A stochastic subgrid model with application to turbulent flow and scalar mixing

Linus Marstorp,[a)] Geert Brethouwer,[b)] and Arne V. Johansson[c)]
Department of Mechanics, Royal Institute of Technology, KTH, Stockholm SE-100 44, Sweden

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A new computationally cheap stochastic Smagorinsky model which allows for backscatter of subgrid scale energy is proposed. The new model is applied in the large eddy simulation of decaying isotropic turbulence, rotating homogeneous shear flow and turbulent channel flow at $\mathrm{Re}_\tau=360$. The results of the simulations are compared to direct numerical simulation data. The inclusion of stochastic backscatter has no significant influence on the development of the kinetic energy in homogeneous flows, but it improves the prediction of the fluctuation magnitudes as well as the anisotropy of the fluctuations in turbulent channel flow compared to the standard Smagorinsky model with wall damping of C_S. Moreover, the stochastic model improves the description of the energy transfer by reducing its length scale and increasing its variance. Some improvements were also found in isotropic turbulence where the stochastic contribution improved the shape of the enstrophy spectrum at the smallest resolved scales and reduced the time scale of the smallest resolved scales in better agreement with earlier observations.
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I. INTRODUCTION

Accurate descriptions of turbulent flows are desirable in many engineering applications and geophysical situations. A promising method in this respect is large eddy simulation (LES) which is an under-resolved numerical simulation of the Navier-Stokes equation whereby the influence of the unresolved, subgrid scales (SGSs) on the large scales has to be modelled. Many approaches to develop subgrid models in LES have been proposed and many of them rely upon some kind of scale similarity assumption between the resolved and unresolved scales, and depend on the resolved scales in a deterministic way. The widely known Smagorinsky model for the unclosed subgrid stress is based on a mixing length hypothesis whereby a SGS velocity scale is constructed from the resolved rate of strain and the filter length scale (Meneveau[1]). The model has several known drawbacks. Two examples are that it does not provide for backscatter of turbulent kinetic energy and that it overpredicts the correlation time of the resolved turbulence as pointed out by He *et al.*[2] The dynamic model Germano *et al.*[3] provides for backscatter but the model constant yields large fluctuations and it can easily become unstable. Applying averaging in homogeneous directions to obtain the constant eliminates the stability problem but the model loses generality and the ability to account for backscatter. In the similarity model developed by Bardina *et al.*[4] the full unfiltered velocity field is estimated by the resolved velocity and the SGS stress is computed using a secondary explicit filtering at equal or larger filter scale. The model does not provide for enough dissipation and is usually used in an *ad hoc* combination with a Smagorinsky term. A recent SGS model by Stolz and Adams[5] is the approximate deconvolution model (ADM), which can be regarded as a generalized similarity model. Deterministic models based on scale similarity have proven to perform well in many situations. However, it is well known that interaction between the resolved scales and subgrid scales give rise to random fluctuations in the SGS stress (Leslie and Quarini[6] and Kraichnan[7]). Consequently, the subgrid scale stress tensor extracted from direct numerical simulation (DNS) contains stochastic noise that cannot be modelled by any deterministic subgrid-scale model. De Stefano *et al.*[8] applied a wavelet denoising technique to isotropic turbulence to separate the incoherent part (close to white noise) from the coherent part of the velocity field. They found a very large amount of incoherent noise in the subgrid stress tensor. These random fluctuations in the SGS stress introduces stochastic backscatter which is a physical process that is missing in many SGS models of today. We believe that the stochastic influence of the subgrid scales on the resolved scales has to be modelled in order to get a correct description of the smallest resolved scales. In this paper we will use a stochastic process to model the stochastic influence of the SGS scales.

The use of stochastic processes in subgrid modelling in LES has been treated by several authors. Bertoglio[9] included a random white noise force to simulate backscatter in homogeneous turbulence. He adjusted the amplitude of the stochastic force to fit the amount of backscatter with EDQNM (Eddy-Damped Quasi-Normal Markovian) predictions. Bertoglio also introduced a finite time scale of the stochastic force and observed some differences in third order statistics due to the finite time scale. Leith[10] supplemented the Smagorinsky model by spatially and temporally uncorrelated random SGS stresses calculated as the rotation of a stochastic vector potential. The backscatter spectrum will then have the

[a)]Electronic mail: linus@mech.kth.se
[b)]Electronic mail: geert@mech.kth.se
[c)]Electronic mail: viktor@mech.kth.se

k^4 slope derived by Kraichnan.[7] Leith[10] found that the backscatter was able to excite growth of resolved energy in a plane mixing layer. Chasnov[11] calculated the Kolmogorov constant from LES of forced isotropic turbulence using a partly stochastic SGS model. The model contained an eddy diffusivity term and a stochastic force derived from the EDQNM stochastic model equation. The inclusion of the stochastic force improved the slope of the energy spectrum in LES. Mason and Thomson[12] proposed a Smagorinsky-stochastic model based on the same stochastic vector potential as Leith and applied the model to the LES of the boundary layer. Their Smagorinsky-stochastic model clearly improved the mean velocity profile and the anisotropy of the velocity fluctuations close to the wall, compared to the standard Smagorinsky model. Schumann[13] also based his model on the Smagorinsky model and added the stochastic behavior of the SGS scales by random SGS stresses for the velocity field, and by random SGS scalar fluxes for the passive scalar field constructed from random velocity and scalar fluctuations. The random fluctuations were solutions to the Langevin equation (Langevin[14]) with a finite time scale and controllable variance. The Schumann model was applied to the LES of decaying isotropic turbulence, where the model was shown to improve the shape of the energy spectrum at the smallest resolved scales. The model also had a small impact on the decay rate of the turbulence kinetic energy. A more recent example of stochastic SGS modelling using the Langevin equation is the Lagrangian stochastic model by Wei *et al.*,[15] who coupled their Lagrangian model to Eulerian LES. Their stochastic model was shown to improve the two time correlation in a Lagrangian frame as well as scalar dispersion.

The purpose of this study is to further develop the work of Alvelius and Johansson[16] and develop a computationally inexpensive stochastic model for the subgrid scales of the velocity field and validate this new model. We also apply the idea of stochastic modelling to the subgrid scales of a passive scalar field.

