Capturing turbulent density flux effects in variable density flow by an explicit algebraic model
I. A. Grigoriev, S. Wallin, G. Brethouwer, and A. V. Johansson

Citation: Physics of Fluids (1994-present) 27, 045108 (2015); doi: 10.1063/1.4917278
View online: http://dx.doi.org/10.1063/1.4917278
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/27/4?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in

Large eddy simulation of flow development and noise generation of free and swirling jets

A realizable explicit algebraic Reynolds stress model for compressible turbulent flow with significant mean dilatation

Modeling of carbon dioxide condensation in the high pressure flows using the statistical BGK approach

Modeling of compressible magnetohydrodynamic turbulence in electrically and heat conducting fluid using large eddy simulation
Phys. Fluids 20, 085106 (2008); 10.1063/1.2969472

Application of the DSMC and NS Techniques to the Modeling of a Dense, Low Reynold’s Number MEMS Device
AIP Conf. Proc. 762, 761 (2005); 10.1063/1.1941627
Capturing turbulent density flux effects in variable density flow by an explicit algebraic model

I. A. Grigoriev, S. Wallin, G. Brethouwer, and A. V. Johansson

1Department of Mechanics, Linné FLOW Centre, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden
2Swedish Defence Research Agency (FOI), SE-164 90 Stockholm, Sweden

(Received 13 November 2014; accepted 25 March 2015; published online 16 April 2015)

The explicit algebraic Reynolds stress model of Grigoriev et al. [“A realizable explicit algebraic Reynolds stress model for compressible turbulent flow with significant mean dilatation,” Phys. Fluids 25, 105112 (2013)] is extended to account for the turbulent density flux in variable density flows. The influence of the mean dilatation and the variation of mean density on the rapid pressure-strain correlation are properly accounted for introducing terms balancing a so-called “baroclinic” production in the Reynolds stress tensor equation. Applying the weak-equilibrium assumption leads to a self-consistent formulation of the model. The model together with a $K-\omega$ model is applied to a quasi-one-dimensional plane nozzle flow transcending from subsonic to supersonic regimes. The model remains realizable with constraints put on the model parameters. When density fluxes are taken into account, the model is less likely to become unrealizable. The density variance coupled with a “local mean acceleration” also can influence the model acting to increase anisotropy. The general trends of the behaviour of the anisotropy and production components under the variation of model parameters are assessed. We show how the explicit model can be applied to two- and three-dimensional mean flows without previous knowledge of a tensor basis to obtain the general solution. Approaches are proposed in order to achieve an approximate solution to the consistency equation in cases when analytic solution is missing. In summary, the proposed model has the potential to significantly improve simulations of variable-density flows. © 2015 AIP Publishing LLC.

I. INTRODUCTION

The explicit algebraic Reynolds stress models (EARSMS) have successfully been applied to many incompressible and weakly compressible flows, i.e., two-dimensional and three-dimensional shear flows and boundary layers, flows with rotation and curved flows, flows with passive scalar transport and active scalars causing buoyancy forces. The ability of EARS to take into account flow rotation, variation of flow conditions, and turbulence anisotropy in free shear and boundary layer flows is an advantage over eddy-viscosity models. Differential Reynolds stress models (DRSMs) can be effectively reduced to EARS in many cases of engineering importance, except in regions where the shear and strain vanishes, by applying the weak equilibrium assumption, i.e., neglecting the advection and transport terms of the anisotropy tensor.

The variable density of a flow has to be correctly accounted for when considering high-speed nozzle flows, thermodynamically supercritical flows, shock-turbulence interaction, and combustion accompanied with substantial heat release. In these cases, both the influence of the mean dilatation $\partial_i U_i$ and mean density variation $\partial_i \bar{\rho}$ may be significant. Though the quantities are related through the mean continuity equation

---

a) Author to whom correspondence should be addressed. Electronic mail: igor@mech.kth.se
\[ \partial_t \bar{\rho} + \delta_j (\bar{\rho} U_i) + \partial_i \rho' u_i' = 0, \]  

(1)

which is written here in the Reynolds averaged form following Grigoriev et al.,\(^9\) they affect the turbulence equations in a different manner. Note that an additional term \( \partial_i \rho' u_i' \) with fluctuating density-velocity correlation appears as well. The \( \rho' u_i' \) correlation representing the turbulence density flux will be of importance as soon as mean density variations are present in turbulent flows, as shown by the mean momentum equation

\[ \partial_t (\bar{\rho} U_i) + \partial_j (\bar{\rho} U_i U_j) + \partial_j R_{ij} + \partial_i P = \partial_j \bar{\tau}_{ij} + \bar{\rho} g_i - \left( \partial_i (\rho' u_i' U_j) + \partial_j (\rho' u_i' U_j) \right). \]

(2)

To eliminate the time derivative on the right-hand side, the expression in parentheses has to be rewritten with the use of \( \rho' u_i' \)-equation (formulated in Sec. II) and modeling assumptions introduced in Appendix A.

In the above and following formulas, \( U_i, \bar{\rho}, \) and \( P \) stand for mean velocity, density, and pressure, respectively, while \( u_i', \rho', \) and \( p' \) denote the corresponding fluctuating parts, respectively, \( u_i = U_i + u_i' , \rho = \bar{\rho} + \rho' , \) and \( p = P + p' \) are the corresponding full non-filtered quantities while \( \tau_{ij} \) is the full viscous stress tensor, \( R_{ij} = \rho u_i' u_j' \) is the Reynolds stress tensor, and \( g_i \) is the acceleration of gravity (or ‘forcing’ per unit mass, in a more general case). The turbulent kinetic energy is defined as \( K = \frac{R_{ij} R_{ij}}{2 \bar{\rho}} \) and the anisotropy tensor \( a_{ij} \) by \( R_{ij} = \bar{\rho} K (a_{ij} + \frac{2}{3} \delta_{ij}) \). The quantities \( \epsilon \) and \( \varepsilon = \bar{\rho} \varepsilon, \) found below, are the turbulence dissipation rate per unit mass and per unit volume, respectively.

To close the system of equations governing the set of primary quantities \( \{U_i, \bar{\rho}, P\} \), we have to consider the equation for the thermodynamic energy of a fluid which can be formulated as \( \rho T (\partial_t s + u_k \partial_k s) = \tau_{ij} \partial_i u_j - \partial_k q_k, \) where \( s(\rho, p) \) is the entropy per unit mass, \( T(\rho, p) \) the temperature, and \( q_k \) the heat flux due to thermal conductivity. For calorically perfect gas \( p \sim \rho T, s \sim \ln p/\rho^γ \) (γ—adiabatic constant) which allows us to write the mean pressure equation explicitly. However, it becomes ill-posed numerically at low Mach number and for CFD (Computational Fluid Dynamics) applications we have to start from the equation \( \partial_t (\bar{\rho} E) + \partial_j \left( [\bar{\rho} E + P] U_j \right) = \partial_j (\tau_{ij} u_j - q_j) \) for the total energy (per unit mass) \( E = \frac{u_i^2}{2} + \frac{\rho' u_i' U_j}{\gamma T} \). The resulting conservational form of the mean total energy equation reads

\[ \partial_t (\bar{\rho} \bar{E}) + \partial_j \left( [\bar{\rho} \bar{E} + P] U_j \right) + \frac{\rho' u_i' U_j}{2} + U_i R_{ij} + \frac{\rho u_i^2 u_j'}{2} + \frac{\gamma p' u_j'}{\gamma - 1} = \partial_j \left( \bar{\tau}_{ij} u_i - \bar{q}_j \right), \]

(3)

where \( \bar{E} \) is the Favre averaged mean total energy per unit mass. \( \bar{E} \) and \( P \) are linearly related with coefficients expressed through the Reynolds averaged quantities \( U_i, \bar{\rho}, K, \) and \( \bar{\rho}' u_i' \) (it is assumed that the equations governing turbulence kinetic energy and turbulence density flux are formulated too). Hence, though theoretically the sets of primary quantities \( \{U_i, \bar{\rho}, P\} \) and \( \{U_i, \bar{\rho}, \bar{E}\} \) are interchangeable, practically the employment of the equation for \( \bar{E} \) is preferable.

The influence of the mean dilatation exhibits itself principally through the rapid pressure-strain correlation \( \Pi^{(r)}_{ij} = \rho^E (\partial_i u_j' + \partial_j u_i' - \frac{2}{3} \partial_k u_k' \delta_{ij})^{(r)} \). The general tensorial form, leading to a consistent model for \( \Pi^{(r)}_{ij} \), has been presented many times in the literature, see, e.g., Speziale and Sarkar,\(^9\) Gomez and Girimaji,\(^11\) but the case with significant mean dilatation, when realizability issues arise, has only recently been given an extensive consideration.\(^9\)

By formulating an algebraic model for the active scalar flux \( \bar{\rho}' u_i' \), it can be shown that \( \bar{\rho}' u_i' \) is proportional to the mean density variation as well as to the (minus) local acceleration \( (D_i U_k - g_k) \). \( D_i \equiv \delta_i + U_i \partial_i \), and therefore often is important. Density-velocity correlation contributes to the production of turbulence kinetic energy through a constituent equal to \( \mathcal{P} = -\rho' u_k' (D_i U_k - g_k) \). In Grigoriev et al.\(^9\) gravity was not taken into account but here we show that our model includes (if
complemented with a low-compressibility equation of state) buoyancy effects studied by Lazeroms et al.\textsuperscript{7}

The dissipation equation can be further generalized by accounting for the influence of varying fluid viscosity\textsuperscript{9} since this may be of importance for varying density flows.\textsuperscript{12} We restrict the consideration to isotropic dissipation rate (see, e.g., Ref. 13 for more advanced approach).

The influence of the density variance can become significant in, e.g., flows under supercritical conditions.\textsuperscript{14} One of the challenges is to prevent unphysical negative values of the density variance governed by the corresponding differential equation.

Various aspects of compressible turbulence have extensively been studied,\textsuperscript{15,16} like the influence of the pressure-dilatation correlation, which becomes relevant when the level of turbulence kinetic energy $K$ is high and the turbulent Mach number $M_t = \sqrt{2K/c_s}$ is of order unity ($c_s$—speed of sound). Though we do not consider here such extreme cases and neglect the quantity $\rho' \partial_i u_i^*$, we will estimate the importance of the term in the paper.

Borée \textit{et al.}\textsuperscript{17} experimentally investigated the generation and breakdown of a compressed tumbling motion while Toledo \textit{et al.}\textsuperscript{18} numerically confirmed their conclusions. Namely, the influence of the curvature and flow separation during intake has been revealed and several types of instability during the breakdown have been identified. Gomez and Girimaji\textsuperscript{19} concluded that the behaviour of the pressure in compressible turbulent flow has three different regimes and proposed a model for the rapid pressure-strain correlation that covers the entire range for homogeneous shear flow. Gomez and Girimaji\textsuperscript{19} also formulated an EARSM for compressible flows based on the same approach but without considering strong mean flow dilatation. Kim and Park\textsuperscript{20} adopted a Girimaji-like compressibility factor function to arrive at an explicit solution for compressible ARSM, with dilatation related terms approximately accounted for. This resulted in a cubic algebraic equation. They examined compressible mixing layer, supersonic flat-plate boundary flow, and planar supersonic wake flow to study the performance of their model. But it must be stressed that the improvements of Kim and Park\textsuperscript{20} are mainly due to compressibility factor function $F$, modifying rapid pressure-strain correlation, which involves the turbulent Mach number $M_t$ and the gradient Mach number $M_g = S/l/c_s$ ($S$—shear rate, $l$—characteristic length scale). At the same time the cases considered by the authors manifest negligible effects associated with mean dilatation and turbulent mass flux.

Only recently has an EARSM been proposed by Grigoriev \textit{et al.}\textsuperscript{9} for flows with large mean dilatation and density variation. In Grigoriev \textit{et al.}\textsuperscript{9}, the dilatation and density variation effects were considered separately, whereas in this paper, we couple the models to arrive at an EARSM self-consistently taking into account both effects. One of our objectives is to examine the realizability of the Reynolds stress tensor of this model. Another purpose is to develop the general tensor invariant three-dimensional solution relevant to EARSM with an active scalar. In Grigoriev \textit{et al.}\textsuperscript{9}, we discussed the equations for the mean momentum, mean density, Reynolds stress tensor, density-velocity correlation, and density variance in Reynolds averaged- and Favre averaged forms.

Though the latter formulation is more concise than the former, this is achieved at the price of employing a Reynolds splitted velocity in the terms containing viscous stress tensor because otherwise first- and second-order spatial derivatives of the density-velocity correlation appear in the equations. Moreover, the turbulence dissipation $\epsilon$ is based on the spatial derivatives of $u_i'$ and is advected by $U_l$ not by $\dot{u} = \bar{\rho}u_i'/\bar{\rho}$. These arguments lead us prefer the Reynolds averaging over Favre averaging. Furthermore, if we want to compare the model with experimental data, Reynolds averaging has an advantage since the recent (optical) measurement techniques like PIV (Particle Image Velocimetry) and LDA (Laser Doppler Anemometry) measure the kinematic velocity (i.e., Reynolds averaged) and not the Favre averaged velocity. In DNS (Direct Numerical Simulation), both Reynolds and Favre averaged quantities can be computed without problems. Note that low-Reynolds-number corrections (in line with Wallin and Johansson\textsuperscript{3} or Wilcox\textsuperscript{21}) are necessary in cases of wall-bounded flows and only in such cases a careful treatment of the viscous part in the momentum equation is really needed. In this paper, we concentrate on the algebraic validation of the method, and in quasi-one-dimensional nozzle flow, we will completely neglect the wall effects.

