to 0$) DNS accurately represents at least *some* of the critical near-wall dynamics (streaks, etc) of high-Re flows
- For some cases (μAVs, very high Ma) DNS *can* reach design conditions

Numerical methods for DNS

▶ Must provide accurate realisation of the flow over wide range or length/time scales, plus accurate time history of the flow, to yield correct turbulence statistics and flow structures

– for *spatial discretisation*, traditional choice is ***spectral methods***… (see '*Primer*' for discussion of *finite-difference methods* and *temporal* discretisation)

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    - **Spectral methods**
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Spatial discretization options:

- Standard (low-order) finite-difference (FD)/finite volume (FV) methods -
  - \+ Allow complex geometries; gridding flexibility
  - – Accuracy often unacceptable (especially for *upwind schemes*)
- Spectral methods – [ideal for simple incompressible flows]
  - \+ Optimal accuracy-to-gridpoint ratio
  - – Limited to simple geometries; difficulties with discontinuities (e.g. shocks)
- High-order FD methods (e.g. Padè schemes) – [popular for computational aeroacoustics, compressible turbulence]
  - \+ Good compromise between accuracy and flexibility
  - – Coding more involved
- Finite-element/spectral-element methods – [emerging techniques for complex DNS: riblets, roughness elements, etc]
  - \+ Arbitrary geometries allowed via unstructured grids
  - – Unstructured grids lead to inefficient solution algorithms
- Viscous vortex methods – [current computational research topic]
  - \+ Can accomodate high-Re bluff-body flows
  - – Extension to 3D not straightforward

Spectral methods

▷ Definition:

Spectral methods ≡ spatial discretization via linear combination ('expansion') of global basis functions that are orthogonal eigenfunctions of an appropriate Sturm-Liouville problem.

$$\underbrace{Q(x)}_{\substack{\text{dep variable} \\ (\rho, u_i, T, \dots)}} = \underbrace{\sum_{n=0}^{N} \underbrace{a_n}_{\text{expansion coefs.}} \underbrace{\phi_n(x)}_{\text{S-L eigenfunctions}}}_{\substack{\text{linear combination of smooth functions, each of} \\ \text{which span entire flow domain}}}$$

Examples:

- $p=1,\ w=1,\ q=0 \Rightarrow \underbrace{\phi_n = e^{ik_n x}}_{\text{Fourier Series}},\quad k_n = \frac{2\pi n}{\Lambda},\ \Lambda = b-a$
- $p=\sqrt{1-x^2},\ w=\frac{1}{\sqrt{1-x^2}},\ q=0 \Rightarrow \phi_n = \underbrace{T_n(x)}_{\text{Chebyshev polynomials}}$
- $p=1-x^2,\ w=1,\ q=0 \Rightarrow \phi_n = \underbrace{L_n(x)}_{\text{Legendre polynomials}}$
- etc...

Sturm-Liouville ODE:

$$\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right] + \Big[\underbrace{\lambda_n}_{\text{eigenvalue}} \underbrace{w(x)}_{\text{weight function}} - q(x)\Big]\phi_n(x) = 0 \qquad (\text{over } a \le x \le b.)$$

where λ_n are countable (discrete) set of eigenvalues

$$\lambda_1, \lambda_2, \dots$$

And for each λ_n there is an associated eigenfunction ϕ_n

$\phi_1, \phi_2, \dots$ which satisfies

orthogonality relation:

$$\int_a^b \phi_n(x)\,\phi_m(x)\,w(x)\,dx = \alpha_n \delta_{nm}.$$

Given $Q(x)$, use orthogonality to find a_n:

$$Q(x) = \sum_{n=0}^{N} a_n \phi_n(x)$$

$$\Rightarrow w(x)\phi_m(x)\,Q(x) = \sum_{n=0}^{N} a_n w(x)\phi_m(x)\,\phi_n(x)$$

$$\Rightarrow \int_a^b w(\sigma)\phi_m(\sigma)\,Q(\sigma)\,d\sigma = \sum_{n=0}^{N} a_n \underbrace{\int_a^b w(\sigma)\phi_m(\sigma)\,\phi_n(\sigma)\,d\sigma}_{\alpha_n \delta_{nm}}$$

$$= \sum_{n=0}^{N} a_n \alpha_n \delta_{nm} = a_m \alpha_m$$

$$\Rightarrow \boxed{a_m = \frac{1}{\alpha_m}\int_a^b w(\sigma)\phi_m(\sigma)\,Q(\sigma)\,d\sigma}$$

$m = 0, 1, \dots, N$

Spectral DNS:

- $u(x,t) \leftarrow u_N(x,t) = \sum_{n=0}^{N} \underset{\substack{\uparrow \\ \text{time}}}{a_n(t)} \, \overset{\substack{\text{space} \\ \downarrow}}{\phi_n(x)}$

- Into N-S eqn $\mathcal{L}(u) = 0$, yields non-zero

  Residual $R_N = \mathcal{L}(u_N) \leftarrow$ (error)

- Minimise error by setting weighted integral of R_N to zero:

$$\int_a^b w(x)\, \phi_m(x)\, R_N \, dx = 0 \quad \text{for } m = 0, 1, \ldots N$$

$$\Downarrow$$

$$[A] \frac{d\vec{a}}{dt} = [B]\, \vec{a} + \vec{f}$$

$$[A] \frac{d\vec{a}}{dt} = [B]\, \vec{a} + \vec{f}$$

$(N+1) \times (N+1)$ matrices (elements defined analytically via orthogonality)

includes nonlinear convective term
∴ pseudo-spectral method (Orszag 1971)

- Apply time-marching algorithm (Runge-Kutta, Crank-Nicholson, ...) to $(N+1)^2$ system to solve for $\vec{a} = [a_0, a_1, \ldots, a_N]^T$ at new time

- Transform $\vec{a}$ to physical space $\vec{u}(\underline{x})$ and compute nonlinear/convective terms

- Transform nonlinear terms to wave space, form RHS of $(N+1)^2$ system

- Repeat...

▷ Features of spectral methods

- If basis-function variations correctly 'fit' dependent-variable variations (as does Fourier series for statistically homogeneous directions), then:

  – the expansion coefficients a_n go to zero as $N \to \infty$,

(cf. FDA: error $\sim (\Delta x)^p \sim (1/N)^p$)

∴ Spectral methods have 'exponential' or '∞-order' convergence.

and – they do so such that discretisation error decreases faster than any(!) power of $1/N$.