II. STOCHASTIC SUBGRID MODELLING

A. Filtered equations

The filtered incompressible Navier-Stokes and passive scalar equations in a rotating frame read (ϵ_{ijk} is the permutation tensor)

$$\frac{\partial \widetilde{u}_i}{\partial t} + \widetilde{u}_j \frac{\partial \widetilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial x_i} + \nu \nabla^2 \widetilde{u}_i - 2\epsilon_{ijk}\Omega_j \widetilde{u}_k - \frac{\partial \tau_{ij}}{\partial x_j},$$

$$\frac{\partial \widetilde{u}_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial \widetilde{\theta}}{\partial t} + \widetilde{u}_j \frac{\partial \widetilde{\theta}}{\partial x_j} = \frac{\nu}{\mathrm{Pr}} \nabla^2 \widetilde{\theta} - \frac{\partial q_j}{\partial x_j},$$

where $\widetilde{u}_i$ and $\widetilde{\theta}$ are the filtered velocity and passive scalar, respectively, and where $\widetilde{p}$ includes both the pressure and the centrifugal force. Ω_i is the system rotation vector and Pr is the Prandtl number. The subgrid stress tensor $\tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j$ and the subgrid flux vector $q_j = \widetilde{\theta u_j} - \widetilde{\theta}\widetilde{u}_j$ have to be modelled to close the equations.

B. A stochastic model

The idea behind the present stochastic subgrid model is to model the random influence of the subgrid stress and flux by a stochastic process with the aim to improve the description of the smallest resolved scales. In this paper we make use of the solution to the Langevin equation

$$dX(x,t) = aX(x,t)dt + b\sqrt{2a}dW(x), \tag{2}$$

where a and b are constants and $dW(x)$ are spatially and temporary independent random numbers with the normal distribution $N(0,\sqrt{dt})$. The solution $X(x,t)$ to (2) is a stationary process with zero mean, $E[X(x_0,t)]=0$, and constant variance, $V[X(x_0,t)]=b^2$. The time scale of the process can be characterized by the decay rate of the correlation $E[X(x_0,t)X(x_0,t_0+t)]/V_X = \exp(-at)$. It follows that the time scale of the process, $\tau_X = 1/a$, decreases with increasing values of a. The Langevin equation was originally a stochastic model for the Brownian motion of particles, see Langevin,[14] but it can also be considered as a model for a fluid particle trajectory in isotropic turbulence. In that case τ_X is the Lagrangian integral time scale and it is possible to choose the model parameters a and b so that the Lagrangian structure function becomes consistent with Kolmogorov theory, see Pope.[17] Our stochastic model is based on the Smagorinsky model in which the eddy viscosity is constructed from the filter scale Δ and a velocity scale $\Delta|\widetilde{S}_{ij}|$. We let $X(x,t)$ represent a random fluctuation in the velocity scale $\Delta(1+X)|\widetilde{S}_{ij}|$ induced by the interactions with the subgrid scales. As a result we obtain a partly stochastic eddy viscosity

$$\nu_T = C_s^2(1+X(x,t))\Delta^2|\widetilde{S}_{ij}|, \tag{3}$$

where C_s is the Smagorinsky constant and Δ is the filter scale. The part corresponding to the Smagorinsky model should generate the right amount of mean dissipation whereas the stochastic part provides for backscatter and creates realistic SGS fluctuations. The model parameter a enables control of the time scale of the stochastic fluctuations and the parameter b enables control of the amount of backscatter.

The subgrid scales are advected by the resolved scales. It follows that X is advected by the flow as well and that τ_x should be computed in a Lagrangian frame of reference. This is, however, beyond the scope of the present study. Instead, we neglect the advection and take τ_x to be an Eulerian time scale. The time scale of the stochastic noise that we intend to model is that of the smallest resolved velocity scales. Therefore, we take τ_x to be proportional to the time scale of the SGS velocity field estimated from the filter scale Δ and the energy transfer. From dimensional arguments we have

$$\tau_X = C\left(\frac{\Delta^2}{\langle \Pi \rangle}\right)^{1/3}, \tag{4}$$

where C is a model constant, and $\Pi = -\tau_{ij}\widetilde{S}_{ij}$ is the SGS dissipation. Because $\tau_X = 1/a$ this defines a. When the Smagorinsky model is applied to (4) τ_X becomes proportional to the shear time scale, $\langle |S_{ij}|^{-3}\rangle^{1/3}$, which is a characteristic time scale of the smallest resolved scales.

Hence, stochastic noise which has a correlation time about as long as the time scale of the subgrid velocity field and which is uncorrelated in space, enters the subgrid stress through the eddy viscosity. The eddy diffusivity for the subgrid scalar flux is modelled using a constant $\mathrm{Pr}_T=0.6$ and thus also contains stochastic noise.

III. SIMULATIONS

LES were carried out for three different test cases: decaying isotropic turbulence, rotating homogeneous shear flow, and fully developed turbulent channel flow. Three different subgrid models were applied: the standard Smagorinsky model, the stochastic model based on the standard Smagorinsky model, and the dynamic model as defined by Lilly[18] (both with model constant averaging in all homogeneous directions and with clipping of large negative values). Negative eddy viscosity in LES has to be treated with care. The stochastic model predicts locally negative total viscosity $(\nu_T+\nu)$ that could lead to numerical instabilities under some circumstances. A large time scale, τ_X, and a large variance of the stochastic process increases the probability for numerical instability. For the LES of turbulent channel flow, using the present choice of parameters, it was necessary to neglect predictions of negative spikes in the transfer beyond −50 times the mean energy transfer. Such negative spikes occurred at less than 0.5% of the nodes and the clipping did not significantly affect the mean energy transfer. The LES of decaying isotropic turbulence and rotating homogeneous shear flow did not need any clipping of negative spikes in the transfer. The model parameters were chosen as $b=2.3$ and $C=0.2$

LES of decaying isotropic turbulence represented on 128^3, 64^3, and 32^3 grid points in a cubic box were performed using a pseudospectral code with a third order Runge-Kutta method for time advancement. The initial velocity field, with the initial turbulence Reynolds number $\mathrm{Re}_T=20000$, was random with a prescribed high Reynolds number shape of the energy spectrum and the Smagorinsky constant was chosen as $C_s=0.17$. The aliasing errors were removed using a combination of phase shifting and truncation.