The paper is organized as follows. In Sec. II, we outline the extension of the model of Grigoriev \textit{et al.}\textsuperscript{9} to the general case of variable density flow. The application of the model to the case of,...
quasi-one-dimensional nozzle flow is presented in Sec. III where both exact and approximate solutions are analyzed. Section IV shows the application of the method to a general two-dimensional case. A tensor-invariant three-dimensional formulation of the model is given in Sec. V. Section VI summarizes the work.

II. EXPLICIT ALGEBRAIC REYNOLDS STRESS MODEL FOR VARIABLE DENSITY FLOWS

In Grigoriev et al.\textsuperscript{9}, we have formulated an EARSM for compressible flow using a new model for the pressure-strain correlation term to account for mean dilatation in a consistent way and prevent spurious unrealizability. Moreover, an algebraic model for the active scalar flux of density-velocity correlation $\rho' u'_k$ was presented. In this section we outline first the derivation of both models whereafter we couple them in a self-consistent way to account for the varying density of the flow.

The following definitions will be used:

$$D = \tau \partial_t U_k, \quad S_{ij} = \frac{\tau}{2} \left( \partial_i U_j + \partial_j U_i \right) - \frac{D}{3} \delta_{ij}, \quad \Omega_{ij} = \frac{\tau}{2} \left( \partial_i U_j - \partial_j U_i \right),$$

$$\xi_i = \frac{\rho' u'_i}{\sqrt{\rho'^2 K}}, \quad \Gamma_i = \tau \frac{\sqrt{K}}{\sqrt{\rho'^2}} \partial_i \bar{\rho}, \quad \Upsilon_i = \tau \frac{\sqrt{\rho'^2}}{\bar{\rho} \sqrt{K}} \left( D_i U_i - g_i \right).$$

with $\tau = K/\epsilon$ the turbulence characteristic time scale and $D$, $S_{ij}$, $\Omega_{ij}$, $\xi_i$, $\Gamma_i$, $\Upsilon_i$ the nondimensionalized dilatation, strain- and vorticity-tensors, density-velocity correlation, mean density gradient, and substantial derivative of the mean velocity with the acceleration of gravity subtracted, respectively. Hence, buoyancy effects are captured indirectly through $\Upsilon_i$ which contains the gravity force $g_i$. This is not further considered in the present paper but interested readers are referred to Lazeroms et al.\textsuperscript{7,8}

A. Density flux model

The equations governing density fluctuation $\rho'$ and velocity fluctuations $u'_i$ are

$$\partial_t \rho' + U_k \partial_k \rho' + \partial_k \left( \rho' u'_k \right) + \rho' \partial_k U_k + u'_k \partial_k \bar{\rho} + \bar{\rho} \partial_k u'_k - \partial_k \rho' u'_k = 0$$

and

$$\partial_t u'_i + U_k \partial_k u'_i + u'_k \partial_k u'_i + U_i \partial_k u'_k + \frac{\partial_i p - \partial_k \tau_{ik}}{\rho} - \frac{\partial_i p - \partial_k \tau_{ik}}{\bar{\rho}} - u'_i \partial_k u'_k = 0.$$

Multiplying (5) by $2 \rho'$ and averaging, we obtain an equation\textsuperscript{22} for the density variance $\rho'^2$,

$$\partial_t \rho'^2 + U_k \partial_k \rho'^2 + \partial_k \rho'^2 u'_k + 2 \rho'^2 \partial_k U_k + 2 \rho' u'_k \partial_k \bar{\rho} + 2 \left( \bar{\rho} \rho' + \rho'^2 / 2 \right) \partial_k u'_k = 0.$$

Multiplying (5) by $u'_i$ and (6) by $\rho'$, adding, and averaging, we will obtain an equation for the density-velocity correlation

$$\partial_t \rho' u'_i + U_k \partial_k \rho' u'_i + \bar{\rho} \partial_k \frac{\rho' u'_i u'_k}{\bar{\rho}} + \rho' u'_i \partial_k U_k + \rho' u'_i \partial_k U_i$$

$$+ R_{ik} \frac{\partial_k \bar{\rho}}{\bar{\rho}} + \bar{\rho} u'_i \partial_k u'_k + \frac{\rho'}{\rho} \left( \partial_i p - \partial_k \tau_{ik} \right) = 0.$$  \hspace{1cm} (8)

Using Eq. (5) for $\rho'^2$ and the equation for the turbulence kinetic energy $K$, we can write the equation for $\xi_i$.\textsuperscript{22}
\[
\partial_t \hat{\zeta}_i + U_k \partial_k \hat{\zeta}_i + \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i u'_k}{\hat{\rho}} \right) - \frac{\hat{\rho}}{2} \left( \partial_t T_k + \frac{\partial_t \rho' u'_k}{\hat{\rho}} \right) = \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i}{\hat{\rho}} \right) - \frac{\hat{\rho}}{2} \left( \partial_t T_k + \frac{\partial_t \rho' u'_k}{\hat{\rho}} \right) + \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i u'_k}{\hat{\rho}} \right)
\]

(9)

with

\[
P = -R_{jk} \partial_k U_j = -e \left( a_{ij} S_{ij} + \frac{2}{3} \mathcal{D} \right), \quad \Psi = \frac{\Psi_{ij}}{2}, \quad \frac{\Psi_{ij}}{\hat{\rho}} = -(\hat{\zeta}_i T_j + \hat{\zeta}_j T_i),
\]

(10)

where \( P \) is the turbulence production by shear and \( \Psi \) the turbulence production due to the density-velocity correlation. The production terms are explicitly given and no modeling is needed. \( \Pi \) is the pressure-dilatation correlation given by

\[
\Pi = \frac{p'_{in}'}{\hat{\rho}} \partial_k u'_k,
\]

(11)

where \( p'_{in} \) is the inertial part of the fluctuating pressure.\(^9\) \( T_{ijk} \) is the flux of the Reynolds stress tensor and \( T_k \) is the flux of turbulence kinetic energy.\(^{22}\)

\[
T_{ijk} = \rho u'_i u'_j u'_k + p'_{in} (u'_i \delta_{jk} + u'_j \delta_{ik}) - \mu \partial_k (u'_i u'_j) + \mu (u'_j \partial_i u'_k + u'_i \partial_j u'_k), \quad T_k = \frac{T_{ikk}}{2}.
\]

(12)

The last term in Eq. (7) and the three last terms on the right-hand side of Eq. (9) unavoidably require modeling. We perform it in Appendix A starting with a modeling expression (A1) for the fluctuating dilatation \( \partial_k u'_k \).

Hence, the following expression for the last term in (7) is adopted:

\[
2 \left( \frac{\hat{\rho} \rho'}{\sqrt{\rho^2 \sqrt{K}}} \partial_k u'_k \right) = \hat{\epsilon}_p \tau^{-1} \rho^2 - 2 \hat{\epsilon}_p \rho' u'_k \partial_k \hat{\rho},
\]

(13)

while the last line in (9) is modeled as

\[
\left( \frac{\hat{\rho} \rho'}{\sqrt{\rho^2 \sqrt{K}}} \partial_k u'_k \right) \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i u'_k}{\hat{\rho}} \right) - \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i}{\hat{\rho}} \right) - \frac{\hat{\rho}}{\sqrt{\rho^2 \sqrt{K}}} \partial_k \left( \frac{\rho' u'_i u'_k}{\hat{\rho}} \right) = \hat{\epsilon}_p \tau^{-1} \hat{\zeta}_i \hat{\zeta}_k + c_m \hat{\zeta}_k \partial_k \hat{U}_i + c_n \hat{\zeta}_k \partial_i \hat{U}_k + c_D \hat{\zeta}_i \partial_k \hat{U}_k - \frac{2}{3} c_{\gamma} \tau^{-1} \hat{T}_i - c_p \tau^{-1} \hat{\zeta}_i \hat{\zeta}_k T_k.
\]

(14)

We may expect that \( c_{\gamma} \) lies in the interval \([0, 1.5]\), as demonstrated in Appendix A. In its turn, the term with \( c_p \) in (14) has the same structure as the fourth before last term in (9) reducing the effect of this nonlinear term. These explicitly nonlinear terms cancel each other when \( c_p \) is equal to 1, which is within a reasonable range of possible \( c_p \)-values. Although taking \( c_p = 1 \) does not render Eq. (9) to become a truly linear equation in \( \hat{\zeta}_i \) because of the \( \hat{\zeta}_i \) dependency in \( \Psi \) resulting in a scalar non-linearity of the form \( \hat{\zeta}_i \hat{\zeta}_k \), we will justify this choice in Sec. II C.

We recall that the formulation of the algebraic model for the density-velocity correlation in Grigoriev et al.\(^9\) is based on applying the weak-equilibrium assumption (i.e., neglecting advection and diffusion of the normalized quantity \( \hat{\zeta}_i \) on the left-hand side of Eq. (9)). With the definitions \( c_S = 1 - c_m - c_n, \quad c_Q = 1 - c_m + c_n, \quad \mathcal{D} = 1 - c_p \), Eq. (9) allows us to formulate the algebraic model for \( \hat{\zeta}_i \),

\[
\left( N \xi \delta_{ik} + c_S S_{ik} + c_D \Omega_{ik} \right) \hat{\zeta}_k = -\left( a_{ik} + \frac{2}{3} \delta_{ik} \right) T_k - \frac{2}{3} c_{\gamma} \hat{T}_i,
\]

(15)

where

\[
N \xi = \frac{1}{2} \frac{P - e + \Pi + \Psi}{e} - c_p + \left( \frac{c_S}{3} - c_D \right) \mathcal{D} - (1-c_p) \hat{\zeta}_k T_k.
\]

(16)

The solution for \( \hat{\zeta}_i \) depends on \( a_{ij} \) through the right-hand side of (15) and through dependence of \( N \xi \) on \( P/e \), the latter is the consequence of coupling to the turbulent kinetic energy equation in the
derivation of (9). So as from (15), we infer that $\dot{\zeta}_i \sim -\partial_i \bar{\rho}$ and from (4) we see that in the main order of magnitude $T_i \sim -\partial_i P$, the density-velocity correlation production $\Psi$ effectively represents the interaction of mean density and mean pressure gradients. For this reason, we will refer to $\Psi$ as to “baroclinic” production.

B. Reynolds stress anisotropy model

The equation for the anisotropy tensor is derived from the exact equations for the Reynolds stress tensor and the turbulence kinetic energy and reads $^9$

$$\bar{\rho} K D_t a_{ij} + \partial_k T_{ijk} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \partial_k T_k = \mathcal{P}_{ij} - \epsilon_{ij} + \Pi_{ij} + \Psi_{ij} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \left( \mathcal{P} - \epsilon + \Pi + \Psi \right),$$

(17)

where the shear production tensor $\mathcal{P}_{ij}$ and “baroclinic” production tensor $\Psi_{ij}$ are given by (10) and need no further modeling. The transport terms are given by (12) and the pressure-dilatation correlation $\Pi$ by (11). Applying the weak-equilibrium assumption and assuming isotropic dissipation $\epsilon_{ij} = \frac{2}{3} \epsilon \delta_{ij}$ and splitting $\Pi_{ij}$ into a rapid part $\Pi_{ij}^{(r)}$, slow part $\Pi_{ij}^{(s)}$, and dilatational part $\frac{2}{3} \Pi \delta_{ij}$, we arrive at

$$- \frac{\Pi_{ij}^{(s)}}{\epsilon} - a_{ij} = \frac{\mathcal{P}_{ij}}{\epsilon} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \frac{\mathcal{P}}{\epsilon} + \frac{\Pi_{ij}^{(r)}}{\epsilon} + \frac{\Pi_{ij}^{(dy)}}{\epsilon} + e^{-1} \left( \Psi_{ij} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \psi - \Pi a_{ij} \right).$$

(18)

The production related part on the right-hand side of (18) can be written as

$$\frac{\mathcal{P}_{ij}}{\epsilon} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \frac{\mathcal{P}}{\epsilon} = \text{tr} \left( \mathbf{a} \mathbf{S} \right) a_{ij} - \frac{4}{3} S_{ij} - \left( a_{ik} \Omega_{kj} + S_{ik} a_{kj} - \frac{2}{3} a_{km} S_{mk} \delta_{ij} \right) + a_{ik} \Omega_{kj} - \Omega_{ik} a_{kj},$$

and the other terms are modeled as

$$\frac{\Pi_{ij}^{(r)}}{\epsilon} = \frac{4}{5} S_{ij} + \frac{7 q_3 - 12}{9} \left( a_{ik} \Omega_{kj} - \Omega_{ik} a_{kj} \right) + q_3 \left( a_{ik} S_{kj} + S_{ik} a_{kj} - \frac{2}{3} a_{km} S_{mk} \delta_{ij} \right),$$

$$\Pi_{ij}^{(dy)} = -c_1 \epsilon a_{ij}, \quad \Pi_{ij}^{(dy)} = -c_\Psi \left( \Pi_{ij} - \frac{2}{3} \Psi \delta_{ij} \right),$$

(19)

where the constant $c_\Psi$ determines the relaxation due to the pressure-strain term $\Pi_{ij}^{(dy)}$, allegedly counteracting the anisotropy tensor production by variable density $\mathcal{P}_{ij}^{(dy)} = \Psi_{ij} - \left( a_{ij} + \frac{2}{3} \delta_{ij} \right) \Psi$.

The expression for the rapid pressure-strain correlation $\Pi_{ij}^{(r)}$ contains three tensor groups. The coefficient before the first one follows from the rapid distortion theory while the input of the second and the third tensor groups depends on the parameter $q_3$. $^9$ Taking $q_3 = 1$, we reduce Eq. (18) for $a_{ij}$ to an equation

$$N a_{ij} = -\frac{6}{5} S_{ij} + \left( a_{ik} \Omega_{kj} - \Omega_{ik} a_{kj} \right) - C_\epsilon \left( \xi_i T_j + \xi_j T_i - \frac{2}{3} T_k \xi_k \delta_{ij} \right), \quad C_\epsilon = \frac{9}{4} \left( 1 - c_\Psi \right),$$

(20)

whose solution is linear in $S_{ij}$ and $\xi_i$, and inversely proportional to $N$. Also the equation for $N$,

$$N = c_\epsilon' + \frac{9}{4} \left( -a_{jk} S_{kj} - \xi_k T_k + \frac{\Pi}{\epsilon} \right), \quad c_\epsilon' = \frac{9}{4} \left( c_1 - 1 \right)$$

(21)

depends linearly on $a_{ij}$ and $\xi_i$.