(apply integration by parts repeatedly to

$$a_n = \frac{1}{\alpha_n}\int_a^b Q(x)\, w(x)\, \phi_n(x)\, dx$$

$$\Downarrow$$

$\boxed{\lim_{N\to\infty} \left(N^{m+1} a_N\right) \to 0}$, where m is order of highest derivative of $Q(x)$

∴ If Q is ∞ly differentiable ($m=\infty$), then $a_N \to 0$ faster than any power p of $1/N$.)

- Verification that $|a_n| \to 0$ as $n \to N$ provides 'a posteriori' check that spatial resolution is sufficient (∴ no grid refinement study needed to ensure spectral DNS results are 'broadly correct')

(BUT see *Lecture VIIb* Case Study)

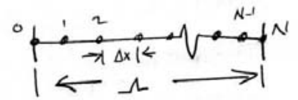
▷ Fourier series [the classic/quintescential spectral method]

Fourier Interpolant : $u(x) = \sum_{l=0}^{N-1} \hat{u}_l e^{i k_l x}$ where $k_l = \frac{2\pi l}{\Lambda}$ ($i = \sqrt{-1}$)

x: continuous variable 'physical space'; $\hat{u}_l$: Fourier coefs 'wave space'

and $u(0) = u(\Lambda)$

At discrete points $x_j = j\Delta x = j\Lambda/N$



we have $u(x_j) \equiv u_j = \sum_{l=0}^{N-1} \hat{u}_l e^{i\frac{2\pi l}{\Lambda} j \frac{\Lambda}{N}}$

$$\Rightarrow \quad \boxed{u_j = \sum_{l=0}^{N-1} \hat{u}_l e^{2\pi i l j/N}}$$

Discrete Fourier Trans.

(1) DFT

'Backward' transf. (wave → phys space)

$$(1) \Rightarrow \sum_{j=0}^{N-1} u_j e^{-2\pi i n j/N} = \sum_{l=0}^{N-1} \hat{u}_l \boxed{\sum_{j=0}^{N-1} e^{2\pi i j(l-n)/N}} \qquad \langle \cdot \rangle$$

Recall Geometric Series : $\sum_{j=0}^{N-1} z^j = \begin{cases} \frac{z^N - 1}{z - 1} & \text{for } |z| \neq 1 \\ N & \text{for } |z| = 1 \end{cases}$

Here we have $z = e^{2\pi i (l-n)/N} = \begin{cases} \neq 1 & l-n \neq pN \\ 1 & l-n = 0 \end{cases}$ $p = 0, \pm 1, \pm 2, \dots$

thus $\boxed{\sum_{j=0}^{N-1} \left[e^{2\pi i (l-n)/N} \right]^j} = \begin{cases} \frac{e^{2\pi i (l-n)} - 1}{e^{2\pi i (l-n)/N} - 1} = 0 & l-n \neq pN \\ N & l-n = 0 \end{cases}$

$$= \boxed{N \delta_{ln}}$$

so that ⟨i⟩ $\Rightarrow \sum_{j=0}^{N-1} u_j e^{-2\pi i n j/N} = \underbrace{\sum_{l=0}^{N-1} \hat{u}_l N \delta_{ln}}_{\text{(discrete form of \underline{orthogonality})}} = N \hat{u}_n$

$$\therefore \boxed{\hat{u}_n = \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-2\pi i n j/N}} \quad (2)$$

Forward DFT (phys => wave)

▷ Fast DFT (FFT) Cooley-Tukey (1965)

Let $N = 2^m \Rightarrow N \in [1, 2, 4, 8, 16, 32, 64 \ldots]$

$$u_j = \underbrace{\sum_{l=0}^{N-1} \hat{u}_l e^{2\pi i l j/N}}_{\mathcal{O}(N^2)} = \sum_{\substack{l=0 \\ \text{even}}}^{N-2} \hat{u}_l e^{2\pi i l j/N} + \sum_{\substack{l=1 \\ \text{odd}}}^{N-1} \hat{u}_l e^{2\pi i l j/N}$$

$\langle l = 2n \rangle \qquad \langle l = 2n+1 \rangle$

$$= \sum_{n=0}^{N/2-1} \hat{u}_{2n} e^{2\pi i 2n j/N} + \sum_{n=0}^{N/2-1} \hat{u}_{2n+1} e^{2\pi i (2n+1) j/N}$$

$\langle$ [2 DFTs of half the length] $\rangle \Rightarrow$
$$= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} \hat{u}_{2n} e^{2\pi i n j/(N/2)}}_{\mathcal{O}[(N/2)^2]} + e^{2\pi i j/N} \underbrace{\sum_{n=0}^{\frac{N}{2}-1} \hat{u}_{2n+1} e^{2\pi i n j/(N/2)}}_{\mathcal{O}[(N/2)^2]}$$

---

wrt DFT…

Note:

- (1) and (2) are a Fwrd/Bkwrd transform pair
- Forward and Backward DFT are both Matrix-Vector Multiply operations $[\mathbf{M}]\{\mathbf{V}\}$ with:

  $M_{nj} = (e^{\pm 2\pi i/N})^{nj}$ and $v_j = \hat{u}_j$ or u_j

- Thus, obtaining the N $\hat{u}_n$'s from the u_j's (or *vice versa*) requires $\mathcal{O}(N^2)$ multiply-add operations

  ∴ **Very expensive…**

Strategy: Further decompose each of smaller DFTs as before until there are N single-pt transforms to perform!

Cost: N (single-pt transform) + a 'reshuffling' (recombination) of N costs for each splitting.