LES of rotating homogeneous shear flow with 128^3, 64^3, and 32^3 grid-points were performed in a periodic box with the dimensions $4\pi\times 3\pi\times 2\pi$. The filtered incompressible Navier-Stokes equations with a constant uniform shear, $U_i=Sx_3\delta_{i1}$, and the passive scalar equation with a mean scalar gradient, $G_i=\delta_{i3}$, were solved using the same pseudospectral code as in the previous test case. Rogallo's method was used to simulate homogeneous shear flow, i.e., the grid moves along with the mean flow to enable the use of periodic boundary conditions, and it is remeshed periodically. Some information is lost during the remeshing. However, the losses are not significant for the evolution of the flow. Fully developed turbulence obtained from isotropic decay was used as an initial condition and the flow is rotating about the spanwise direction, $\Omega_i=\Omega\delta_{i2}$, at the nondimensional rotation numbers $R=2\Omega/S=0$, $-1/2$, and -1. The initial scalar field was without any fluctuations. The initial nondimensional shear rate was chosen as $S\widetilde{K}/\widetilde{\epsilon}=3.38$, and the initial turbulence Reynolds number was $\mathrm{Re}_T=\widetilde{K}_0^2/(\widetilde{\epsilon}_0\nu)=1500$. The LES results were compared to DNS data of homogeneous turbulent shear flow represented on $960\times 512\times 576$ grid-points in a box with the dimensions $4\pi\times 2\pi\times 2\pi$. The DNS data was filtered to the resolutions 32^3, 64^3, and 128^3 using both a spectral cutoff filter and the Gaussian filter used by Cerutti and Meneveau.[19] In the DNS the initial Reynolds number, $\mathrm{Re}_T=135$, is lower than in the LES. Moreover, the mean scalar gradient in the DNS is directed in the streamwise direction instead of the transverse direction as in the LES. However, we believe that it is appropriate to compare data on the amount of backscatter with the LES results. We will also use the filtered data to compare the intermittency of the SGS energy dissipation to the intermittency of the SGS scalar variance dissipation.

LES of turbulent channel flow at the wall friction Reynolds number $\mathrm{Re}_\tau=360$ represented on $64\times 97\times 48$ grid-points was performed in a box with the dimensions $4\pi\delta\times 2\delta\times 4\pi\delta/3$, using $C_s=0.12$ and van Driest damping

$$C^2 = C_s^2(1-e^{-y^+/25})^2, \tag{5}$$

where y^+ is the wall distance normalized in wall units, when the Smagorinsky and stochastic model were employed. The time scale τ_X was calculated according to (4) without wall damping of C_s. The grid spacing is $\Delta x^+=71$, $\Delta z^+=35$ and an averaged grid spacing of $\langle\Delta y^+\rangle=7.4$. The numerical code is spectral in the streamwise (x) and spanwise (z) directions and uses Chebychev representation in the wall normal direction (y). The LES results were compared to filtered DNS data obtained from Alvelius and Johansson.[16] The DNS (see Alvelius and Johansson[16] or Grundestam *et al.*[20]) is represented on a $384\times 257\times 240$ mesh and was filtered to a 64×48 mesh in the x and z directions using a spectral cutoff filter. There was no explicit filtering in the y direction. We have also employed a sharp cutoff filter in Chebychev space as an explicit filter in the wall normal direction, but it is omitted here since it only had a very small impact.

IV. DECAYING ISOTROPIC TURBULENCE

A. Turbulent kinetic energy and enstrophy

Figure 1 shows the time development of the turbulence kinetic energy for the 64^3 LES of isotropic turbulence and the spectra of $\widetilde{K}$ and the enstrophy at, $T=\widetilde{\epsilon}_0 t/\widetilde{K}_0=5$, where $\widetilde{K}_0$ and $\widetilde{\epsilon}_0$ represent the initial values. The time development of $\widetilde{K}$ in the LES with the Smagorinsky model and stochastic model differ little showing that the stochastic term has a very small influence on the mean dissipation rate. The kinetic energy decays with an asymptotic power law scaling $\widetilde{K}(t)\sim t^{-1.35}$. The only difference due to the present stochastic model is that the LES follows the power law decay rate $\widetilde{K}(t)\sim t^{-1.35}$ a bit longer than with the Smagorinsky model. This is substantially different from the behavior of the stochastic model by Schumann,[13] which contributes to the mean