C. Solution strategies

The system of Eqs. (15) and (20) becomes closed when supplemented with two consistency relations (16) and (21). To arrive at a full description of variable density effects, $a_{ij}$ and $\xi_i$ first have
to be expressed in \( S_{ij}, \Omega_{ij}, D, \Gamma_i, T_i, N, \) and \( N_c. \) By solving the system of algebraic equations (16) and (21) for two unknowns \( N_c \) and \( N, \) we completely answer the formulated problem.

To reveal the degree of the coupling and to identify possible strategies to solve the system, we remind that both \( N \) and \( N_c \) depend on \( a_{ij} \) and \( \zeta_i. \) Rewriting (16) as

\[
N_c = \frac{2}{9} N - \frac{c_1}{2} - c_p + (c_s - 1 - 3 c_D) \frac{D}{3} - (1 - c_p) \zeta \Gamma_k, \tag{22}
\]

we see that the dependence of \( N_c \) on \( a_{ij} \) is completely contained in \( N \) while the dependence on \( \zeta_i \) is comprised in \( N \) partly because an additional term proportional to \( \zeta \Gamma_k \) exists. We choose \( c_p = 1 \) (analogous to Wikström et al.\(^6\)) for which case this term vanishes. Then, \( N_c \) becomes a linear function of \( N \) and \( D, \) and only one consistency Eq. (21) is needed to close the system of Eqs. (20) and (15) which is quasilinear in both \( a_{ij} \) and \( \zeta_i. \)

The formal solution of (15) in terms of \( S_{ij}, \Omega_{ij}, a_{ij}, \Gamma_i, \) and \( N \) can be obtained for the general three-dimensional case as shown by Wikström et al.\(^6\) By substituting \( \zeta_i \) into (20), we can obtain an equation for \( a_{ij} \) only, but the emerging matrix equation is, typically, extremely difficult to solve. To work around the complexity, Lazeroms et al.\(^7\) for a similar problem resulting from the algebraic modeling of buoyancy driven flow employed the formal representation of \( a_{ij} \) and \( \zeta_i \) in linear combinations of independent tensor groups. Unfortunately, in some cases the tensor basis can become degenerate causing associated artificial problems.

In Secs. III–V, we propose a method to couple the models which avoids this drawback. We will here focus on two-dimensional mean flows but this method can be extended to general three-dimensional mean flows as we will show later. We start with the proposed direct solution of (20) for \( a_{ij}, \) which turns out to be linearly dependent on \( \zeta_i \) expressed as a vector of parameters. Substituting the obtained solution into (15), we can formulate an algebraic equation for \( \zeta_i \) and derive its solution and, consequently, a solution for \( a_{ij}. \) Both involve only one undetermined quantity \( N, \) which allows us to derive an algebraic equation for \( N. \) Finding a proper approximation to it (exact solutions are possible only for specific geometries or if drastic assumptions about model parameters are made), we can solve the problem.

In the case of a general two-dimensional mean flow, Eq. (20) with \( \zeta_i \equiv 0 \) gives a fourth-order algebraic equation for \( N \) (with \( \alpha_3 \equiv 0), \)

\[
N^4 - c_1' N^3 - \left(2 H_\Omega + \frac{27}{10} H_S\right) N^2 + 2 c_1' H_\Omega N + \frac{9}{10} H_\Omega D^2 = \Delta_3, \quad H_S = S_{jk} S_{kj}, \quad H_\Omega = \Omega_{jk} \Omega_{kj}, \tag{23}
\]

whose root can be determined univocally.\(^9\) In that study we applied fixed-point analysis to homogeneously sheared and strained flow, fixed the magnitudes of the strain-rate \( \sigma \equiv \omega \) or production-to-dissipation ratio \( P/\varepsilon, \) and varying the dilatation \( D. \) It was found that in the entire range of admissible values of \( D, \) the model remains realizable with the eigenvalues of the anisotropy tensor satisfying \(-2/3 \leq \lambda_{1,2,3}. \) It has been used together with a \( K - \omega \) model to investigate the behaviour of a quasi-one-dimensional plane nozzle flow, Fig. 1(a), in which case only a quadratic equation needs to be solved. For any initial conditions, the anisotropy tensor components turned out to be realizable during the spatial evolution.

Fortunately, for plane quasi-one-dimensional nozzle flow, the system of Eqs. (15), (20), and (21) yields a fourth-order algebraic equation for \( N, \) making it amenable to finding an exact solution. For this reason, we begin with this geometry to study the behaviour of the new model. We do not have DNS data on compressible flow which simulates a flow both with significant mean dilatation and significant turbulence density flux, therefore quasi-one-dimensional nozzle flow is attractive as representing both effects jointly. We will demonstrate the possibility of calibrating the model by choosing model parameters which give realizable behaviour for the same set of parameters in different turbulence regimes.
FIG. 1. (a) Quasi-one-dimensional plane nozzle flow configuration. (b) Real roots of quartic equation (28) as function of $F$. Blue, red, green, and magenta curves are the four roots $N^{(1)}, N^{(2)}, N^{(3)},$ and $N^{(4)}$ of the quartic equation given by (B2), respectively. Black horizontal line is solution to the quadratic equation $N^2 - c_1^2 N - \frac{9}{2} H S = 0$ for the $\xi_1 \equiv 0$ case. Only values of $F$ to the left of the black vertical line are admissible. $N^{(2)}$, indicated by an arrow, is the physical root.

III. QUASI-ONE-DIMENSIONAL PLANE NOZZLE FLOW

The quasi-one-dimensional plane nozzle flow used to assess the developed model is illustrated in Fig. 1(a). The mean flow is close to irrotational and incoming mean streamwise velocity $U$ is uniform with negligible cross-stream variation, while cross-stream velocity $V$ in Fig. 1(a). The mean flow is close to irrotational and incoming mean streamwise velocity $U$ is uniform with negligible cross-stream variation, while cross-stream velocity $V$ is negligibly small but its $y$-derivative is finite. This case is the same as the one considered in Grigoriev et al.\textsuperscript{9} The Mach number for this geometry is given by $M(x) = 0.25 + 1.5 x - 2 (x - 0.5)^3$ and goes from $M = 0.5$ at inlet ($x = 0$) through $M = 1$ at the throat ($x = 0.5$) to $M = 1.5$ at the outlet ($x = 1$) indirectly implying the area ratio through the isentropic relations.\textsuperscript{22}

A. Exact solution

The only non-zero quantities in Eqs. (15) and (20) for the quasi-one-dimensional plane nozzle flow described above are $\Gamma_1$, $T_1$, and the diagonal components of $a_{ij}$ and $S_{ij}$. This simplifies the model system to

$$\begin{align*}
\left( N_\varepsilon + c_s S_{11} \right) \xi_1 &= -a_{11} \Gamma_1 - \frac{2}{3} \left( \Gamma_1 + c_v T_1 \right), \\
N \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \end{pmatrix} &= -6 \frac{5}{9} \begin{pmatrix} S_{11} \\ S_{22} \\ S_{33} \end{pmatrix} - \frac{2}{3} c_\varepsilon T_1 \xi_1 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \\
N &= c_1^2 + 9 \left( -a_{kk} S_{kl} - \xi_1 T_1 + \frac{\Pi}{\varepsilon} \right),
\end{align*}$$

(24)

where

$$N_\varepsilon = -c_s S_{11} + \frac{2}{9} (N + F), \quad F = \frac{9}{2} \left[ -c_\rho - \frac{c_1}{2} + (c_s - 1 - 3 c_B) \frac{D}{3} + c_s S_{11} \right].$$

(25)

From the second equation in (24), we derive $a_{11} = -N^{-1} \left( \frac{9}{5} S_{11} + \frac{2}{3} c_\varepsilon T_1 \xi_1 \right)$ which after substitution into the first equation gives a solution to the whole system

$$\begin{align*}
\xi_1 &= -3 \frac{\left( \Gamma_1 + c_v T_1 \right) N - \frac{9}{5} S_{11} \Gamma_1}{N^2 + F N - 6 c_\varepsilon \Pi \Gamma}, \\
A_{ij} &= \beta_1 S_{ij} + \beta_2 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \\
\beta_1 &= -6 \frac{1}{5} N, \quad \beta_2 = -2 c_\varepsilon \frac{1}{3} N \xi_1 T_1 = 2 \frac{C_\varepsilon}{N} \left( \Pi \frac{\Gamma \varepsilon + c_v \Pi \Gamma}{5} \right) N - \frac{9}{5} H S \frac{\Gamma \varepsilon}{5}.
\end{align*}$$

(26)

Note that when $\xi_1 = 0$, the total production of turbulence kinetic energy is given by $(\mathcal{P} + \Pi + \Psi)$. Following Grigoriev et al.,\textsuperscript{9,22} we split $\mathcal{P}$ as
and refer to the first part on the right-hand side as “incompressible” and to the second part as “dilatational.” This split is based on the notion that the quantity \( J = \frac{1}{2} \left( \partial_x U - \partial_y V \right) \) dominates in the incompressible regime while dilatation \( D \) dominates in the compressible regime. Recall that we refer to \( \Psi = -\xi^i T_i^j \sim II_{\gamma \gamma} \) as “baroclinic” production because essentially it represents the interaction of the mean density gradient with the mean pressure gradient. \( II \) is referred to as “pressure-dilatation” production.

Using relation (21) and

\[
\frac{\rho}{e} = -\left( 2 J \beta_1 + 3 \beta_2 \right) J - \left( \beta_1 D/6 + \beta_2/2 + \frac{2}{3} \right) D
\]  

(27)

we obtain an equation for \( N \),

\[
N^4 - \left( c_i' - F \right) N^3 - \left\{ \frac{27}{10} II_S + c_i' F + 6 C_\xi II_{\gamma \gamma} + \frac{27}{4} \left( II_{\gamma \gamma} + c_\gamma II_T \right) \right\} N^2 - \left( \frac{27}{10} \right) II_S F - 6 c_i' C_\xi II_{\gamma \gamma} - \frac{9}{5} \frac{27}{4} III_S \gamma \gamma - \frac{27}{2} C_\xi \left( III_{\gamma \gamma} + c_\gamma III_{\gamma \gamma} \right) + \frac{27}{2} \frac{1}{3} S_{i1} - II_S \right\} \right. \]

(28)

where the invariants are

\[
II_{\gamma \gamma} = \Gamma_k \mathcal{T}_k, \quad III_{\gamma \gamma} = \Gamma_k S_{ki} \mathcal{T}_l, \quad III_{\gamma \gamma 2} = \mathcal{T}_k S_{ki} \mathcal{T}_l, \quad II_T = \mathcal{T}_k \mathcal{T}_l.
\]  

(29)

We can expect that \( II/e \) is negligible, but if the pressure-dilatation correlation cannot be neglected, we can redefine \( c_i' \) by including \( II/e \) into it, which formally does not change the formulas (note, that relation (22) also remains the same).

In contrast to the zero density-velocity correlation case \( \xi_i = 0 \) investigated in Grigoriev et al., we give a simple quadratic equation \( N^2 - c_i' N - 27/10 II_S = 0 \), we there need to investigate quartic equation (28) involving parameters \( c_{\gamma \gamma}, c_S, c_D, c_{\mu} \) and \( c_\gamma \). The dependence on the density variance \( \rho^2 \) entering through the definition of \( \Gamma \) and \( \gamma \) in (4) cancels for all terms but \( II_T \) and \( III_{\gamma \gamma 2} \). Hence, until Sec. III C we put \( c_\gamma = 0 \) to avoid the need to solve additional differential equation (7) for the density variance \( \rho^2 \). The solution to equation (28) is given in Appendix B and Fig. 1(b) shows the typical behaviour of the four roots as function of \( F \) defined in (25) assuming that the other variables \( (II_S, III_{\gamma \gamma}, \text{and} S_{i1}) \) are constant (straight line corresponds to the solution with \( \xi_i = 0 \)).

To complement the model we employ a \( K - \omega \) model

\[
\partial_t \left( \bar{\rho} K \right) + \partial_k \left( \bar{\rho} K U_k \right) + \partial_k T_k^{(K)} = \frac{P}{e} - \varepsilon + \Pi + \Psi,
\]

\[
\partial_t \omega + U_k \partial_k \omega + \frac{1}{\bar{\rho}} \partial_k T_k^{(\omega)} = C_{\mu} \left( \epsilon_{\gamma \gamma} - 1 \right) \frac{\rho}{e} \omega^2 - C_{\mu} \left( \epsilon_{\gamma \gamma} - 1 \right) \omega^2 + \tau^{-1} \frac{2}{3} \left( C_{\mu} \epsilon_{\gamma \gamma} - 1 \right) \frac{\rho}{e} \omega^2 + \tau^{-1} \frac{2}{3} \left( C_{\mu} \epsilon_{\gamma \gamma} - 1 \right) \frac{\rho}{e} \omega^2
\]  

(30)

where \( \omega = \frac{\xi_i}{c_{\gamma \gamma}^K} \) is the turbulence frequency and the terms in the last line are constructed according to Grigoriev et al. and extend common incompressible \( K - \omega \) model to the consideration of variable density effects. The first term on the last line accounts both for variable viscosity of the flow and for the modification of turbulence production due to non-zero mean dilatation \( D \), while the influence of the pressure-dilatation correlation and the “baroclinic” production is represented by the second and the third terms, respectively.