Since $N = 2^m$ ($\Rightarrow m = \log_2 N$) there are a total of $\log_2 N$ splittings

∴ total cost of FFT scales as $\boxed{N \cdot \log_2 N}$. —A

(Makes Fourier-spectral DNS feasible)

▷ Periodicity of FT → negative wave numbers

$$u_j = \sum_{l=0}^{N-1} \hat{u}_l e^{2\pi i j l/N} = \sum_{l=0}^{N/2-1} \hat{u}_l e^{2\pi i j l/N} + \underbrace{\sum_{l=N/2}^{N-1} \hat{u}_l e^{2\pi i j l/N}}$$

(let $p = l - N$, $l = p + N$)

$$\sum_{p=-N/2}^{-1} \hat{u}_{N+p} e^{2\pi i j (p+N)/N}$$

$$= \sum_{p=-\frac{N}{2}}^{-1} \hat{u}_{N+p} \overbrace{e^{2\pi i j}}^{1} e^{2\pi i j p/N} = \sum_{p=-\frac{N}{2}}^{-1} \hat{u}_p e^{2\pi i j p/N}$$

Notation Convention: $\hat{u}_{N+p} \leftarrow \hat{u}_p$ for $-\frac{N}{2} \le p \le -1$

NB. The highest wavenumber that can be captured on a given uniform grid is $\boxed{k_{max} = \frac{\pi}{\Delta x}}$

(cf. Nyquist frequency from signal processing)

$$\frac{2\pi(N/2)}{\Lambda} = \pi N/\Lambda = \pi/\Delta x$$

$$\therefore u_j = \sum_{l=0}^{N/2-1} \hat{u}_l e^{2\pi i j l/N} + \sum_{p=-N/2}^{-1} \hat{u}_p e^{2\pi i j p/N}$$

or: $u_j = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_l e^{2\pi i j l/N}$ ← Conventional form.

Negative wave numbers allow straightforward connection between $|l|$ (ie $|k_l| = \left|\frac{2\pi l}{\Lambda}\right|$) and number of oscillations

And if grid spacing Δx is too coarse for the spatial variations in $0 \le x \le \Lambda$ (i.e. if k_{max} of the grid is smaller than the actual max k), then the Fourier Coefficients that are supported by the grid will be corrupted by Aliasing Errors.

Implication for DNS:

Nonlinear/convective terms in N-S eqns increase the span of wavenumbers from

$$\underbrace{-\frac{N}{2} \le n \le \frac{N}{2}}_{\text{(span reqd to capture all dep. var. variations)}} \quad \text{to} \quad \underbrace{-N \le n \le N}_{\text{(Span of NL terms)}}$$

Must be resolved by numerical scheme too!

Cure:

Increase resolution of grid when computing the nonlinear terms: if <u>1.5×N</u> collocation/quadrature points are used, results free from aliasing errors (AKA '$\frac{3}{2}$-rule').

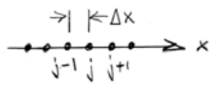
(For <u>compressible</u> flows, for which $(NL)_i = \partial \rho u_i u_j / \partial x_j$ and RHS of N-S Eqns involve ρ', the situation is much worse, and cannot be exactly cured, by say $(\frac{3}{2})^2$-rule.)

▷ Fourier series can be used to analyze <u>other spatial discretizations</u> via <u>modified</u>/effective <u>wavenumber k^*</u>

<u>Example</u>: consider <u>upwind FDA</u> $\left.\frac{\Delta u}{\Delta x}\right|_{x_j} = \frac{u_j - u_{j-1}}{\Delta x}$

and use it to approximate derivatives of <u>homogeneous/periodic</u> field on <u>uniform</u> grid Δx

<u>Actual</u> field $u(x) = \sum_k \hat{u}(k)\, e^{ikx}$



$x_j = j\Delta x$ ← constant

Apply upwind FDA:

$$\left.\frac{\Delta u}{\Delta x}\right|_{x_j} = \frac{u_j - u_{j-1}}{\Delta x} = \frac{1}{\Delta x}\sum_k \hat{u}(k)\left[e^{ikx_j} - e^{ik\overset{x_j - \Delta x}{x_{j-1}}}\right]$$

$$= \frac{1}{\Delta x}\sum_k \hat{u}(k)\, e^{ikx_j}\left[1 - e^{-ik\Delta x}\right]$$

$$\therefore \left.\frac{\Delta u}{\Delta x}\right|_{x_j} = \sum_k \underbrace{\frac{1}{\Delta x}\left(1 - e^{-ik\Delta x}\right)}_{ik^*} \hat{u}(k)\, e^{ikx_j}$$

cf. <u>Actual</u> $\left.\frac{du}{dx}\right|_{x_j} = \sum_k ik\,\hat{u}(k)\, e^{ikx_j}$

define <u>modified wavenumber k^*</u>

For <u>this</u> FDA:

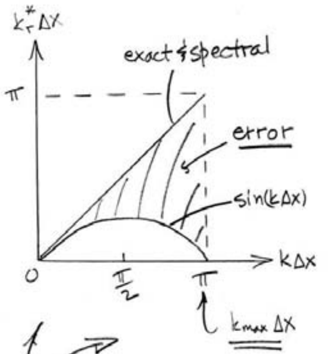
$$ik^* = \frac{1}{\Delta x}\left[1 - \underbrace{(\cos(k\Delta x) - i\sin(k\Delta x))}_{e^{-ik\Delta x}}\right]$$

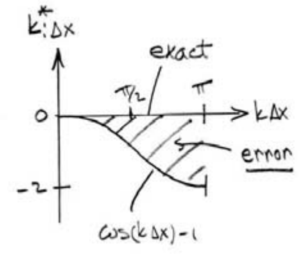
$$= \frac{1}{\Delta x}\left[1 - \cos(k\Delta x) + i\sin(k\Delta x)\right]$$

thus $k^*\Delta x = \underbrace{\sin(k\Delta x)}_{k_r^*\Delta x} + i\underbrace{(\cos(k\Delta x) - 1)}_{k_i^*\Delta x}$

<u>Real</u> part k_r^* gives <u>dispersion (phase) error</u>

<u>Imag</u> part k_i^* gives <u>dissipation error</u>





⚠* Implies only lowest wave numbers $k < \pi/2\Delta x$ (i.e. smoothest, large-scale varaitions) are treated correctly by this upwind scheme

Note: Modified wavenumber analysis can also be used to quantify errors associated with FDA of other derivatives (e.g. 2nd derivative):

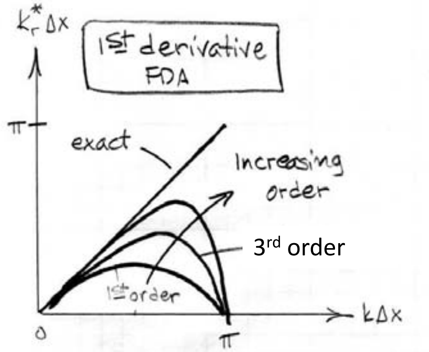
exact: $\widehat{\dfrac{d^2u}{dx^2}} = -k^2\hat{u}$

$\updownarrow$ ⚠*

FDA: $\widehat{\dfrac{\Delta^2 u}{\Delta x^2}} = -k_*^2\hat{u}$

defines modified wavenumber associated with 2nd derivative FDA used.