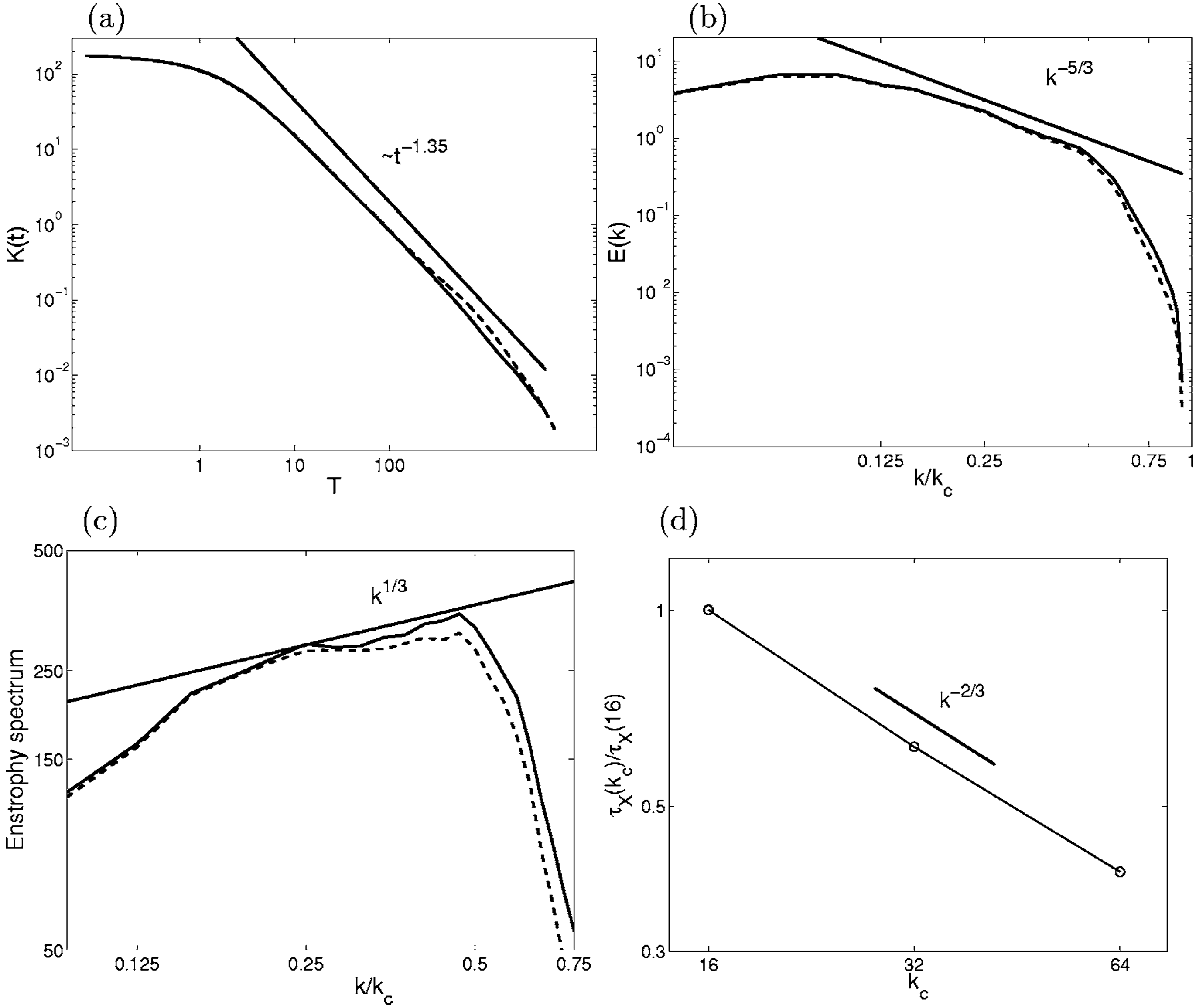

FIG. 1. (a) Time development of the turbulence kinetic energy. (b) Energy spectrum at $T=5$. (c) Enstrophy spectrum at $T=5$. (d) Filter scale dependence of τ_X at $T=5$. Stochastic model, solid line; Smagorinsky model, dashed line.

energy transfer and decreases the decay rate of $\widetilde{K}(t)$ in the LES of isotropic turbulence. The energy spectrum at $T=5$ [Fig. 1(b)] reveals that there is an increased activity at the smallest resolved scales due to the stochastic noise, but the difference is small. The increased small scale activity is more pronounced in the enstrophy spectrum in Fig. 1(c). The stochastic model yields better agreement with the Kolmogorov inertial range scaling $k^{1/3}$ at the smallest resolved scales compared to the Smagorinsky model. This should be regarded as an improvement since the smallest resolved scales lie within the inertial range.

Figure 1(d) shows the time scale τ_X with a constant $C=0.2$ at $T=5$ as a function of cutoff wave number in LES of decaying isotropic turbulence. The time scale decreases as $\tau_X \sim k_c^{-2/3}$ which is the correct behavior for an eddy time scale within the inertial subrange according to Kolmogorov theory. Hence, there is no need for a filter scale dependence of C. The value $C=0.2$ can be explained from the fact that $(C_s^2)^{1/3} \approx 0.2$. This is desired because then we have $\tau_X \approx \langle |S_{ij}|^{-3} \rangle^{1/3}$, which is appropriate when using the Smagorinsky model.

B. Backscatter

Backscatter is local energy transfer from the SGS to the resolved scales, see Piomelli *et al.*[21] and Cerutti and Meneveau.[19] In the present model backscatter is represented by intermittently negative eddy viscosity assuming that the SGS stress is aligned with the resolved rate of strain. This is of course only a simple model for the real backscatter phenomenon. It is well known that backscatter represented by a smooth eddy viscosity gives a wrong scaling of the backscatter spectrum. However, the stochastic eddy viscosity proposed in this paper is spatially delta-correlated, and we can follow the steps by Schumann[13] to show that the model has the required k^4 scaling of the backscatter spectrum. The time

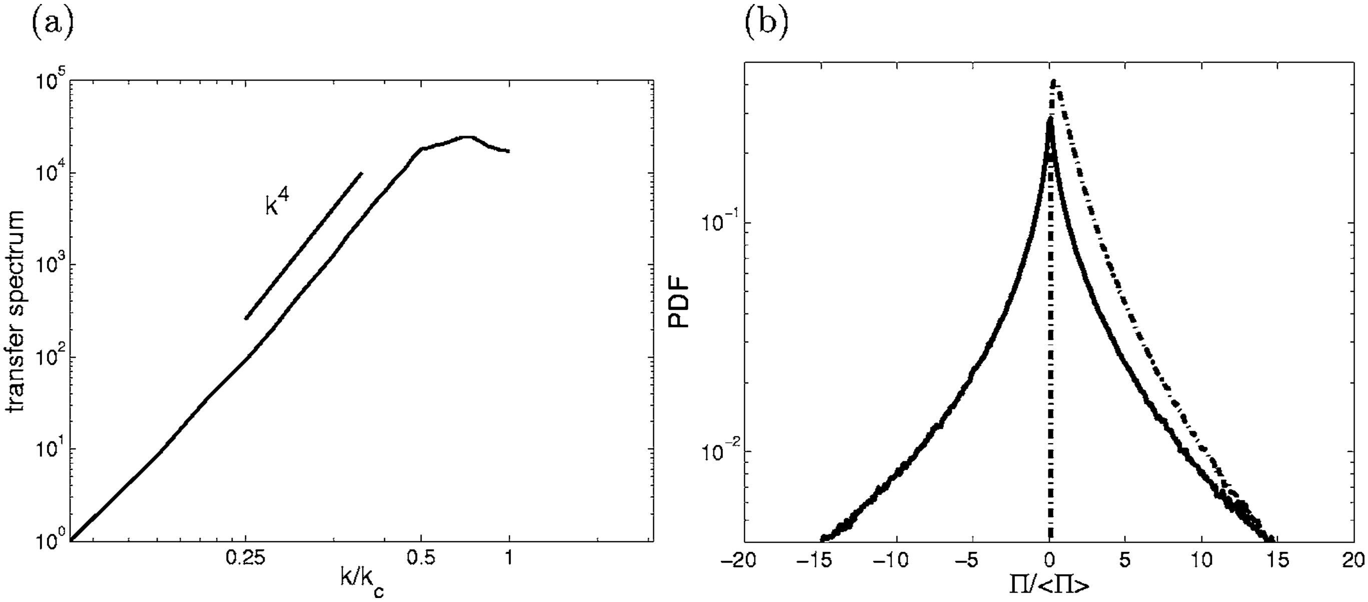

FIG. 2. (a) Spectrum of the stochastic energy transfer. (b) PDF of the subgrid dissipation according to the stochastic model. Π_X, solid line; Π_s, dashed-dotted line.