Here, we take the standard values of \( C_{\mu} = 0.09, C_{\epsilon_1} = 1.56, C_{\epsilon_2} = 1.83, \gamma = 1.4, \text{and} n = 2/3 \) for the thermodynamic parameters (\( \gamma \) is a specific heat ratio, \( n \) is a power in the viscosity law \( \mu \sim T^n \)). As will be shown below, \( \Psi \) is positive in the nozzle flow. If \( C_{eb} < 1 \), the normalized strain-rate becomes stronger (\( \tau \) is enhanced) while if \( C_{eb} > 1 \), the normalized strain-rate becomes weaker (\( \tau \) is suppressed). Here, we take \( C_{eb} = 1 \) since we do not know how \( \Psi \) affects \( \omega \). We assume that \( \Pi \approx 0 \) and \( K \) is low enough for turbulence-related terms not to effect mean density and mean
momentum equations as well as to make $\tau^{(K)}_k$ and $T^{(\omega)}_k$ negligible. At $x = 0$, the velocity gradients are zero, subsequently $D = S_{11} = S_{22} = S_{33} = 0$, and we are on the root $A^{(2)}$ shown with the arrow in Fig. 1(b). If $F$ becomes larger than some critical value (which depends on variables $II_{S}, II_{F_{G}},$ and $S_{11}$) and crosses the black vertical line in Fig. 1(b), the physical root (as well as its conjugate) acquires an imaginary part indicating that no fix point solution exists for the full transport model. Such situation cannot be captured by an algebraic model and calls for DRSM for consistent description.

To avoid the need for DRSM, we have to choose the model parameters properly, i.e., the more $c_{F_{G}}$ deviates from 1.0, the smaller $c_{S}$ must be chosen to ensure that the physical root remains real during the spatial evolution. A set of parameters $c_{F_{G}}, c_{D}, c_{S},$ and $c_{F_{G}}$ that gives a marginally realizable solution produces sharp peaks in the spatial evolution of all the quantities near the point where the physical root $N_{2}$ touches the black vertical line in Fig. 1(b) due to a strong dependence on $F$. Table I illustrates this for various turbulence regimes ($S_{0}^* \to 0, S_{0}^* = 1.22, S_{0}^* = 6.68, S_{0}^* = 11.3,$ and $S_{0}^* \to \infty$) characterized by the “total strain-rate” $S_{0}^* = \tau_{x=0} | \partial_{i} U_{j} + \partial_{j} U_{i} |^{2}/2 |_{x=0.5},$ where turbulence properties are taken at the inlet and kinematic properties at the throat of the nozzle. Recall from Grigoriev et al.\textsuperscript{9} that when choosing an asymptotically low $S_{0}^*$ (achieved by $\tau \to 0$), we arrive at a universal state characterized by a finite strain-rate with the maximum magnitude of $S^* = \tau \sqrt{(| \partial_{i} U_{j} + \partial_{j} U_{i} |^{2}/2)}$ close to unity. On the other hand, sufficiently increasing $\tau_{x=0} \to \infty$, we arrive at a state with fixed curves of anisotropy tensor components regardless of the further increase in $\tau_{x=0}$. This is the case of the asymptotically high strain-rate $S_{0}^* \to \infty$.

Table I shows that when $c_{F_{G}}$ and $c_{D}$ are close to zero, an admissible set of parameters $c_{S}$ and $c_{F_{G}}$ for high strain-rate cases is too restrictive for cases with low strain-rate. Increasing parameters $c_{F_{G}}$ and $c_{D}$ from zero to higher values substantially relieves the limitations for $c_{S}$ if $c_{F_{G}} \geq 0.3$ and strain-rate is not too high ($S_{0}^* \leq 12$). Therefore, restricting ourselves to turbulence regimes with moderate strain-rates, we are able to choose a universal set of parameters giving physical results for any particular case.

Though we do not have DNS data to calibrate the model, we will demonstrate that it is possible to choose the parameters which allow for a reasonable behaviour of the model. We consider here two cases corresponding to those considered in Grigoriev et al.\textsuperscript{9}—low-strain rate case $S_{0}^* = 1.22$ and high strain-rate case $S_{0}^* = 11.3$. Adding the density-velocity correlation effects does not change $S_{0}^*$ because $\tau_{x=0} | \partial_{i} U_{j} + \partial_{j} U_{i} |$ is fixed and the mean velocity $U_{i}$ is governed by isentropic relations due to low level of turbulence kinetic energy $K$. Fig. 2 shows the spatial evolution of the turbulence production components, anisotropy tensor components, and normalized density flux, while Fig. 3 shows the spatial evolution of the total strain-rate and turbulence kinetic energy for these two cases and compares our model for different choices of parameters with the model by Grigoriev et al.\textsuperscript{9} assuming $\xi_{i} = 0$. Two illustrative sets of parameters have been chosen here—one with $c_{F_{G}} = c_{D} = -c_{S} = 0.9$ and the other with $c_{F_{G}} = c_{D} = -c_{S} = 0.5$ while $c_{F_{G}} = 0.7$ is taken for both.

<table>
<thead>
<tr>
<th>$S_{0}^* \to 0$</th>
<th>$S_{0}^* = 1.22$</th>
<th>$S_{0}^* = 6.68$</th>
<th>$S_{0}^* = 11.3$</th>
<th>$S_{0}^* \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{F_{G}} = c_{D} = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_{F_{G}} = 1.0$</td>
<td>$-0.11$</td>
<td>$-0.43$</td>
<td>$-0.72$</td>
<td>$-0.79$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.5$</td>
<td>$-0.41$</td>
<td>$-0.83$</td>
<td>$-1.27$</td>
<td>$-1.33$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.3$</td>
<td>$-0.54$</td>
<td>$-1.0$</td>
<td>$-2.53$</td>
<td>$-2.91$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.0$</td>
<td>$-0.76$</td>
<td>$-2.5$</td>
<td>$-4.6$</td>
<td>$-5.12$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.5$</td>
<td>$1.16$</td>
<td>$0.38$</td>
<td>$-0.18$</td>
<td>$-0.28$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.5$</td>
<td>$0.97$</td>
<td>$0.22$</td>
<td>$-0.28$</td>
<td>$-0.55$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.3$</td>
<td>$0.86$</td>
<td>$-0.05$</td>
<td>$-1.65$</td>
<td>$-2.03$</td>
</tr>
<tr>
<td>$c_{F_{G}} = 0.0$</td>
<td>$0.65$</td>
<td>$-1.44$</td>
<td>$-3.61$</td>
<td>$-4.15$</td>
</tr>
</tbody>
</table>
FIG. 2. (a) and (b) Spatial evolution for low strain-rate $S_0^* = 1.22$ and (c) and (d) for high strain-rate $S_0^* = 11.3$. (a) and (c) Normalized production $P_1 \Psi$: red shows the “incompressible” part, blue—the “dilatational” part, green—the “baroclinic” (corresponding to the density-velocity correlation) part, black—total production, respectively. (b) and (d) Anisotropy tensor components: red—$a_{11}$, blue—$a_{22}$, green—$a_{33}$. Thick solid lines denote the solution for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.5$; thin solid lines—for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.9$; dashed lines—for the model with $\zeta_1 \equiv 0$, respectively.

Figs. 2(a) and 2(c) illustrate that in our model with $\zeta_1 \neq 0$ the “baroclinic” production $\Psi = -\Upsilon_1 \zeta_1$ is positive in the geometry of quasi-one-dimensional nozzle flow ($\Upsilon_1 > 0$, $\zeta_1 < 0$) and can become comparable in magnitude with the “incompressible” and “dilatational” components of production both of which become somewhat suppressed. As a result the total production and turbulence kinetic energy $K$ are enhanced while (assuming $C_{\epsilon b} = 1$ in (30)) turbulence time scale and, subsequently, total strain-rate increase too as illustrated in Fig. 3. Note also that the “incompressible” production now attains slightly negative values in a region near the exit.

Accounting for the influence of the density-velocity correlation causes the anisotropies in Figs. 2(b) and 2(d) to become less likely to achieve unrealizable values. Indeed, positive $\Psi$ contributes to the increase in $N$ and, hence, suppresses the anisotropy. From the second equation in (24), it follows that positive $\Psi$ directly (i.e., not only through the modification of $r$ and $N$) increases $a_{11}$ and decreases $a_{22}$ and $a_{33}$. Therefore $a_{11}$, the only anisotropy tensor component which remains negative everywhere during the spatial evolution in the model with $\zeta_1 \equiv 0$ and attains the largest negative values, now becomes less negative and for $c_S = -0.5$ even changes sign near the nozzle exit (slightly for $S_0^* = 1.22$ and moderately for $S_0^* = 11.3$). In its turn $a_{22}$ reduces and achieves slightly smaller values after becoming negative (it experiences a weaker influence of the density-velocity correlation than the two other anisotropies). Substantial reduction in $a_{33}$ is observed but it stays positive. By increasing $c_S$ or by decreasing $c_\Psi$, we can cause $a_{33}$ to change sign at some point, but the resulting behaviour would be characterized by unphysically sharp peaks and oscillations in all the values.

Finally, the general trend is that by reducing $c_\rho$, $c_D$, $c_\Psi$, or by increasing $c_S$, we dampen “incompressible” and “dilatational” productions but the “baroclinic” component grows substantially increasing the total production while $a_{11}$ increases and $a_{22}$, $a_{33}$, $\zeta_1$ decrease (the exception is that decrease in $c_\Psi$ suppresses total production, especially in “incompressible” regime). Fig. 2
FIG. 3. Spatial evolution: (a) and (b)—total strain-rate $S^*$, (c) and (d)—turbulence kinetic energy $K$. (a) and (c) Low strain-rate $S_0^* = 1.22$, (b) and (d)—high strain-rate $S_0^* = 11.3$. Red lines denote solutions for $c_\Psi = 0.7, c_\rho = c_D = -c_S = 0.5$; blue lines—for $c_\Psi = 0.7, c_\rho = c_D = -c_S = 0.9$; green lines—for the model with $\tilde{\zeta}_i \equiv 0$, respectively.

confirms this trend for two sets of $c_\rho = c_D = -c_S$ with $c_\Psi$ fixed. In Sec. III C, we will discuss how the trend is modified when $c_\Psi \neq 0$.

B. Iterative procedure

For our configuration $N$-equation (28), derived from model (24), is fourth order which allows us to find the exact solution. Generally, the algebraic equation for $N$ resulting from systems (15), (20), and (21) is usually higher than fourth order, and an efficient numerical procedure is required to determine $N$. The simplest approach is to employ an iterative sequence

$$N_n = c_1' + \frac{9}{4} \left( -a_{jk}(N_{n-1}) S_{kj} - \hat{\zeta}_k(N_{n-1}) \right)$$

starting with a zero-order approximation $N_0$ calculated in some manner. Good approximation $N_0$ can, in principle, even give a first-order approximation $N_1 = c_1' + \frac{9}{4} \left( -a_{jk}(N_0) S_{kj} - \hat{\zeta}_k(N_0) \right)$ which is very close to the exact solution. We mention here two methods to obtain $N_0$.

The first method is to reduce the order of the equation for $N$ by putting $c_S, c_D, C_\zeta = 0$, and to define $N_0$ as the solution to the resulting equation. Using expressions for $a_{ij}(N)$ and $\hat{\zeta}_i(N)$, which will be derived in the general two-dimensional and three-dimensional flow geometries in Sec. IV and Sec. V, respectively, we can calculate $a_{ij}(N_0)$ and $\hat{\zeta}_i(N_0)$, and start iterative sequence (31). The second method is to define $N_0$ as the solution to the equation for $N$ corresponding to the model with $\tilde{\zeta}_i \equiv 0$ by Grigoriev et al.\textsuperscript{9} Though fourth-order equation (23) with $A_3 \equiv 0$ describes a general two-dimensional flow with zero density-velocity correlation, its solution can also be used as $N_0$ in three-dimensional flows (which are described by a sixth-order algebraic equation arising when we include $A_3 = \frac{81}{5} \frac{N^2-14N+2}{N^2-14N+2}$, $A = V_1 N^2 + \frac{2}{3} I_{II} IV_1 + IV_1^2$, $IV_1 = IV - \frac{2}{6} II_{II}$, $\tilde{\zeta}_i \neq 0$).
To illustrate the application of the iterative sequence, we again consider the quasi-one-dimensional plane nozzle flow for which case the exact solution is available. Starting with the first method we note that taking $c_S = 0$ in (28) does not affect the order of the equation because due to our simple geometry, $F$ is the only quantity depending on $c_S$. However, when even a slight deviation from quasi-one-dimensionality is accounted for, non-zero $c_S$ (as well as non-zero $c_P$) adds to the order of $N$-equation. Therefore to reveal the possible problems when applying this method to a more advanced case, we will assume $c_S = 0$. Putting $C_\xi = 0$ reduces (28) to a third order equation for $N_0$.

$$N_0^3 - (c'_1 - F) N_0^2 - \left\{ \frac{27}{10} S I_S + c'_1 F + \frac{27}{4} (II_F V + c_\gamma I H_V) \right\} N_0 - \left\{ \frac{27}{10} S I_S F - \frac{9}{5} \frac{27}{4} S I_S F V \right\} = 0. \tag{32}$$

The dependence of the three roots of Eq. (32) can be described by a “root diagram” analogous to Fig. 1(b) with one of the four roots being zero for any value of $F$. However, solving (32) using the same values of $c_P$ and $c_D$ which (along with true values of $c_S$ and $C_\xi$) admit physical solution to (28), we find that at some $x$ discriminant of (32) becomes negative. This corresponds to $F$ crossing at this $x$ the region right to the black vertical line in Fig. 1(b) when physical root gets an imaginary part. To remedy this we propose along with taking $c_S$, $C_\xi = 0$ to use larger values of $c_P$ and $c_D$ to reduce the magnitude of $F$ while calculating $N_0$ by solving (32). For example, considering the high strain-rate case $S_0 = 11.3$ with $c_P = 0.7$, $c_P = c_D = -c_S = 0.9$, we will assume that (32) is solved at $c_P \to 1.5 c_P$, $c_D \to 1.5 c_D$ which helps to achieve a continuous behaviour of $N_0$. Actually, at this particular choice of parameters $c_P$ and $c_D$ need to be increased only by factor 1.2 but we have to assume a larger factor to ensure that at moderate variation of parameters, Eq. (32) would return physical results too. As for the second method of choosing $N_0$, we see that Eq. (23) reduces to a quadratic equation $N^2 - c'_1 N_0 - \frac{27}{10} S I_S = 0$ in our case and its solution always exists. Fig. 4 shows that in both cases of parameter variation discussed above, the iterative sequence is convergent.