In general:



Example (cont'd)...

... Apply upwind scheme to 1D wave eqn (linear version of LHS of N-S eqns):

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} + \underset{\text{const.}}{c} \dfrac{\partial u}{\partial x} = 0 \\ u(x,t) = \sum_k \hat{u}(k,t)e^{ikx} \end{array}\right] \Rightarrow \sum_k \frac{d\hat{u}}{dt}e^{ikx} + \sum_k cik\hat{u}e^{ikx} = 0$$

$$\Rightarrow \frac{d\hat{u}}{dt} + ick\hat{u} = 0 \Rightarrow \boxed{\hat{u}(k,t) = \hat{u}(k,0)e^{-ickt}}$$

exact

⚠

For FDA: $\dfrac{\partial u}{\partial t} + c\dfrac{\delta u}{\delta x} = 0 \rightarrow \boxed{\hat{u}(k,t) = \hat{u}(k,0)e^{-ick^*t}}$

Note: this analysis considers only spatial FDA errors - No account taken of errors due to time-marching scheme. (see 'Primer')

Since $k^* = k_r^* + i k_i^*$ we have

$$\hat{u}(k,t) = \hat{u}(k,0)\, e^{-\alpha^* t}\, e^{-i\omega^* t}$$

with <u>decay rate</u> $\alpha^* = -c k_i^* = c\left[\dfrac{1-\cos(k\Delta x)}{\Delta x}\right] = \alpha^*(k)$

and <u>modified/effective phase speed</u> $\omega^* = c k_r^* = c\left[\dfrac{\sin(k\Delta x)}{\Delta x}\right] = \omega^*(k)$

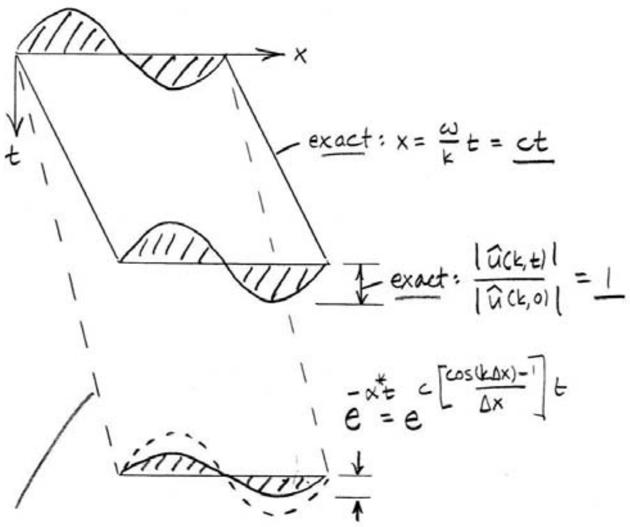
<u>Note</u> that decay rate and phase speed introduced by FDA both depend on wavenumber.

<u>Dispersion/phase error</u>:

<u>exact</u>: $x = ct$

<u>FDA</u>: $x = ct\,\dfrac{\sin(k\Delta x)}{k\Delta x} \rightarrow \begin{cases} ct & \text{as } k \to 0 \quad (\text{OK}) \\ 0 & \text{as } k \to \pi/\Delta x \quad (\ddot{\frown}) \end{cases}$

Thus the <u>higher</u> the k (smaller the wavelength, spatial scales) the <u>slower</u> it <u>convects</u>. <u>Dispersion error</u>



Upwind FDA:

$$x = \frac{\omega^*}{k} t = ct\,\frac{\sin(k\Delta x)}{k\Delta x}$$

<u>Dissipation error</u>:

<u>exact</u>: $e^{-\alpha t} = 1$

<u>FDA</u>: $e^{-\alpha^* t} = e^{c\left[\frac{\cos(k\Delta x)-1}{\Delta x}\right]t} \rightarrow \begin{cases} 1 & \text{as } k \to 0 \quad (\text{OK}) \\ 0 & \text{as } k \to \frac{\pi}{\Delta x} \quad (\ddot{\frown}) \end{cases}$

$\therefore$ the <u>higher</u> the k, the <u>faster</u> it <u>decays</u>. <u>Dissipation error</u>

Conclusions

- Upwind schemes do not represent the largest resolvable wavenumbers correctly
  - Much finer grid required to match spectral methods
  - Apply to turbulence simulation with caution!
- Do not judge the accuracy/fidelity of a DNS (or *any* CFD solution) solely by the total number of grid points

Near-wall similarity and DNS:

1. **The resilience of the logarithmic law to pressure gradients – Evidence from DNS**
2. **The hunt for κ**

Near-wall similarity and DNS

Lecture VIIb

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Engineering and Physical Sciences Research Council

CAPABILITY COMPUTING

Issues

- How ***Robust*** and ***universal*** is the ***log law***?...

  ...Effect of:
  - Flow type/boundary conditions?
  - Reynolds number?
  - Pressure gradients (i.e. $d\tau/dy \neq 0$)?