scale of X is not relevant for the spectral behavior because X is white noise in space with a Dirac type spatial autocorrelation at any fixed time. The stochastic part of the modelled SGS stress is linear in X and will essentially have the same delta-autocorrelation as X. White noise has a k^2 slope of the spectrum in three dimensions. It follows that the backscatter spectrum, which scales as the power spectrum of $\partial X/\partial x_j$, has the desired k^4 behavior. Following Schumann the random SGS stresses transfers energy at the rate $2C_s^2\Delta^2 u_i \partial(X|S|S_{ij})/\partial x_j$. The spectrum of negative events of stochastic energy transfer at $T=5$ is plotted in Fig. 2(a). Evidently, it scales as k^4 in the inertial subrange. Most of the transfer comes from $k>0.5k_c$ (85%) and (45%) comes from $k>0.75k_c$.

The subgrid dissipation Π of the stochastic model can be decomposed into two parts

$$\Pi = C_s^2\Delta^2|\tilde{S}_{ij}|^3 + XC_s^2\Delta^2|\tilde{S}_{ij}|^3 \equiv \Pi_S + \Pi_X, \tag{6}$$

where Π_S is the standard Smagorinsky contribution and Π_X is the stochastic contribution which is linear in X. Figure 2(b) shows the probability density function (PDF) of Π_S and Π_X at $T=5$ respectively. From the PDF of Π_X we see that it is symmetric about its mean value $\langle\Pi_X\rangle=0$. Again, this shows that the stochastic contribution to the eddy viscosity does not change the mean dissipation in a direct manner. Π_S has no backscatter of turbulent kinetic energy and has a positive mean value. The combination of the Smagorinsky model and the stochastic term thus accounts for a combination of forward energy flux (if $1+X>0$) and backscatter (if $1+X<0$).

One basic constraint on the backscatter is that the energy that is transferred to the larger scales must not exceed the SGS kinetic energy. To prevent such events it is required to have a small and controllable time scale of the backscatter, since a short time scale of the backscatter decreases the amount of energy transferred to the larger scales. The subgrid kinetic energy in the present LES, i.e., the trace of the SGS stress, is unknown and is lumped together with the pressure when solving the LES equations. It is, however, possible to show that the energy transferred by the stochastic model during the time $\tau_X \sim |S_{ij}|^{-1}$ scales as

$$K_{\mathrm{tr}} \approx \Delta^2 C_s^2(1+X)|S_{ij}|^3\tau_X \sim \Delta^2|S_{ij}|^2.$$

This is the same scaling as one obtains by constructing a model for the SGS kinetic energy from the squared Smagorinsky velocity scale $K_{\mathrm{SGS}} \sim (\Delta|S_{ij}|)^2$. Hence, by choosing the proposed time scale of X and a constant model parameter b, the energy transferred to the larger scales during the time τ_X will decrease in the same way as the SGS kinetic energy within the inertial subrange. Note that this scaling for the energy backscatter cannot be guaranteed with a dynamic Smagorinsky model, because the time scale of the backscatter event cannot be controlled. A controllable time scale that decreases with filter scale is also important for the stability of the LES, since a long correlation time of negative eddy viscosity leads to numerical instabilities.

C. Correlation time

Time correlations are important in LES of sound generation, see He *et al.*[2] He *et al.*[2] showed that in LES with the Smagorinsky model the time scales are overpredicted at all turbulent length scales, especially close to the filter scale, and suggested a stochastic force to solve the problem. The present stochastic model adds such a stochastic SGS force to the velocity field. We have chosen to study the time correlations in decaying isotropic turbulence. The results of the correlation time scales are affected by the decay of the turbulence intensity, but it is appropriate for comparison of the Smagorinsky and stochastic models because the mean turbu-

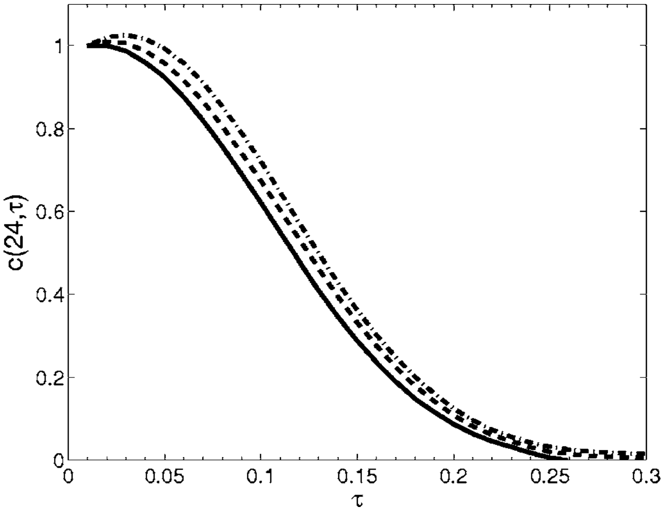

FIG. 3. The time correlation coefficient for $k=0.75k_c$ at $T=5$. Standard Smagorinsky model, dashed-dotted line; stochastic model $C=0.8$, dashed line; $C=0.2$, solid line.