FIG. 4. Iterative approach for high strain-rate case $S_0 = 11.3$ with $c_P = 0.7$, $c_P = c_D = -c_S = 0.9$. Spatial evolution: (a) and (b)—anisotropy tensor components: red—$a_{11}$, blue—$a_{22}$, green—$a_{33}$. In (a) and (b), thick lines show exact solution, dashed lines—the first-order approximation. (c) and (d) $N$: black line—the exact solution; green, blue, and red lines—zero-order, first-order and second-order approximations, respectively. (a) and (c) $N_0$ calculated from (32) with $c_P \to 1.5 c_P$, $c_D \to 1.5 c_D$; (b) and (d) $N_0$ calculated from $N^2 - c'_1 N - \frac{27}{10} S I_S = 0$ for $\xi_1 \equiv 0$ model.
approaches $N_1$ is close to the exact solution while $N_2 = c'_r + \frac{2}{3} [ -a_{jk} (N_i) S_{jk} - \hat{\xi}_k (N_i) \Gamma_k ]$ almost coincides with it, which means that higher order approximations quickly converge to $N$.

Lazemos et al.\textsuperscript{8} demonstrate in the context of buoyancy-driven flows how to further improve zero-order approximation. It can be achieved by separating the effect of strain and the effect of active scalar with subsequent combination of the effects.

### C. Influence of the density variance

In the general three-dimensional flow, the values of $\hat{\xi}_i T_j, a_{ij},$ and $N$ do not depend on the density variance $\rho'^2$ when $c_T = 0$. For quasi-one dimensional nozzle flow, this is illustrated by formulas (26) and (28) which show that $\hat{\xi}_i T_j, a_{\alpha\alpha},$ and $N$ at $c_T = 0$ depend on the normalized density gradient and normalized substantial derivative of mean velocity only through a combination $H_{\Gamma T} = \Gamma_1 T_1 = \tau_2 D_i U_x \partial_x \hat{\rho} / \hat{\rho}$. Allowing $c_T$ to be non-zero, we have to account for the values of $H_T = \Gamma_1 \sim \rho'^2$ and $H_{\Sigma T} = S_{11} H_T$.

Causch-Schwarz inequality states that the condition $\sqrt{\rho'^2} \geq \frac{\rho'^2}{\sqrt{\kappa}}$ has to be fulfilled everywhere. Algebraic model for density velocity correlation (15) at $c_p = 1$ shows that $\hat{\xi}_i$ is the sum of a term proportional to $N^{-1} \Gamma_k \sim 1/\sqrt{\rho'^2}$ and a term proportional to $N^{-1} c_T \partial_k / \sqrt{\rho'^2}$. Therefore, the order of $\sqrt{\rho'^2}$ has to be not less than the order of $| N^{-1} \tau \sqrt{\kappa} \partial_k \hat{\rho} |$ and not larger than the order of $N \rho' \sqrt{\kappa}/c_T \tau (D_i U_k - g_k)$. \footnote{\textsuperscript{1}}

For this reason while $\rho'^2$ is chosen so that $| \hat{\xi}_i | \leq \sqrt{\kappa}$, the order of $\Gamma_i$ is not less than the order of $N^{-1} c_T \tau_2 D_i U_k \partial_i \hat{\rho} / \rho$ and not higher that the order of $N$. But the density variance is described by Eq. (7) which we rewrite here as

$$\partial_t \rho'^2 + U_k \partial_t \rho'^2 + \partial_k \rho'^2 u'_k = -\tau^{-1} \rho'^2 \left( 2 D + 2 (1 - c_u) \hat{\xi}_i \Gamma_i + c_p \right),$$

where modeling (A2) (with $\partial_D = 0$) is used. All the quantities in parentheses on the right-hand side are limited. Hence in the case of steady quasi-one-dimensional nozzle flow (with low $K$ to be able to neglect the diffusion term), we have exponential decay (or growth) of the density variance: $\partial_x \ln \rho'^2 \sim - (\tau U)^{-1} (c_p + 2 D + 2 (1 - c_u) \hat{\xi}_i \Gamma_i)$.

Considering the expansion from subsonic to supersonic regimes with $\Gamma_i < 0, \Gamma_T > 0$, we recall (Sec. III A) that at $c_T = 0$, normalized density-velocity correlation $\xi_i$ is always negative and $\rho'^2$ exponentially decays. Positive $c_T$ increases $\xi_i$ and when $H_T$ is comparable in magnitude to $H_{\Gamma T}$, density-velocity correlation can even become positive. Then, the “baroclinic” production becomes negative and the second term in (33) becomes positive. A possible restriction from above on the value of $\rho'^2|_{x=0}$ is to ensure that $| \Gamma_i |$ is not larger than $| \Gamma_i |$. We find that at $\rho'^2|_{x=0} = 0.4 (\bar{\rho}^3 K/U^2)|_{x=0}$, the maximum values are $| \Gamma_i |_{\text{max}} \approx | \Gamma_i |_{\text{max}} \approx 1.5$ for the case of low-strain rate and $| \Gamma_i |_{\text{max}} \approx | \Gamma_i |_{\text{max}} \approx 7.5$ for the case of high-strain rate. Fig. 5 illustrates the behavior of the density variance and normalized density velocity correlation at $\rho'^2|_{x=0} = 0.4 (\bar{\rho}^3 K/U^2)|_{x=0}$ at the same model parameters as in the previous figures. Now, we present also the results with non-zero $c_T$ (dashed lines).

The general trend is that by increasing $c_T$, we enhance “incompressible” and “dilatational” productions but the “baroclinic” component decreases substantially reducing the total production while $a_{11}$ decreases and $a_{22}, a_{33}$, and $\xi_1$ increase (with exception that total production may increase a bit in “incompressible” regime when $c_T$ deviates from unity too much). This is the same trend as if we would increase $c_p, c_D, c_T$, or reduce $c_S$, as described in the end of Sec. III A. However, in the regions where $\xi_i$ is positive (due to $c_T \neq 0$), the components of anisotropy and production, $\xi_i$, and total production react in the opposite way to the increase in $c_p, c_D, c_T$ and decrease in $c_S$. Note that the model is very sensitive in $c_T$ and increasing this parameter, we get a risk to achieve unrealizable values of $a_{11}$ (less than $-2/3$).

A short discussion of the generalization of model (15) to $c_p \neq 1$ follows. In this case in the limit $\rho'^2 \rightarrow 0$, the normalized density-velocity correlation does not go to infinity but approaches
Spatial evolution: (a) and (b)—density variance $\frac{\rho'\xi}{\rho'^2}(0)$. (c) and (d)—normalized density-velocity correlation $\xi_t$. (a) and (c) Low strain-rate $S_0^2 = 1.22$, (b) and (d)—high strain-rate $S_0^2 = 11.3$. Red lines denote solutions for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.5$; blue lines—for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.9$; green lines—for the model with $\bar{\xi}_t = 0$, respectively. Solid lines correspond to $c_\gamma = 0$; dashed lines—to $c_\gamma = 0.3$.

FIG. 5. Spatial evolution: (a) and (b)—density variance $\frac{\rho'\xi}{\rho'^2}(0)$, (c) and (d)—normalized density-velocity correlation $\xi_t$. (a) and (c) Low strain-rate $S_0^2 = 1.22$, (b) and (d)—high strain-rate $S_0^2 = 11.3$. Red lines denote solutions for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.5$; blue lines—for $c_\Psi = 0.7$, $c_\rho = c_D = -c_S = 0.9$; green lines—for the model with $\bar{\xi}_t = 0$, respectively. Solid lines correspond to $c_\gamma = 0$; dashed lines—to $c_\gamma = 0.3$.

a finite value. In the case of quasi-one-dimensional nozzle flow, $\hat{\xi}_t \rightarrow \langle \hat{\xi}_t \rangle_{\text{lim}} = -\sqrt{\frac{2(1-N^{-1}S_{11})}{3(1-c_p)}}$, where $N$ is determined from quadratic equation $N^2 - c'_1 N - \frac{\gamma - 2}{10} II_S = 0$. For example, at $c_p = 0.5$, the magnitude of $\langle \hat{\xi}_t \rangle_{\text{lim}}$ is less than $\sqrt{2}$ for all turbulence regimes in our geometry and is of order 1. Then, the term $\hat{\xi}_t I_i$ (positive and proportional to $1/\sqrt{\rho'^2}$) becomes dominant in equation (33) and quasilinearly drives the square root of the density variance to negative values like $\partial_x \sqrt{\rho'^2} \sim -(\tau U)^{-1}(1-\hat{c}_u)\hat{\xi}_t I_i \sqrt{\rho'^2}$. To prevent the unphysical behaviour of the density variance, we can damp the $\hat{\xi}_t I_i$ term by requiring a positive value of $\bar{c}_D$ in the region where $\hat{\xi}_t I_i$ attains large positive values. One of the options is to suggest $\hat{c}_u = 1 - e^{-\delta \max(0,\hat{\xi}_t I_i)}$, where $\delta$ is a constant. Another pragmatic solution assumes an eddy-viscosity type of modeling like $(1-\bar{c}_u)\hat{\xi}_t = -\bar{c}_D \tau \sqrt{K} \rho^{-1} \partial_i \ln \sqrt{\rho'^2}$ with non-dimensional constant $\bar{c}_D$. Thus, the extension of model (15) to $c_p \neq 1$ makes the lower limit on $\rho'^2$ less stringent. Recall that this is achieved at the price of the need to solve additional consistency equation (21).

If the flow conditions allow us to state that the fluctuations of thermodynamic quantities are approximately adiabatic with $p' \approx c'_2 \rho'$, then the modeling of $\Pi = p' \partial_k u'_k$ can be related to the modeling of $\rho' \partial_k u'_k$ performed in Appendix A. Hence, from (A2) (we retain $\bar{c}_D$ to illustrate the possible effects), it follows

$$\frac{\Pi}{\varepsilon} \approx c'_2 \frac{\rho' \partial_k u'_k}{\varepsilon} = \left( \hat{c}_D - 2 \hat{c}_D D - 2 \hat{c}_D \hat{\xi}_k \Gamma_k \right) M_i^{-2} \rho'^2 / \bar{\rho}^2.$$  

(34)

$\Pi/\varepsilon$ enters only $N$-equation (21) and its effect can be accounted for by redefining $c'_1 \rightarrow c'_1 + \frac{9}{4} \frac{\Pi}{\varepsilon}$. The only combinations in the model which depend on the density variance are $\Pi$ and $c_\Psi^2 \Gamma_1 \Gamma_1$, and we have to estimate the latter quantity too. We will compare it with $\Pi \Gamma \Gamma = \Gamma_k \Gamma_k$ which itself is
of the order of \( M^2 |\tau \partial_t U_i| \), where \( M = U / c_s \) is the flow Mach number and plays a role only at sufficiently large \( M \). Approximating \((D_i U_i - g_i)\) as \(-\partial_t P / \bar{\rho}\) and assuming that the gradients of \( P \) and \( \bar{\rho} \) are also related to each other adiabatically, we arrive at

\[
\frac{|c_T T_j|}{|H T_j|} \approx 2 |c_T| M^{-2} \frac{\rho'^2}{\bar{\rho}^2}.
\] (35)

Therefore, \( \Pi, c_T H_T, \) and \( c_T H T_T \) are, in principle, of similar relative importance specified by a factor \( M^{-2} \frac{\rho'^2}{\bar{\rho}^2} \). In the paper we concentrate on the algebraic model for the density flux and neglect \( H \) completely (to justify or reject modeling (34) would require a lengthy discussion).

Note that if we apply the adiabatic relation to the mixed mean-fluctuating quantities, we will obtain \( \rho'/\bar{\rho} \approx -M^2 u'_k U_k/U^2 \) which leads to the incorrect modeling like \( \rho'^2/\bar{\rho}^2 \approx M^4 M^2 / 3 \left( 1 + \frac{2}{3} a_{11} \right) \).