Approach

- Fully spectral DNS of:
  - Plane Couette-Poiseuille flow
  - Pressure-driven Ekman layer (3DBL)

ISL Assumptions: $U = U(y, \tau_w, \nu) \overset{DA}{\Rightarrow} \frac{dU}{dy} = \frac{u_\tau}{y} f(y^+)$

where $u_\tau^2 = \tau_w/\rho$

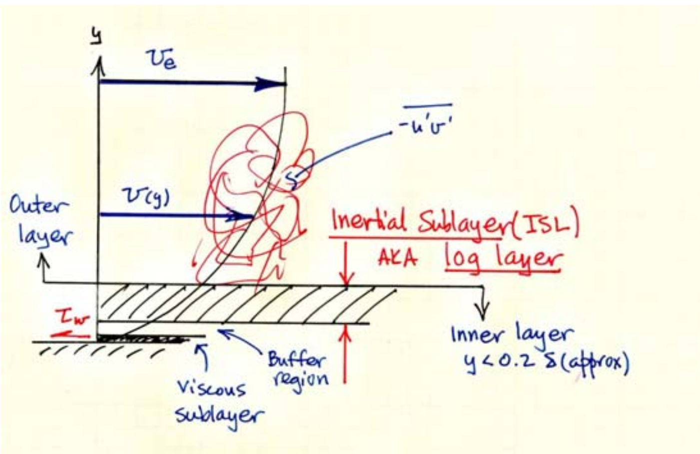
and $y^+ = y u_\tau/\nu$

As $y^+ \to$ 'large enough', effect of ν becomes negligible, and $f(y^+) \to \frac{1}{\kappa}$, where κ is (presumably) a universal constant

$$\therefore \frac{dU}{dy} = \frac{u_\tau}{\kappa y} \Rightarrow U = \frac{u_\tau}{\kappa} \ln y^+ + C$$

Option (1): $\boxed{\frac{u_\tau}{dU/dy} = \kappa y}$ "(ISL1)"

Review: *Near-wall similarity* of low-speed zero-pressure-gradient (ZPG) turbulent boundary layers:



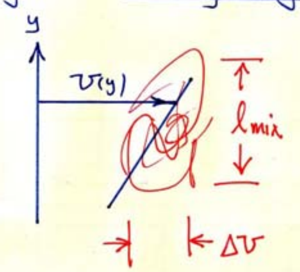
Recall Eddy viscosity: $-\overline{u'v'} = \nu_T \frac{dU}{dy}$

$$\Rightarrow \nu_T = \frac{-\overline{u'v'}}{dU/dy}$$

ISL Assumptions: $-\overline{u'v'} = u_\tau^2$, $dU/dy = u_\tau/\kappa y$ $\Rightarrow \nu_T = u_\tau \kappa y$

$\rightarrow$ Option (2): $\boxed{\frac{-\overline{u'v'}/u_\tau}{dU/dy} = \kappa y}$ "(ISL2)"

Algebraic mixing length:



Assume $u' \sim \Delta V \equiv \ell_{mix} \left| \frac{dV}{dy} \right|$

and $v' \sim -u' = -\Delta V$

$\Rightarrow -\overline{u'v'} \sim (\Delta V)^2$

$\therefore \boxed{-\overline{u'v'} = \ell_{mix}^2 \left(\frac{dV}{dy} \right)^2}$

$\Rightarrow \boxed{\ell_{mix} \equiv \frac{\sqrt{-\overline{u'v'}}}{dV/dy}}$

and $\boxed{\nu_T = \ell_{mix}^2 \frac{dV}{dy}}$

Note: $-\overline{u'v'} = u_\tau^2 \Rightarrow \boxed{\frac{d\tau}{dy} = 0}$ (i.e. ZPG BL only)

Recap: Near-wall/ISL similarity:

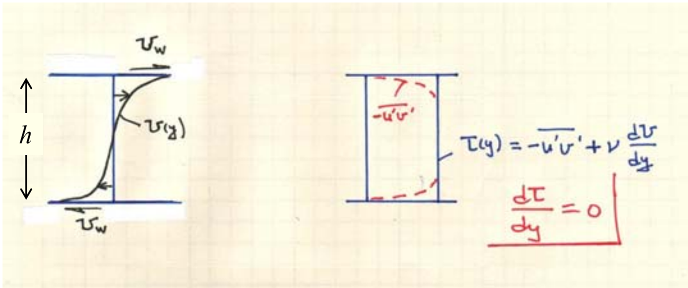
$$\underbrace{\frac{u_\tau}{dV/dy}}_{\text{log law}} = \underbrace{\frac{-\overline{u'v'}/u_\tau}{dV/dy}}_{\text{eddy viscosity}} = \underbrace{\frac{\sqrt{-\overline{u'v'}}}{dV/dy}}_{\text{mixing length}} = \kappa y \quad \text{for ZPG}$$

(1) (2) (3)

ISL Assumptions: $\left.\begin{array}{l} \ell_{mix} \sim y \\ -\overline{u'v'} = u_\tau^2 \\ \frac{dV}{dy} = \frac{u_\tau}{\kappa y} \end{array}\right] \Rightarrow \ell_{mix} = \frac{u_\tau}{u_\tau/\kappa y} = \kappa y$ $\left(\ell_{mix} \equiv \frac{\sqrt{-\overline{u'v'}}}{dV/dy} \right)$

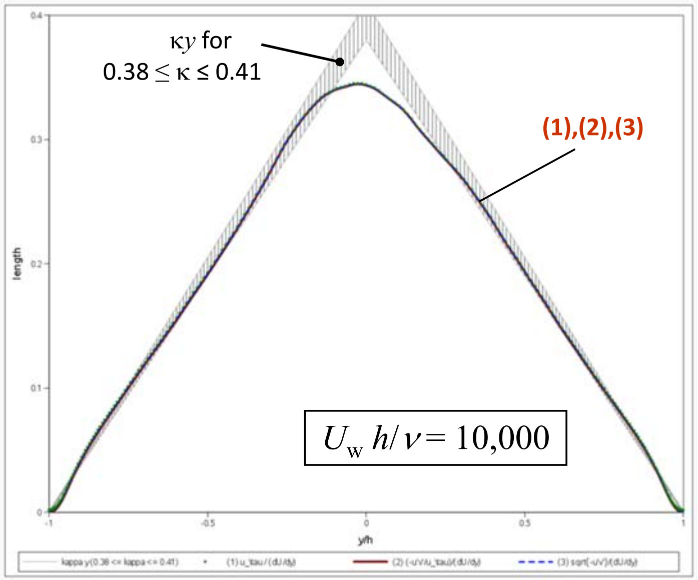
$\rightarrow$ Option (3): $\boxed{\frac{\sqrt{-\overline{u'v'}}}{dV/dy} = \kappa y}$ "(ISL 3)"

Compare with **DNS of plane turbulent Couette flow**:



(Results courtesy of R Johnstone, U Southampton, 2009)

Couette flow DNS ($\mathrm{d}P/\mathrm{d}x=0$)



(✓)

Options:

(1) $\dfrac{u_\tau}{d\bar{U}/dy} \stackrel{?}{=} \kappa y$

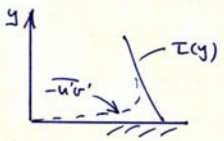
(2) $\dfrac{-\overline{u'v'}/u_\tau}{d\bar{U}/dy} \stackrel{?}{=} \kappa y$

$\Big\}$ depend on u_τ (i.e. τ_w)

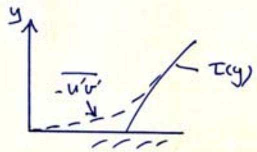
(3) $\dfrac{\sqrt{-\overline{u'v'}}}{d\bar{U}/dy} \stackrel{?}{=} \kappa y$ ← local quantity

What happens when $dP/dx \neq 0$?