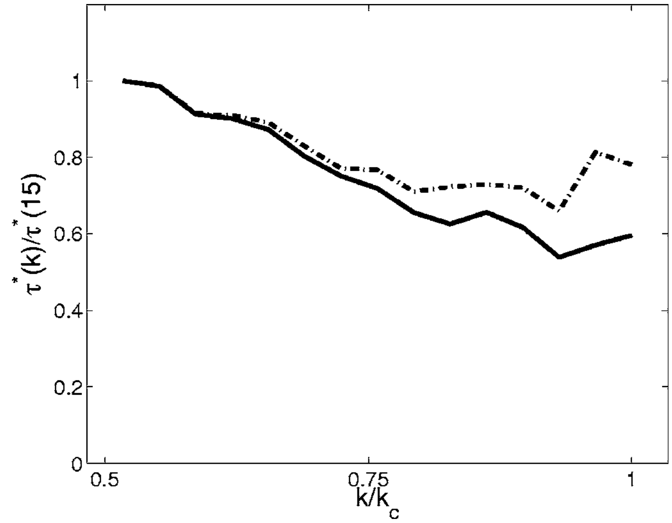

FIG. 4. Time scale, $\tau^*(k)$, as a function of the wave number. Standard Smagorinsky model, dashed-dotted line; stochastic model with $C=0.2$, solid line.

lent kinetic energy develops in the same way for both models as shown before. In correspondence with He *et al.*, we define the correlation coefficients as

$$c(k,t,\tau) = \frac{\langle u_i(k,t)u_i(-k,t+\tau)\rangle}{\langle u_i(k,t)u_i(-k,t)\rangle}, \tag{7}$$

where the average operator denotes averaging over a spherical shell in wave number space. In Fig. 3 the time correlation coefficients at $k=0.75k_c$ of the present stochastic model decay faster in time than the coefficients of the Smagorinsky model. In the case $C=0.2$ the correlation coefficients decrease faster than at $C=0.8$ which is expected because the time scale of X is smaller at $C=0.2$. The correlation time scale as a function of k is computed as

$$\tau^*(k,t_1) = \int_0^\infty c(k,t_1,\tau)d\tau \tag{8}$$

and is shown in Fig. 4, for $t_1=5$. The stochastic model predicts shorter correlation times than the Smagorinsky model for all k, but in particular at large k the reduction of τ^* is large and this is where it is most desired, as was also pointed out by He *et al.* Similar improvements of the behavior of the correlation time scale were reported by Wei *et al.*[15] using a Lagrangian stochastic model coupled with Eulerian LES.

V. ROTATING HOMOGENEOUS SHEAR FLOW

A. Large scale statistics

Figure 5 shows the time development of the turbulent kinetic energy in rotating homogeneous shear flow. The flow is strongly destabilized by rotation at $R=-0.5$. At $R=-1$, $\widetilde{K}$ still grows but at much slower rate than at $R=0$ in agreement with Brethouwer and Matsuo.[22] The growth has a pronounced exponential part, $\widetilde{K}=\widetilde{K}_0 e^{\alpha \mathrm{St}}$ at $R=0$ and $R=-1/2$. At $R=0$ the exponential growth rate is in agreement with the experiment by Tavoularis and Corrsin.[23] $C_s=0.10$ is used in the standard Smagorinsky and stochastic model, which is equal to the value $C_s\approx 0.10$ predicted by the dynamic model. This value is also close to the value $C_s=0.11$ suggested by Canuto and Cheng[24] for homogeneous shear flow, but much smaller than the value $C_s=0.19$ used by Bardina.[4] The time development of $\widetilde{K}$ (and the Reynolds stresses) are very similar for the LES with the stochastic model and the dynamic model according to the figure. The results for the Smagorinsky model and stochastic model were indistinguishable, as should be expected.

Next, we turn our attention to the scalar mixing. The direction of the turbulent flux is defined as

$$\alpha_f = \operatorname{atan}\left(\frac{\langle \widetilde{\theta}' \widetilde{u}_3' \rangle}{\langle \widetilde{\theta}' \widetilde{u}_1' \rangle}\right), \tag{9}$$

where $\widetilde{u}_i'$ and $\widetilde{\theta}'$ are the fluctuating velocity and scalar components, and where $\langle \widetilde{\theta}' \widetilde{u}_1' \rangle$ and $\langle \widetilde{\theta}' \widetilde{u}_3' \rangle$ are the mean turbulent scalar fluxes in the x_1 and x_3 direction, respectively. According to Fig. 5(b), the development of α_f depends on the rotation number. The angle α_f in the LES with the standard, dynamic and stochastic Smagorinsky model approach the same equilibrium values and the results were found to yield good agreement with DNS results of Brethouwer and Matsuo and Rogers *et al.*[25] Also here, the results of the stochastic model are very close to those of the Smagorinsky model. The ratio of the mean turbulent diffusivity to the molecular diffusivity is about 50 showing that the similarities between the predictions of the stochastic and the Smagorinsky model are not owed to a low Reynolds number.

B. Backscatter

Despite the small differences in large scale statistics in the LES with the standard Smagorinsky and stochastic model, there are significant differences at the smaller scales. One element is the backscatter of kinetic energy, which is absent in LES with the Smagorinsky model but can be modelled by employing the stochastic model. If we assume that it