IV. TWO-DIMENSIONAL MEAN FLOW

A. General model

We now consider the case of two-dimensional mean flow with \( U_z = \Gamma_z = \bar{\Gamma}_z \equiv 0 \) and \( \partial_z \equiv 0 \), and define two-dimensional traceless analogs of the strain- and anisotropy- tensors, while the vorticity tensor can be treated as two-dimensional (\( \Omega_{3i} \equiv 0 \)),

\[
S_{ij}^{(2D)} = S_{ij} - \frac{D}{2} \left( \delta_{ij}^{(2D)} - \frac{2}{3} \delta_{ij} \right) = \left( \begin{array}{cc} \mathcal{J} & \sigma \\ -\sigma & \mathcal{J} \end{array} \right), \quad a_{ij} = a_{ij}^{(2D)} + \beta_0 \left( \delta_{ij}^{(2D)} - \frac{2}{3} \delta_{ij} \right),
\]

\[
\Omega_{ij}^{(2D)} = \Omega_{ij} = \left( \begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right). \quad S_{ik}^{(2D)} \Omega_{kj} = -\Omega_{ik} S_{kj}^{(2D)} = \omega \left( \begin{array}{cc} -\sigma & \mathcal{J} \\ \mathcal{J} & \sigma \end{array} \right), \quad \delta_{ij}^{(2D)} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\] (36)

where

\[
\mathcal{J} = \frac{\tau}{2} (\partial_x U - \partial_y V), \quad \sigma = \frac{\tau}{2} (\partial_y U + \partial_x V), \quad \omega = \frac{\tau}{2} (\partial_y U - \partial_x V), \quad (U, V) = (U_x, U_y).
\] (37)

Since \( S_{ij}^{(2D)} \) and \( a_{ij}^{(2D)} \) have the same structure, the relation \( \Omega_{ik} a_{kj}^{(2D)} = -a_{ik}^{(2D)} \Omega_{kj} \) is also valid. Employing this, we rewrite (20) as

\[
\left( N \delta_{ik}^{(2D)} + 2 \Omega_{ik} \right) a_{kj}^{(2D)} + N \beta_0 \left( \delta_{ij}^{(2D)} - \frac{2}{3} \delta_{ij} \right) = -\frac{6}{5} S_{ij}^{(2D)} - C \left( \bar{\xi}_j T_j + \bar{\xi}_j T_{ij} - T_k \bar{\xi}_k \right) - \left( \frac{3}{5} \mathcal{D} + C \bar{\xi}_k \right) \delta_{ij}^{(2D)} - \left( \frac{3}{5} \mathcal{D} + C \bar{\xi}_k \right) \delta_{ij}^{(2D)}
\]

\[
- \left( \frac{3}{5} \mathcal{D} + C \bar{\xi}_k \right) \delta_{ij}^{(2D)} - \left( \frac{3}{5} \mathcal{D} + C \bar{\xi}_k \right) \delta_{ij}^{(2D)} - \left( \frac{3}{5} \mathcal{D} + C \bar{\xi}_k \right) \delta_{ij}^{(2D)}
\] (38)

where \( \beta_0 \) is univocally determined by equating the three-dimensional tensor groups on both sides of the equation. The remaining part is purely two-dimensional and we multiply (38) from the left by \( \left( N \delta_{mi}^{(2D)} - 2 \Omega_{mi} \right) \) to obtain (after changing \( m \rightarrow i \))

\[
\left( N^2 - 2 \mathcal{I} \Omega \right) a_{ij}^{(2D)} = -\frac{6}{5} \left( N S_{ij}^{(2D)} + 2 S_{ik}^{(2D)} \Omega_{kj} - C \left( \bar{\xi}_j T_j + \bar{\xi}_j T_{ij} - T_k \bar{\xi}_k \right) \right)
\]

\[
\left( N^2 - 2 \mathcal{I} \Omega \right) a_{ij}^{(2D)} = -\frac{6}{5} \left( N S_{ij}^{(2D)} + 2 S_{ik}^{(2D)} \Omega_{kj} - C \left( \bar{\xi}_j T_j + \bar{\xi}_j T_{ij} - T_k \bar{\xi}_k \right) \right)
\] (39)

Now, we introduce two-dimensional \( \Gamma T \)-strain and \( \Gamma T \)-rotation rate tensors \( \delta_{ij}^{(2D)} \) and \( \dot{\Omega}_{ij} \),

\[
\delta_{ij}^{(2D)} = \frac{\Gamma_i T_j + \Gamma_j T_i - 2 \mathcal{I} \Gamma T \delta_{ij}^{(2D)}}{2}, \quad \dot{\Omega}_{ij} = \frac{\Gamma_i T_j - \Gamma_j T_i}{2}, \quad \Gamma_i T_j = \frac{\mathcal{I} \Gamma T}{2} \delta_{ij}^{(2D)} + \dot{\Omega}_{ij}.
\] (40)

The following relations can be shown to be valid:
After performing the manipulations shown in Appendix C, we arrive at an equation for $\hat{\xi}_i$,

$$A_{ij} \hat{\xi}_j = \left( N^2 - 2 H_\Omega \right) \frac{1}{5} \left( N S_{(j)}^{(2D)} + 2 S_{(i)k}^{(2D)} \Omega_{kj} \right) \Gamma_j + N^{-1} \frac{D}{5} \Gamma_i - \frac{2}{3} \left( \Gamma_i + c_T T_i \right),$$

which is solved by inverting the matrix $A_{ij}$ defined by

$$A_{ij} = \tilde{N}_{\xi} \delta_{ij}^{(2D)} + c_S S_{ij}^{(2D)} + \hat{c}_S \tilde{S}_{ij}^{(2D)} + \hat{c}_\Omega \Omega_{ij} + \hat{c}_\Omega \tilde{\Omega}_{ij},$$

$$\tilde{N}_{\xi} = N_{\xi} + c_S \frac{D}{6} - C_\xi \frac{N^{-1}}{6} H_{1\Gamma \gamma} - C_\xi \left( N^2 - 2 H_\Omega \right) \left( N H_{1\Gamma \gamma} + 2 H_\Omega \right) \frac{1}{N}, \quad \hat{c}_S = -C_\xi \frac{N^{-1}}{3},$$

$$\hat{c}_\Omega = c_\Omega + 2 C_\xi \left( N^2 - 2 H_\Omega \right) \frac{1}{N} H_{1\Gamma \gamma}, \quad \hat{c}_\Omega = -C_\xi \frac{N^{-1}}{3} + 2 C_\xi N \left( N^2 - 2 H_\Omega \right)^{-1}.$$  

The inverted matrix $A_{ij}^{-1}$ is

$$A_{ij}^{-1} = \left( \text{det } A \right)^{-1} \left( \tilde{N}_{\xi} \delta_{ij}^{(2D)} - c_S S_{ij}^{(2D)} - \hat{c}_S \tilde{S}_{ij}^{(2D)} - \hat{c}_\Omega \Omega_{ij} - \hat{c}_\Omega \tilde{\Omega}_{ij} \right),$$

$$\text{det } A = \tilde{N}_{\xi}^2 - \frac{1}{2} \left( c_S \frac{D}{6} + 2 c_S \tilde{S}_{12D} + 2 c_\xi \frac{D}{6} \tilde{S}_{12D} + c_\xi \frac{D}{6} \tilde{S}_{12D} + c_\xi \frac{D}{6} \tilde{S}_{12D} + c_\xi \frac{D}{6} \tilde{S}_{12D} + c_\xi \frac{D}{6} \tilde{S}_{12D} \right).$$

Expressing $\hat{\xi}_i$ from (42) and subsequently computing $a_{ij}$ with the use of (39) and (36), we substitute the quantities into consistency relation (21). For the general two-dimensional case, the resulting $N$-equation is higher than fourth-order and has to be solved iteratively, for example, by the iteration sequence outlined in Sec. III B. To arrive at $N_0$, we can reduce the order of the $N$-equation by putting $c_S = c_\Omega = C_\xi = 0$. For two-dimensional flows, this results in fifth-order algebraic equation, as shown in Appendix C. The other method of obtaining $N_0$ is to use Eq. (23) with $\Delta_3 = 0$, corresponding to the $\xi_3 \equiv 0$ model.

**B. Nozzle flow**

We can easily obtain $\hat{\xi}_i$ for the quasi-one-dimensional plane nozzle flow from Sec. III using the general two-dimensional approach explained in Sec. IV A. Indeed,

$$\tilde{N}_{\xi} = N_{\xi} + c_S \frac{D}{6} - \frac{7}{6} C_\xi N^{-1} H_{1\Gamma \gamma}, \quad S_{11} = J + \frac{D}{6}, \quad S_{22} = -J + \frac{D}{6},$$

$$\text{det } A = \tilde{N}_{\xi}^2 - \left( c_S J - C_\xi \frac{N^{-1}}{6} H_{1\Gamma \gamma} \right)^2 = \left( N_{\xi} + c_S S_{22} - C_\xi N^{-1} H_{1\Gamma \gamma} \right) \left( N_{\xi} + c_S S_{11} - \frac{4}{3} C_\xi N^{-1} H_{1\Gamma \gamma} \right),$$

$$\hat{\xi}_i = \text{det}^{-1} A \left[ \tilde{N}_{\xi} \delta_{ik}^{(2D)} - c_S S_{ik}^{(2D)} + C_\xi \frac{N^{-1}}{3} S_{ik}^{(2D)} \right] \left[ 6 N^{-1} S_{kj}^{(2D)} \Gamma_j + N^{-1} \frac{D}{5} \Gamma_k - \frac{2}{3} \left( \Gamma_i + c_T T_i \right) \right] = \left( N_{\xi} + c_S S_{11} - \frac{4}{3} C_\xi N^{-1} H_{1\Gamma \gamma} \right)^{-1} \left[ 6 N^{-1} S_{11} \Gamma_i - \frac{2}{3} \left( \Gamma_i + c_T T_i \right) \right] \frac{1}{0},$$

which is equivalent to solution (26). This simplification is possible because the first multiplicator in the denominator ($N_{\xi} + c_S S_{22} - C_\xi N^{-1} H_{1\Gamma \gamma}$) is analytical cancelled by the first matrix multiplicator in the numerator ($\tilde{N}_{\xi} \delta_{ik}^{(2D)} - c_S S_{ik}^{(2D)} + C_\xi \frac{N^{-1}}{3} \tilde{S}_{ik}^{(2D)}$) in the quasi-one-dimensional limit. However, when the general two-dimensional model of Sec. IV A is applied on quasi-one-dimensional flows, the terms obviously remain. The problem is that these terms might change sign resulting in a singularity.
During the spatial evolution of the quasi-one-dimensional plane nozzle flow, the quantity \( (N_x + c_S S_{22} - C_z N^{-1} \Gamma \gamma) \) can change sign twice. While for low strain-rate cases, it is easy to choose the parameters so that the multiplier always stays negative, for high strain-rate cases this assumes a marginal and stringent choice of parameters. Hence, the quasi-one-dimensional approach to the nozzle flow does not allow us to exclude singularities in determinant \( \det A \) for all turbulence regimes. It is unlikely that the flow would rearrange so to cancel the singularities by the numerator. Such singularities must, hence, be eliminated in the general formulation for practical use. One possibility could be to make an approximation like \( \det A \approx \det A/ (\det^2 A + \chi^2) \), where \( \chi \) is a small number.

We must here stress that the possible singularity is purely an artifact from the algebraic assumption of an equilibrium state. This only means that the corresponding full transport model has no possibility could be to make an approximation like \( \det A \approx \det A/ (\det^2 A + \chi^2) \), where \( \chi \) is a small number.

We must here stress that the possible singularity is purely an artifact from the algebraic assumption of an equilibrium state. This only means that the corresponding full transport model has no possibility could be to make an approximation like \( \det A \approx \det A/ (\det^2 A + \chi^2) \), where \( \chi \) is a small number.

V. THREE-DIMENSIONAL MEAN FLOW

While many important flow cases (with and without active or passive scalars) are two-dimensional, their consideration in the framework of large-eddy simulation requires general three-dimensional formulation of subgrid-scale model. Besides two-dimensional flows, there are also important three-dimensional flow cases. In this section, we develop a direct method to find the general three-dimensional tensor invariant solution to our coupled model for variable-density flows represented by Eqs. (20), (15), and (21).

Equation (20) can be stated as

\[
N a_{ij} = -S_{ij} + \left( a_{ik} \Omega_{kj} - \Omega_{ik} a_{kj} \right),
\]

where \( S_{ij} = \frac{2}{3} S_{ij} + C_z S_{ij} \) and

\[
S_{ij}^{(s)} = \hat{\xi}_i T_j + \hat{\xi}_j T_i - \frac{2}{3} \hat{\gamma}_{ik} \hat{\xi}_k \delta_{ij},
\]

We can apply the general three-dimensional approach based on the use of independent tensor groups to solve this equation for \( a_{ij} \) in terms of \( S_{ij} \), \( \Omega_{ij} \), and \( N \)

\[
a = \beta_1 \tilde{T}^{(1)} + \beta_3 \tilde{T}^{(3)} + \beta_4 \tilde{T}^{(4)} + \beta_6 \tilde{T}^{(6)} + \beta_9 \tilde{T}^{(9)}, \quad \tilde{T}^{(1)} = \tilde{S}, \quad \tilde{T}^{(3)} = \tilde{\Omega} \left( \Omega^2 - \frac{3}{N} \Omega \right), \quad \tilde{T}^{(4)} = \tilde{S} \Omega - \Omega \tilde{S}, \quad \tilde{T}^{(6)} = \tilde{S} \Omega^2 + \Omega^2 \tilde{S} - \frac{2}{3} \tilde{S} \Omega \Omega, \quad \tilde{T}^{(9)} = \Omega \tilde{S} \Omega^2 - \Omega^2 \tilde{S} \Omega, \quad \tilde{S} = \text{tr} \left( \tilde{S} \Omega \right),
\]

\[
\beta_1 = -N (2 N^2 - 7 \Omega \Omega), \quad \beta_3 = -12 N \Omega^{-1}, \quad \beta_4 = -2 (2 N^2 - 2 \Omega \Omega) Q^{-1}, \quad \beta_6 = -6 N Q^{-1}, \quad \beta_9 = 6 Q^{-1}, \quad Q = (N^2 - 2 \Omega \Omega) (2 N^2 - 2 \Omega \Omega).
\]

Equation (48) shows that the anisotropy tensor depends linearly and additively on \( S_{ij} \) and \( S_{ij}^{(s)} \) (we include invariant \( IV \) into the definition of the third tensor group, not into the coefficient before it). Thus, one can formally split \( a_{ij} \) into two parts \( a_{ij}^{(0)} + a_{ij}^{(s)} \), where the former can be readily calculated and depends on \( S_{ij}, \Omega_{ij}, N \) while the latter depends on \( \Omega_{ij}, \hat{\xi}_i, T_i, N \).