FPG: $\dfrac{d\tau}{dy} < 0$

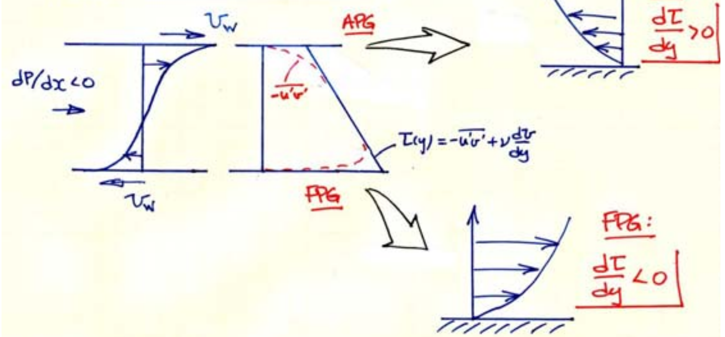


APG: $\dfrac{d\tau}{dy} > 0$

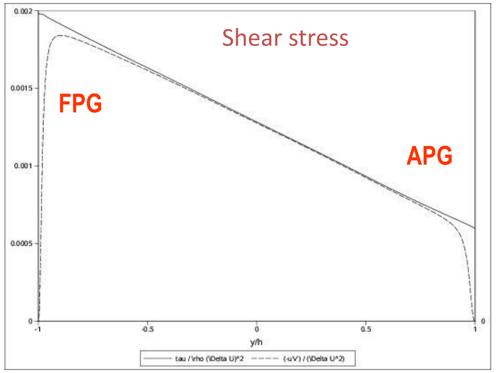


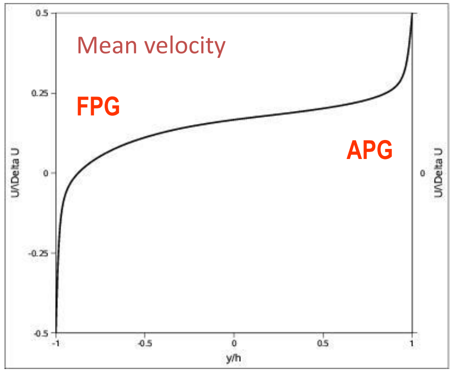
Motivates DNS of:

Plane Poiseuille-Couette Flow:



Couette-Poiseuille Flow

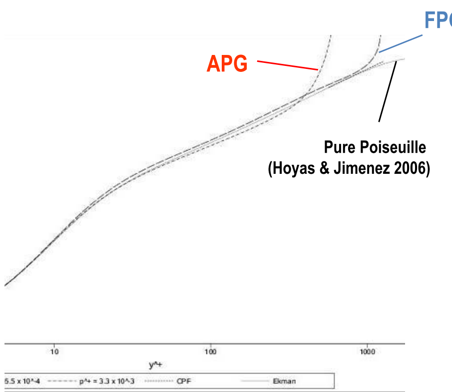


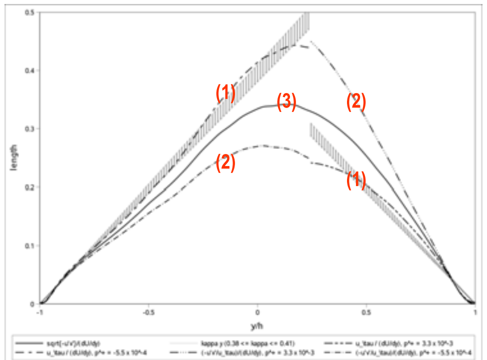


Conclusion:

The ***log law is more resilient to pressure gradients*** than eddy-viscosity and mixing length (Johnstone *et al* 2010; cf Galbraith, Sjolander & Head 1977, and Huang & Bradshaw 1995)

Couette-Poiseuille: The Bottom Line





- LHS: Both walls "trying hard" to have log law…
- RHS: Three similarity candidates vs κy…
  - Theory with $0.38 \le \kappa \le 0.41$
  - The Log Law (1) is clear winner
  - Qualitative (rather than sharp, asymptotic) but important result

Recap: Log law characteristics:

Q1. How robust?

A. Quite!

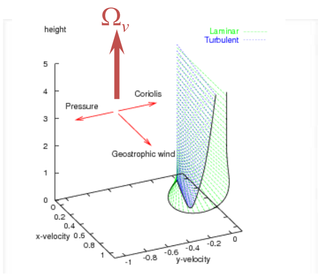
Q2. How universal? (i.e. is κ ***constant***?)

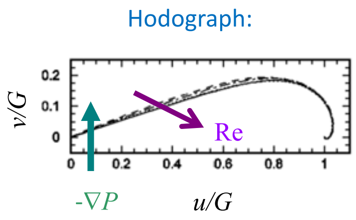
A. TBD…

…What does DNS imply?...