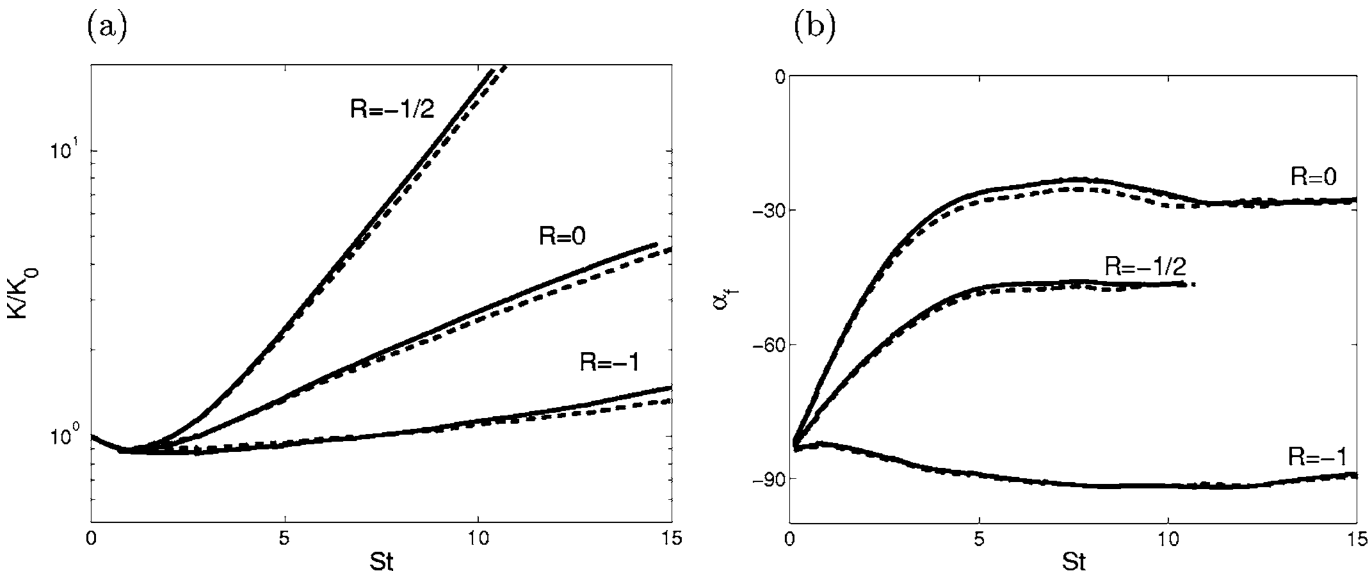

FIG. 5. (a) The time history of the turbulent kinetic energy. (b) The direction of turbulent scalar flux. Dynamic model, dashed line; stochastic and Smagorinsky model, solid line.

is appropriate to model all backscatter by the stochastic eddy viscosity, we can adjust the parameter b so that the modelled amount of backscatter matches observations. Piomelli *et al.*[21] found that the probability of backscatter is about 50% for a sharp spectral cutoff filter and about 30% with a Gaussian filter at all locations in turbulent channel flow. The present stochastic model is not able to predict 50% backscatter probability unless $b \to \infty$. We separate the mean SGS dissipation into averaged forward $\langle \Pi^+ \rangle$ and backscatter $\langle \Pi^- \rangle$ contributions

$$\langle \Pi^+ \rangle + \langle \Pi^- \rangle = \langle \Pi \rangle \tag{10}$$

and calibrate the model with a Gaussian filter as reference. In the filtered DNS data using a Gaussian filter the ratio $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle$ is typically 0.4 with a slight increase at smaller filter scales. A constant b in the stochastic model implies a constant ratio $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle$. We choose $b=2.3$ which implies $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle \approx 0.4$, which yielded a fair overall agreement with the DNS results as shown in Fig. 6. The spectral cutoff filter implies higher ratios, typically $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle \approx 0.7$. A constant fraction of backscatter is not completely well justified. Piomelli *et al.*[21] showed that there is an increasing trend for the backscatter with the Reynolds number. The estimated value is therefore not universal, but a further study would be necessary to address this question. However, we believe that $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle \approx 0.4$ is a fair approximation for a Gaussian filter for a wide range of Reynolds numbers. With a constant b the amount of backscatter scales with the mean energy transfer. This is consistent with the EDQNM theory for isotropic turbulence with an infinite inertial subrange where the scaling constant depends on the type of filter. Leslie and Quarini[6] found a forward to backward transfer ratio of $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle = 0.254$ for a Gaussian filter and $-\langle \Pi^- \rangle / \langle \Pi^+ \rangle = 0.578$ for spectral cutoff filter. The rotation number and the model parameter a did not have any significant effect on the amount of backscatter in the LES.

The stochastic model accounts also for backscatter of scalar variance in the same manner as for the turbulent kinetic energy. This is shown in Fig. 7(a), where the PDF of the total scalar variance dissipation,

$$Q = -2 q_i \partial \tilde{\theta}' / \partial x_i \tag{11}$$

is plotted together with the PDF of the total SGS dissipation, for the LES with a resolution of 128^3. In agreement with the DNS data in Fig. 7(b), the peak of scalar PDF is more narrow than for the velocity field indicating that the scalar field is more intermittent than the velocity field. The amount of scalar variance backscatter $-\langle Q^- \rangle / \langle Q^+ \rangle$ in Fig. 6, agrees reasonably well with that of the DNS data despite the fact that we

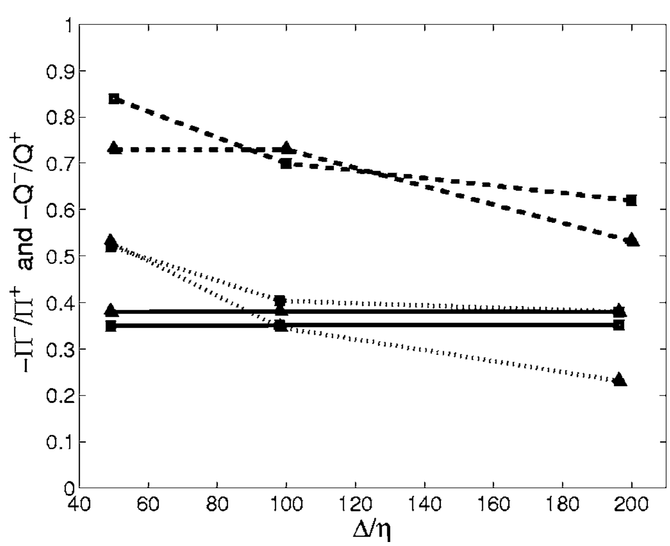

FIG. 6. Amount of backscatter in the LES with different resolution and DNS data with different filter widths. Stochastic model, solid lines; DNS data with Gaussian filter, dotted lines; DNS data with spectral cutoff filter, dashed lines. Energy backscatter, square; scalar backscatter, triangle.