Aiming at a proper treatment of Eq. (15), we note that the term \( a_{ij}^{(s)} \) can be algebraically transformed into \( C_z \tilde{T}_{ij}^{(s)} \) with \( L_{ij}^{(s)} \) depending on the known quantities \( \Omega_{ij}, T_i, T_j, N \) and one unknown quantity \( N \). For example, the \((+)\)-part of \( \tilde{T}_{ij}^{(s)} \) transforms as

\[
\tilde{T}_{ij}^{(s)} = C_z \tilde{T}_{ij}^{(s)} + L_{ij}^{(s)} N^{-1} \tilde{T}_{ij}^{(s)},
\]
Performing similar transformation for the \((+)-\)parts of all tensor groups as shown in Appendix D, we write Eq. (15) as

\[
\left( \tilde{T}^{(+)}_{ij} \right) = C_\xi \left( \tilde{\xi}_i \Gamma_j + \xi_j \Gamma_i - \frac{2}{3} \tilde{\xi}_k \Gamma_k \delta_{ij} \right) \Gamma_j = C_\xi \left( I \Gamma \Gamma \delta_{ij} + \Gamma_j \Gamma_i - \frac{2}{3} \Gamma_j \Gamma_j \right) \tilde{\xi}_j.
\]

(49)

We solve this equation by inverting the matrix on the left-hand side to find \(\tilde{\xi}_i\) in terms of \(S_{ij}, \Omega_{ij}, \Gamma_i, \Gamma_j, N\). Subsequently, (48) serves to express \(a_{ij}\) through \(S_{ij}, \Omega_{ij}, \Gamma_i, \Gamma_j, N\). Finally, the consistency relation \(N = \frac{3}{4} \left( c_1 - a_{ij} S_{ij} - \tilde{\xi}_k \Gamma_k \right)\) provides us with an algebraic equation for \(N\). Generally, this equation is extremely complex and even assuming \(c_S = c_\Omega = C_\xi = 0\), we cannot obtain a tractable equation for a zero-order approximation \(N_0\). One of the few exceptions is the irrotational \(II_{\Omega}\) case when \(N_0\) at \(c_S, c_\Omega, C_\xi = 0\) is governed by a cubic equation

\[
N_0^3 - (c_1 - F_0) N_0^2 - \left( c_1' F_0 + \frac{27}{10} I S + \frac{27}{4} (I I \Gamma \Gamma + c_\Gamma I I \Gamma) \right) N_0 - \left( \frac{27}{10} I S F_0 - \frac{27}{4} \frac{9}{5} III S \Gamma \Gamma \right) = 0.
\]

(51)

In Sec. III B, we proposed to use the solution to quartic equation (23) with \(\Delta_3 = 0\) as \(N_0\) in general three-dimensional flow despite the fact that in the three-dimensional model with zero density-velocity correlation, the \(N\)-equation is sixth-order. Therefore, employing iterative sequence (31), we treat corrections due to three-dimensionality and to non-zero \(\tilde{\xi}_i\) on an equal footing.

VI. CONCLUSION

We have developed an explicit algebraic Reynolds stress model for variable density turbulent flows that describes both the effects of mean dilatation and density-velocity correlation. Mean dilatation is accounted for by a proper extension of the rapid pressure-strain correlation model for incompressible flows to compressible flows. An algebraic model for the density-velocity correlation which allows to linearly relate the density-velocity correlation to the mean density variation as well as to the substantial derivative of the mean velocity has been developed. Due to the density-velocity correlation, the pressure-strain correlation acquires an additional term proportional to the density-velocity correlation which acts as a relaxation of the “baroclinic” component of the production in the Reynolds stress tensor equation.

The solution strategies for the coupled algebraic equations for the anisotropy tensor and the density-velocity correlation have been analyzed. To illustrate the applications of the model in combination with a \(K - \omega\) model, we considered the case of quasi-one-dimensional plane nozzle flow. In this case, the complexity of our coupled model is reduced to the need to solve a quartic equation for \(N\). A physical root exists everywhere in the nozzle if certain constraints on the model parameters are imposed. Assuming that we restrict ourselves from above by moderately high strain-rates \((\bar{S}^{0} \lesssim 12)\), it is possible to choose a universal set of parameters that admits a physical root to the quartic equation and realizable behaviour of the model in different turbulence regimes. We found that the behaviour of the anisotropy tensor is less likely to become unrealizable in comparison to a model that neglects the turbulence density fluxes. The influence of positive “baroclinic” production \(\mathcal{P}\) is to make turbulence more isotropic. Another remarkable effect is that while the “incompressible” and “dilatational” components of production are reduced due to the influence of the density flux, the “baroclinic” component grows substantially and causes the total production to grow too.

On the contrary, the density variance through the coupling with “local mean acceleration” acts to increase the anisotropy and suppress the density flux and turbulence production. The effect can change the direction of the density flux. The components of the anisotropy tensor and production
react in opposite ways on the variation of the model parameters in the regions with different signs of the density flux.

General two-dimensional and general three-dimensional mean flows also admit explicit solutions for the density-velocity correlation and anisotropy tensor. This is achieved by noting that right-hand side in the anisotropy tensor equation is linear in both strain tensor and density-velocity correlation which allows us to formally solve the equation. Substituting this solution into the equation for the density-velocity correlation, we transform it so that it can be explicitly solved for the density-velocity correlation depending only on one unknown quantity \( N \). Deriving the equation for \( N \) from the consistency relation, we note that its order is smaller or equal to four only in specific flow cases and an iteration procedure is needed to solve it. A zeroth-order approximation can be chosen as an exact solution to the \( N \)-equation after reducing its order by putting several parameters to zero or as an exact solution to \( N \)-equation corresponding to two-dimensional model with zero density-velocity correlation. We have demonstrated that iteration sequence in the case of quasi-one-dimensional plane nozzle flow converges rapidly with first-order approximation very close to the exact solution.

The new model allows to self-consistently consider a wide class of turbulent variable density flows such as combustion engine flows and jet, nozzle, and supercritical flows. We have initiated studies to validate the model against the DNS results in reacting turbulent flames with heat release and in a supercritical flow with heat transfer in a gravity field. Importantly, tensor invariant three-dimensional formulation gives a possibility to employ it not only in RANS (Reynolds-averaged Navier-Stokes) framework but also in LES (Large Eddy Simulation) framework too.

ACKNOWLEDGMENTS

Support from the Swedish Research Council VR through Grant Nos. 2010-3938, 2013-5784, and 2014-5700 is gratefully acknowledged.

APPENDIX A: MODELING OF THE TERMS IN THE DENSITY VARIANCE AND DENSITY VELOCITY CORRELATION EQUATIONS

The first and second terms in the last of line of equation (9) represent the interaction of fluctuating density and fluctuating velocity, respectively, with fluctuating dilatation of the flow which originates from the penultimate term in Eq. (5). To get insight into the influence of \( \partial_k u_k' \), we rewrite the equation for the fluctuating density as

\[
D_t \rho' = -\rho' \partial_k U_k - u_k' \partial_k \bar{\rho} - \bar{\rho} \partial_k u_k' - \partial_k (\rho' u_k') + \partial_k \rho' u_k',
\]

where the term \( \partial_k (\rho' u_k' - \bar{\rho} u_k') \), quadratic in fluctuating quantities, is neglected. We make here an assumption that the right-hand side of this equation may effectively be represented as a sum of terms which linearly depends on non-differentiated fluctuating quantities \( u_i' \), \( \rho' \) and may linearly depend on the spatial derivatives of the mean quantities \( U_i, \bar{\rho} \). Constructing the term \( \bar{\rho} \partial_k u_k' \), we notice that \( D_t \rho' \) allegedly has a “relaxational” component proportional to \( -\tau^{-1} \rho' \) which leads to the corresponding constituent in \( \bar{\rho} \partial_k u_k' \). Physically, the constituent proportional to \( \rho' \) has to be of the form \( \rho' \delta_k U_i \) while the constituent proportional to \( u_k' \)—of the form \( u_k' \partial_k \bar{\rho} \). We arrive at the following modeling of the fluctuating dilatation:

\[
\partial_k u_k' = \frac{\rho'}{\bar{\rho}} \left( \frac{C_\rho}{2} \tau^{-1} - C_{lm} \partial_l U_m \right) - C_u u_k' \partial_k \bar{\rho} - \land  
\]

\[
\text{(A1)}
\]

The parameters \( C_\rho, C_{lm}, C_u \) are postulated to be of the form \( C = c \cos(\phi) \) with constant moduli \( c \) and fluctuating random phases \( \phi \). Neglecting the correlations of third-order in \( \rho' \), we obtain

\[
\begin{align*}
\bar{\rho} \rho' + \rho'^2/2 \partial_k u_k' &= \bar{c}_D \tau^{-1} \rho'^2/2 - \bar{c}_D \rho'^2 \partial_k U_k - \bar{c}_k \rho' u_k' \partial_k \bar{\rho} \\
\text{(A2)}
\end{align*}
\]
for the half of the last term in density-variance equation (7) and the corresponding term in the density-velocity correlation equation is

\[
(\bar{\rho}^2 + \rho^2/2) \partial_t u_k^c \xi_i = \bar{\rho}^2 (\bar{\rho} u_k^c \partial_t \xi_i) - \rho^2 (\bar{\rho} u_k^c \partial_t \xi_i).
\]

Dilatation \( \partial_t U_k \) is the only linear invariant of the tensor \( \partial_t U_k \), therefore only phase fluctuations in \( C_{lm} \) can violate the condition \( C_{lm} = c D_{lm} \). This allows to model the correlation \( \rho^2 u_k^c C_{lm} \partial_t U_m \) not only as \( c_D \rho^2 u_k^c \partial_t U_k \) but also to add terms proportional to \( \rho^2 u_k^c \partial_t U_k \) and \( \rho^2 u_k^c \partial_t U_k \).

\[
\frac{\bar{\rho}}{\sqrt{\rho^2 + \bar{\rho}^2}} u_k^c \partial_t U_k = c_D \rho^2 \xi_i/2 - c_m \xi_k \partial_t U_i - c_n \xi_k \partial_t U_k - c_D \xi_i \partial_t U_k - c_n \xi_i \partial_t U_k - c_D \xi_i \partial_t U_k - c_n \xi_i \partial_t U_k - \left( a_{ik} + \frac{2}{3} \delta_{ik} \right) \Gamma_k.
\]

The difference between \( u_k \) and \( U_k \) has been neglected in the last formula. Parameters \( c_D \) and \( c_n \) are defined as \( c_D \rho^2 = \rho^2 C_{D} \) and \( c_n \rho^2 u_k^c = \rho^2 u_k^c C_{n} \) and can be different due to different phase correlations of \( C_{D} \) with \( \rho^2 \) and \( u_k^c \). Similarly, \( c_D \neq c_D \) and \( c_n \neq c_n \) in general.

Expanding the remaining third term in the last line of (9), we see that

\[
\frac{1}{\sqrt{\rho^2 + \bar{\rho}^2}} \rho^2 (\partial_t \rho - \partial_t \tau_{ik}) \equiv \frac{\rho^2 (\partial_t u_i + u_k \partial_t u_i - g_i)}{\bar{\rho}} + \frac{1}{\sqrt{\rho^2 + \bar{\rho}^2}} \rho^2 (\partial_t u_i - \partial_t \tau_{ik}).
\]

We note that the first part on the right-hand side (rewritten by using the momentum equation) is closely approximated by \( \tau^{-1} \xi_i \) while straightforward calculation of the second part gives \( \left( -\tau^{-1} \xi_i \partial_k U_i \right) \). Thus, we can rewrite the last term in (9) as

\[
\frac{1}{\sqrt{\rho^2 + \bar{\rho}^2}} \rho^2 (\partial_t \rho - \partial_t \tau_{ik}) \equiv \frac{-\xi_k \partial_t U_i}{\sqrt{\rho^2 + \bar{\rho}^2}}.
\]

From this we may conclude that the second part in (A5) cancels the first part \( \tau^{-1} \xi_i \) and modifies the constant \( c_m \) when added to (A4). Although the second term on the right-hand side of (A6) is completely untractable and its modeling would have to be of ad hoc character, we may infer that this term can contribute a relaxation term proportional to \( \tau^{-1} \xi_i \). On the other hand, \( \rho^2 \rho^2 (\partial_t \rho - \partial_t \tau_{ik}) \) in (A5) may be interpreted as a mean acceleration (with minus) that the flow acquires due to the interaction of fluctuating hydrodynamic forces and fluctuating density. This acceleration may be viewed as influenced by the emerging density flux \( \rho^2 u_k^c \), whose direction may be characterized by tensor \( \xi_i \xi_k \). \( \xi_i / \sqrt{2} \) are the correlation coefficients between fluctuating density and fluctuating velocity and are typically substantially less than unity (note that \( |\xi_i| < \sqrt{2} \) due to Cauchy-Schwarz inequality). From (9), it follows that the density flux itself is driven mainly (if not exclusively) by the term proportional to \( -R_{ik} \partial_k \bar{\rho} / \bar{\rho} \). Inspired by this analogy, we conclude that the acceleration can be modeled as \( -\xi_k \xi_i K \partial_k \bar{\rho} / \bar{\rho} \). Combining the modeling of the last term in (9) based on (A5) and (A6), we conclude that it may contain a term proportional to \( \tau^{-1} \xi_i \), a term proportional to \( \xi_k \partial_k U_i \) (which we account for by modifying the constant \( c_m \) in (A4)), a term proportional to \( \tau^{-1} \xi_i \), and a term proportional to \( \tau^{-1} \xi_i \xi_k \). Introducing constants \( c_{\xi} \), \( c_T \), and \( c_{\xi} \), we arrive at

\[
\frac{1}{\sqrt{\rho^2 + \bar{\rho}^2}} \rho^2 (\partial_t \rho - \partial_t \tau_{ik}) = c_T \tau^{-1} \xi_i + \frac{2}{3} c_T \tau^{-1} \xi_i + c_T \tau^{-1} \xi_i \xi_k \Gamma_k.
\]

We may expect that \( c_{\xi} \) is positive while comparing the two approaches to modeling of (A5) we may conclude that \( c_T \) ranges from 0 to 1.5. The choice of the value of the sum \( c_T + c_T \) is discussed in the main text. In this paper, we assume \( c_T = 0 \), which means that the last term in Eq. (7) does not give a contribution analogous to the fourth term and also \( c_T = 0 \) not to modify the production term due to \( \Gamma_k \) in Eq. (9) (third term on the second line). In Sec. III C, we prove that positive value of
\( \bar{c}_u \) can be necessary in Eq. (7) to prevent the density variance \( \bar{\rho}^2 \) from achieving negative values. With the definitions \( c_{p} = \bar{c}_{p}/2 - c'_{p}/2 - c''_{p} \) and \( c_{p} = \bar{c}_{p} + \bar{c}_u \), we arrive at relations (13) and (14).