Turbulent (pressure-driven) Ekman layer:

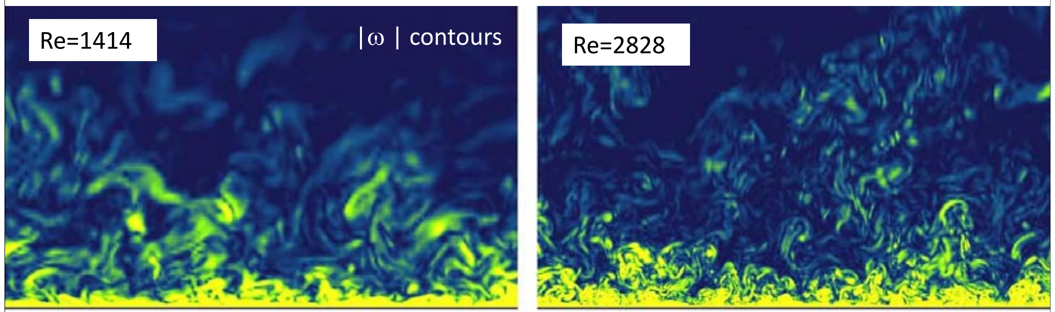
- Balance between pressure gradient, Coriolis and "friction"
  $\Rightarrow$ ***3D boundary layer***…





- Defining parameter: **Reynolds number** <u>Re=GD/ν</u> , where
  - G ← freestream/geostrophic wind speed
  - $D = (2\nu/f)^{1/2}$ ← viscous boundary-layer depth
  - $f = 2\Omega_v$ ← Coriolis/rotation parameter
  - $\nu = \mu/\rho$ ← kinematic viscosity

Ekman Layer DNS (Spalart *et al.* 2008)



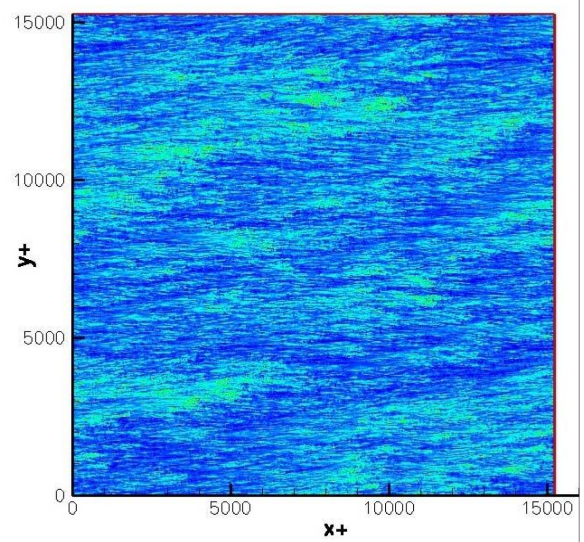
- Relevance for near-wall similarity?...

Ekman-layer DNS (Spalart, Coleman & Johnstone 2008, *Phys Fluids*)

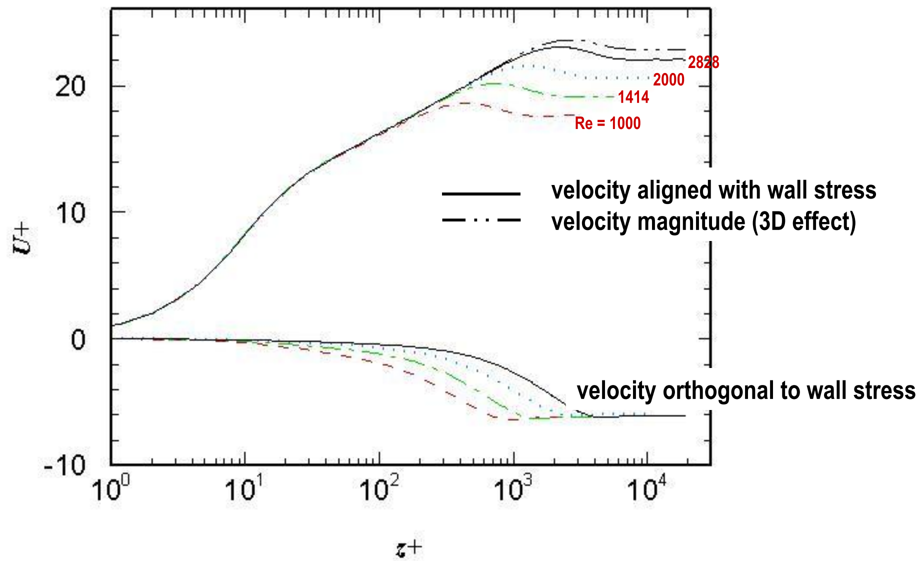
- Re = 1000, 1414, 2000 and 2828

Ekman-Layer DNS at Re = 2828

- Coriolis term allows BL homogeneous in x, y <u>and</u> t
- Pressure gradient, equivalent to channel at $Re_\tau = 1250$
- Boundary-layer thickness
  $\delta \approx 5000\nu/u_\tau$
- Fully spectral Jacobi/Fourier BL code
- 768 x 2304 x 204 (=360M) quadrature/collocation points
- Patch over $15{,}000^2$ in wall units, i.e. 150 streaks side-by-side
- Note the "mega-patches"

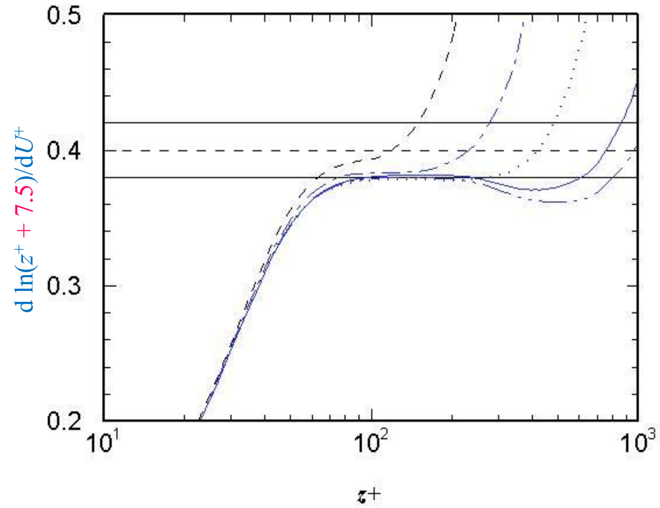


Log Law in Ekman-Layer DNS?



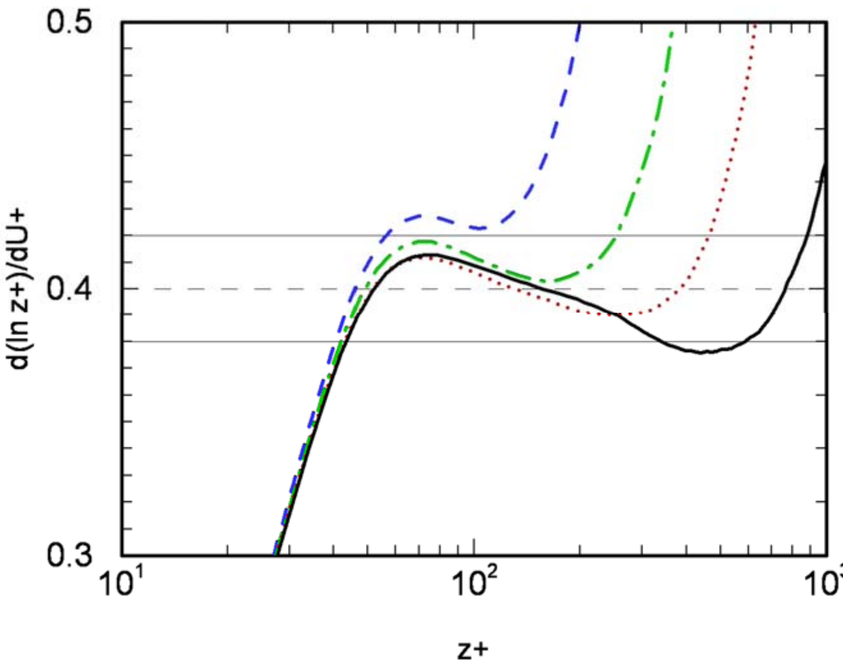
Ekman Reynolds numbers from 1000 to 2828: δ^+ scales like $Re^{1.6}$

Karman Measure in Ekman-Layer DNS with Shift



- Shifting to ln (z^+ + 7.5) creates a plateau at 0.38! (Main result of 2008 Phys Fluids paper.)

Karman Measure in Ekman-Layer DNS



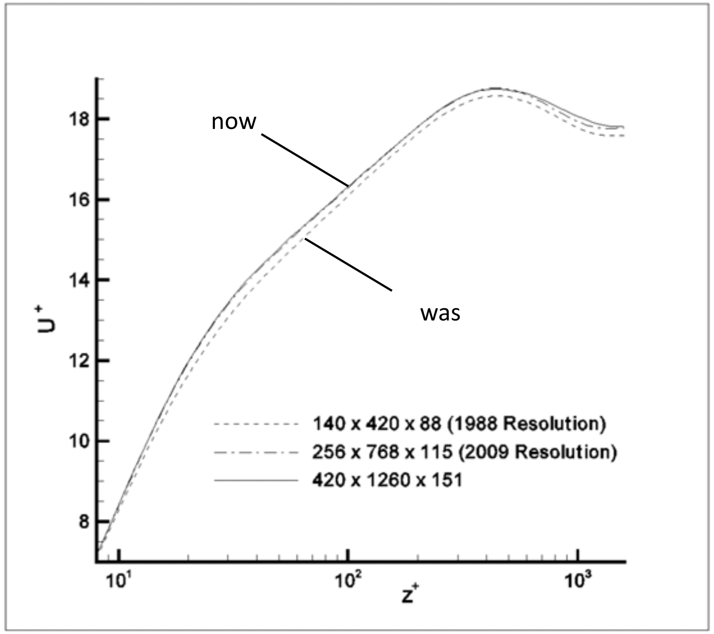
BUT, it was all too good to be true…

- Subsequent work has revealed 2008 Ekman DNS used Rules of Thumb for resolution (from Spalart's 1988 TBL) *that are not adequate*:
  - 1988 Resolution:

    $\Delta x^+ = 20$, $\Delta y^+ = 7$, n=10 at $z^+ = 10$
  - 2009 Resolution:

    $\Delta x^+ = 12$, $\Delta y^+ = 4$, n=10 at $z^+ = 6$

Resolution check: Ekman DNS at Re=1000



New Ekman DNS:

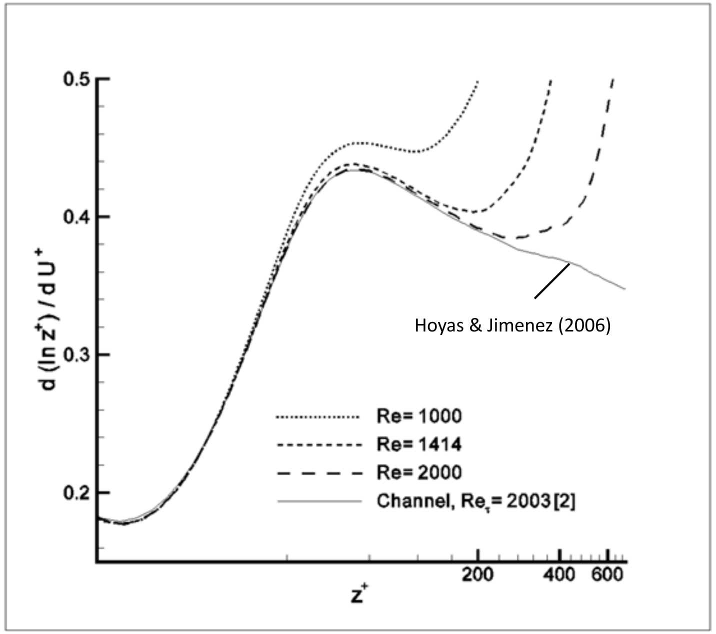
☺ Agrees with channel, implies unique Law of the Wall $U^+=f(z^+)$.

☹ No sign of plateau in Karman Measure (and would require implausibly large offset to create one).

😐 Correction in Phys. Fluids **21**, 109901 (2009).

☹ Determination of κ via DNS now appears many years away.

Karman measure for new Ekman DNS



Summary

- Couette-Poiseuille DNS implies the Log Law is very robust.
- Is it universal? *We'd like to thinks so, but time will tell…*
  - (An order of magnitude over Kim *et al* (1987) has been gained in Re, but κ is less certain than ever!)
  - NB. Recent "Variable κ" proposals = *death* of Log Law.
- Lessons from Ekman-layer DNS:
  - The general *Law of the Wall* (i.e. $U^+=f(z^+)$) is unique.
  - The Log Law is established only for $z^+ > 200$ at best (conventional ideas about Viscous, Buffer and beginning of Log layer need revision).
  - Using Standard Rules of Thumb regarding resolution requires caution, and *does not in-and-of-itself guarantee perfection at every level.*