Applying the above modeling to Eq. (8), we can write the terms with the density flux \( \bar{\rho}' u_j' \) in Eq. (2) as

\[
- \left( \partial_t \bar{\rho}' u_i' + \partial_j (\bar{\rho}' u_i' U_j) + \partial_j (\bar{\rho}' u_j' U_i) \right) = -U_i \partial_j \bar{\rho}' u_j' + \bar{\rho} \partial_j \frac{\bar{\rho}' u_j'}{\bar{\rho}} + \frac{2}{3} c_{p} \frac{\bar{\rho}^2}{\bar{\rho}} (D_i U_i - g_i) +
\]

\[
+ \left[ (\bar{c}'' \rho + c_{p}/2) \tau^{-1} - c_{d} \partial_l U_l \right] \delta_{ij} - c_{m} \partial_j U_i - c_{n} \partial_i U_j \right) \bar{\rho}' u_j' + \left( \bar{c}_{p} \frac{\bar{\rho}' u_i' \bar{\rho}' u_j'}{\bar{\rho}^2} + \frac{R_{ij}}{\bar{\rho}} \right) \partial_j \bar{\rho}.
\]

(A8)

**APPENDIX B: SOLUTION TO QUARTIC EQUATION**

Any attempt to apply explicit algebraic modeling to complex geometries or to account for the influence of an active scalar evokes the need to solve an algebraic equation for \( N \) higher than third-order. While tools for solving such equations are offered by iterative methods, we may expect that employing exact solutions to quartic equations emerging in different flow cases can be useful too. In Grigoriev et al., we have shown that by an appropriate choice of solution method it is possible to univocally identify the physical root of quartic equation arising in the case of a general two-dimensional compressible flow with zero density-velocity correlation. Here, we will give the solution to Eq. (28) describing quasi-one-dimensional nozzle flow. Although the general solution to quartic equation is well known, for reference purposes we present it here in the form which allows us to identify the physical root.

We rewrite (28) as

\[
N^4 - q N^3 - AN^2 - BN - C = 0, \quad q = c_{p}' - F,
\]

\[
A = \frac{27}{10} II_S + c_{p}' F + 6C_{\zeta} II_{\Gamma \Gamma} + \frac{27}{4} \left( II_{\Gamma \Gamma} + c_{p} II_{\Gamma} \right), \quad C = \frac{81}{5} C_{\zeta} II_{\Gamma \Gamma} \left( \frac{3}{2} S_{11}^2 - II_S \right),
\]

\[
B = \frac{27}{10} II_S F - 6c_{p}' C_{\zeta} II_{\Gamma \Gamma} - \frac{9}{5} \frac{27}{4} II_{\Gamma \Gamma} II_{\Gamma} - \frac{27}{2} C_{\zeta} \left( II_{\Gamma \Gamma} + c_{p} II_{\Gamma} \right). \tag{B1}
\]

The solution to (B1) can be easily extracted from

\[
N = \frac{S}{2} + \frac{q}{4} \left( S \mp \sqrt{O} \right)^2 = -O + w \pm \frac{q^3 + 4 A q + 8 B}{4 \sqrt{O}},
\]

where square root is chosen to be positive and \( O \) is determined by

\[
O^3 - w \sqrt{O} - a O - b = 0, \quad w = \frac{3}{4} q^2 + 2A,
\]

\[
a = -\left( \frac{3}{16} q^4 + A q^2 + A^2 + B q + 4 C \right), \quad b = \frac{1}{4} \left( \frac{q^3 + 4 A q + 8 B}{4} \right)^2.
\]

We choose the following root \( O_1 \):

\[
O_1 = \frac{1}{3} \left\{ w + \left( Q + \sqrt{D} \right)^{1/3} + \left( Q - \sqrt{D} \right)^{1/3} \right\} \text{ for } D \geq 0,
\]

\[
O_1 = \frac{w}{3} + \frac{2}{3} \sqrt{P} \cos \left( \frac{1}{3} \arccos \frac{Q}{\sqrt{P^2}} \right) \text{ for } D < 0,
\]

\[
D = Q^2 - P^2, \quad P = w^2 + 3 a, \quad Q = w^3 + \frac{9}{2} a w + \frac{27}{2} b,
\]

and noting that

\[
\left( S \pm \sqrt{O_1} \right)^2 = -O_1 + w \pm \frac{q^3 + 4 A q + 8 B}{4 \sqrt{O_1}},
\]
\[ D^+ = -O_1 + w + \frac{q^3 + 4 A q + 8 B}{4 \sqrt{O_1}}, \quad D^- = -O_1 + w - \frac{q^3 + 4 A q + 8 B}{4 \sqrt{O_1}} \]

we write solution to (28)

\[ N^{(1),(2),(3),(4)} = \frac{q}{4} + \frac{S^{(1),(2),(3),(4)}}{2}. \]  

**APPENDIX C: DETAILS OF THE GENERAL TWO-DIMENSIONAL MODEL**

Multiplying both sides of (39) by \( \Gamma_j \) and noting that

\[
\left( N \delta_{ik}^{(2D)} - 2 \Omega_{ik} \right) \left( \hat{\xi}_k \hat{T}_j + \hat{\xi}_j \hat{T}_k - \hat{\gamma}_i \hat{\xi}_l \delta_{ik}^{(2D)} \right) = \left( N \delta_{ik}^{(2D)} - 2 \Omega_{ik} \right) \left( \hat{\xi}_k \hat{T}_j + \hat{\xi}_j \hat{T}_k - \hat{\gamma}_i \hat{\xi}_l \delta_{ik}^{(2D)} \right)
\]

we arrive at

\[
-\left( N^2 - 2 \Pi \Omega \right) \delta_{ij} = \frac{6}{5} \left( N S_{ij}^{(2D)} + 2 \delta_{ij}^{(2D)} \Omega_{ik} \Omega_{kj} \right) \Gamma_j + C_\varepsilon \left( N \Pi \Omega \Omega_{ij} + N \Omega_{ij} \right) \hat{\xi}_j.
\]

Using \( a_{ij} = \frac{\delta_{ij}^{(2D)} + \beta_0 (\delta_{ij}^{(2D)} + \frac{2}{5} \delta_{ij}) \varepsilon_i \varepsilon_j}{\frac{2}{3}} \) with \( \beta_0 \) from (38), we rewrite Eq. (15) as

\[
\left[ N \delta_{ij} + c_S \frac{D}{6} + c_S S_{ij}^{(2D)} + c_\varepsilon \Omega_{ij} - C_\varepsilon \frac{N^{-1}}{3} \Gamma_j \right] \hat{\xi}_i = -a_{ij} \delta_{ij}^{(2D)} \Gamma_j + N^{-1} D \frac{2}{3} \Gamma_j + \frac{2}{3} \left( \Gamma_j + c_\varepsilon \Omega_{ij} \right).
\]

Collecting the terms in this purely two-dimensional equation, we arrive at (42).

We will derive zeroth-order approximation of \( N \)-equation with \( c_S = c_\varepsilon = C_\varepsilon = 0 \) keeping in mind the first variant of iterative method proposed in Sec. III B and illustrated there for the case of quasi-one-dimensional nozzle flow. Defining \( F_0 \) as \( N \hat{\xi}_i \hat{T}_i = N \hat{\xi}_i \hat{T}_i - 2 \Pi \Omega N^{-1} \frac{D}{6} \Pi \Omega + 2 IV_{S^2D \Omega \Gamma} \Omega \Omega - \frac{2}{3} \left( \Pi \Omega \Omega + c_\varepsilon \Pi \Omega \right) \Omega \Omega, \)

Using it and \( a_{ij} = \frac{\delta_{ij}^{(2D)} - \frac{3}{5} D \delta_{ij}^{(2D)} - \frac{2}{3} \delta_{ij}}{\frac{2}{3}} \) \( \delta_{ij}^{(2D)} \) is determined by (39) with \( C_\varepsilon = 0 \), we write (21) as

\[
N = c'_i + \frac{\frac{9}{4} \left( N^2 - 2 \Pi \Omega \right)^{-1} \frac{6}{5} \left( N III \Pi \Omega \Omega \Pi \Omega \Pi \Omega + N^{-1} D \frac{2}{3} \Pi \Omega \Omega + 2 IV_{S^2D \Omega \Gamma} \Omega \Omega \right) - \frac{2}{3} \left( \Pi \Omega \Omega + c_\varepsilon \Pi \Omega \right) \Omega \Omega, \)
\]

which can be transformed to the fifth-order in \( N \) equation

\[
N_0^4 - \left( c'_i - F_0 \right) N_0^3 - \left( c'_i F_0 + \frac{27}{10} \Pi \Omega I + 2 II \Omega + \frac{27}{4} \left( \Pi \Omega \Omega + c_\varepsilon \Pi \Omega \right) \Omega \Omega \right) N_0^2 - \left( \frac{27}{10} \Pi \Omega I + 2 II \Omega \right) N_0 + II \Omega \left( \frac{9}{10} D^2 + 2 c'_i F_0 + \frac{27}{2} \left( \Pi \Omega \Omega + c_\varepsilon \Pi \Omega \right) \Omega \Omega \right) +
\]

\[ + \frac{243}{10} IV_{S^2D \Omega \Gamma \Omega} + \frac{9}{10} II \Omega \delta \left( F_0 - \frac{9}{10} II \Omega \delta \right) N_0^{-1} = 0, \quad IV_{S^2D \Omega \Gamma \Omega} = \Gamma_1 S_{ij}^{(2D)} \Omega_{ik} \hat{T}_j. \]
In the case of zero dilatation $D$, the equation becomes of fourth order, and in the case of no rotation $(II_{ij} = 0)$ – of third order.

**APPENDIX D: THREE-DIMENSIONAL SOLUTION OF $\zeta_i$-EQUATION**

We can perform the transformations similar to (49) for all the $(\pm)$-parts of tensor groups (48),

$$L_{ij}^{(r)} = \beta_1 L_{ij}^{(1)} + \beta_3 L_{ij}^{(3)} + \beta_4 L_{ij}^{(4)} + \beta_6 L_{ij}^{(9)}, \quad \left(\tilde{T}_{ij}^{(n)} \Gamma_j\right)^{(+)c} = C_L L_{ij}^{(n)} \hat{p}_j,$$

$$L_{ij}^{(1)} = II_{ij} \delta_{ij} + \frac{2}{3} \Gamma_i \Gamma_j, \quad L_{ij}^{(2)} = 2 \Omega_{ij} \Omega_{kl} \left[\frac{II_{ij} \Gamma_l}{3} - \frac{II_{ij} \Gamma_l}{3} \right] \left[\frac{II_{ij} \Gamma_m}{3} - \frac{II_{ij} \Gamma_m}{3} \right],$$

$$L_{ij}^{(3)} = \frac{4}{3} \left[II_{ij} \Omega_{kl} \Gamma_k + \frac{II_{ij} \Omega_{kl} \Gamma_k}{3} \right] + \frac{4}{9} II_{ij} \Gamma_i \Gamma_j,$$

$$L_{ij}^{(4)} = (\Omega^2)_{ij} \frac{II_{ij} \Omega_{kl} \Gamma_k + \frac{II_{ij} \Omega_{kl} \Gamma_k}{3} \right] + \frac{4}{9} II_{ij} \Gamma_i \Gamma_j,$$

$$L_{ij}^{(9)} = (\Omega^2)_{ij} \frac{II_{ij} \Omega_{kl} \Gamma_k + \frac{II_{ij} \Omega_{kl} \Gamma_k}{3} \right] + \frac{4}{9} II_{ij} \Gamma_i \Gamma_j.$$

Then the solution to (50) is obtained by inverting the matrix $(\tilde{N}_{ij} \delta_{ij} + M_{ij})$ before $\tilde{\zeta}_i$,

$$M_{ij} = c_S S_{ij} + c_0 \Omega_{ij} + c_1 \tilde{L}_{ij}(\Gamma^{(r)}), \quad \text{tr} M = 0, \quad II_M = \text{tr} M^2, \quad III_M = \text{tr} M^3,$$

$$(\tilde{N}_{ij} \delta_{ij} + M_{ij})^{-1} = \det^{-1} A \left[\left(\tilde{N}_i^2 - II_M / 2\right) I_3 - \tilde{N}_i \text{M} + M^2\right],$$

$$\det A = \tilde{N}_i^3 - II_M M^2 / 3.$$

---

22. Errata for Grigoriev et al.: The last term on the right-hand side of (A.5) has wrong sign; the signs of the third and the fourth terms of $T_{ijk}$ in (A.12) are incorrect; the second part in the chain of equalities (56) has wrong sign; also the signs in the first, second, and third terms in the unnumbered formula between (54) and (55) are incorrect.