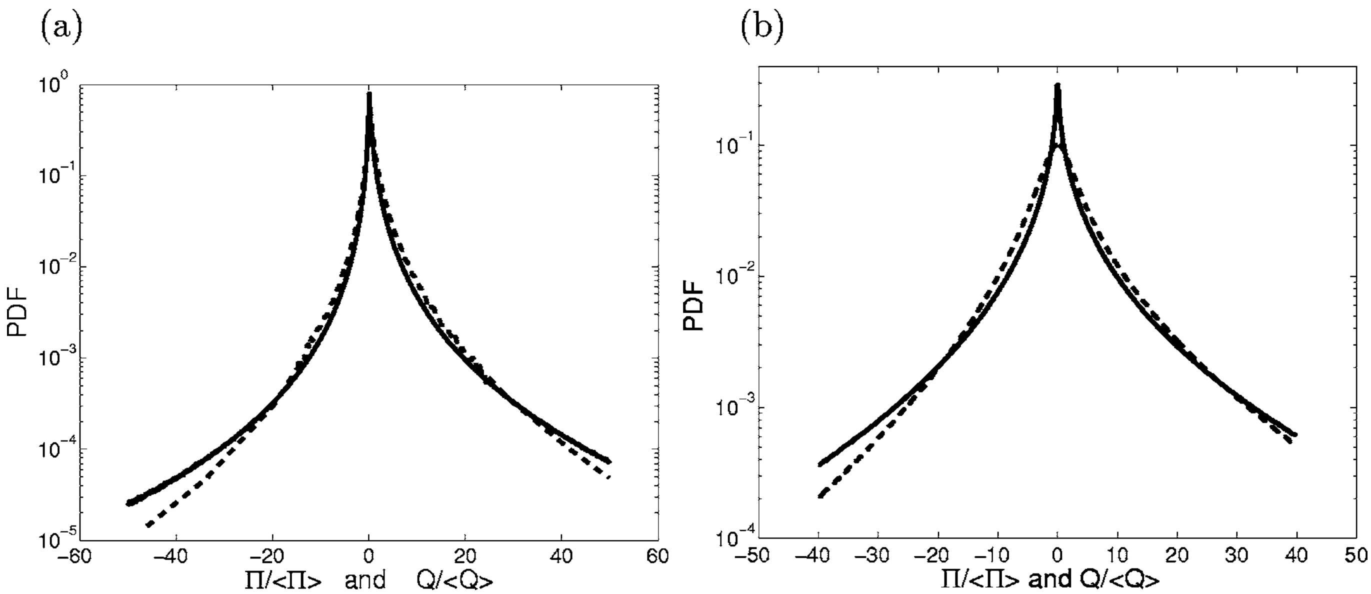

FIG. 7. PDF of the energy SGS dissipation and the scalar variance SGS dissipation according to (a) the stochastic Smagorinsky model; (b) the filtered DNS data using a spectral cutoff filter. Energy SGS dissipation, dashed line; scalar variance SGS dissipation, solid line.

calibrated the constant b to fit the data for the velocity field. It seems appropriate to use the same process X and parameter b as for the velocity field.

C. Intermittency of SGS dissipation

The flatness factor of the subgrid dissipation

$$F = \frac{\langle(\Pi - \langle\Pi\rangle)^4\rangle}{\langle(\Pi - \langle\Pi\rangle)^2\rangle^2} \tag{12}$$

is a measure of the intermittency. Cerutti and Meneveau[19] compared the flatness factor of the subgrid dissipation predicted by various subgrid stress models using a velocity field obtained from DNS. They found that the dynamic model without spatial averaging predicts an intermittent SGS dissipation, i.e., large values of F, and that the Smagorinsky model predicts SGS dissipation that is about as intermittent as the real SGS dissipation. The flatness factor of Π, at $R=0$ according to the LES with the stochastic model, the clipped dynamic model ($C_s^2 > -0.01$) and the standard Smagorinsky model are plotted in Fig. 8(a).

The constant a in the stochastic model is varied by changing C in Eq. (4). We can see that the flatness predicted by the stochastic model with $C=0.2$ is of the same order of magnitude as for the standard Smagorinsky model, whereas

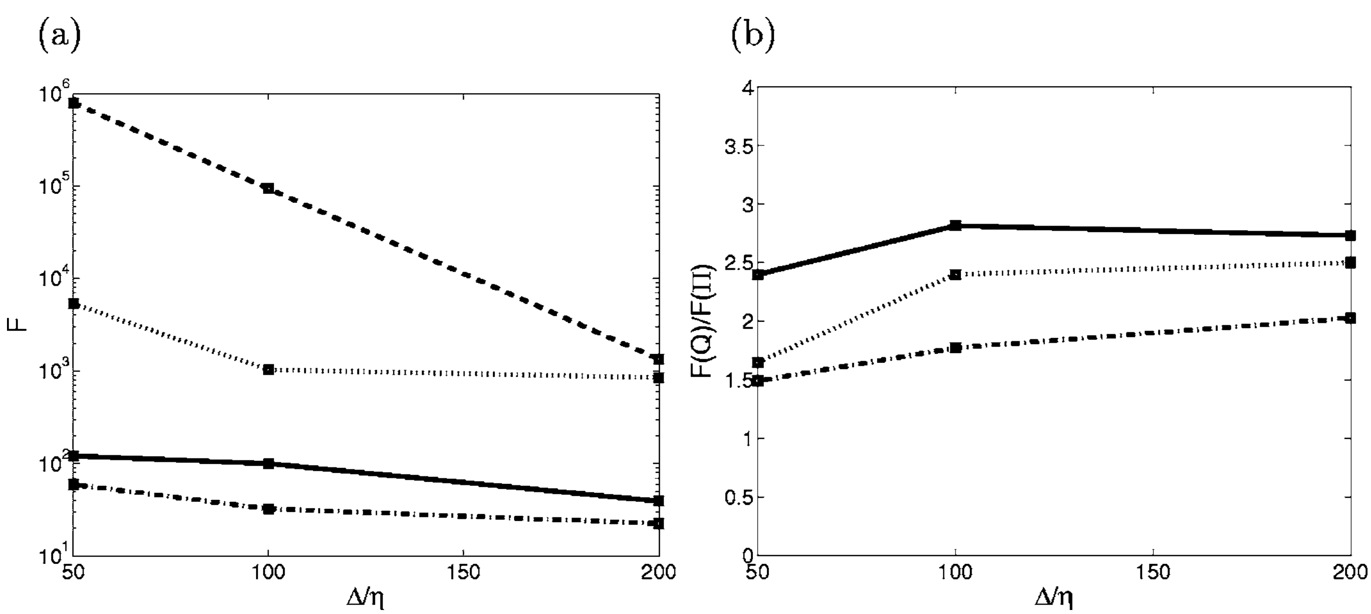

FIG. 8. (a) Flatness factor $F(\Pi)$ at different filter scales. Stochastic model with $C=0.2$, solid line; stochastic model with $C=0.8$, dotted line; dynamic model ($C_s^2 > -0.01$), dashed line; standard Smagorinsky model, dashed-dotted line. (b) $F(Q)/F(\Pi)$. Stochastic model, solid line; standard Smagorinsky model, dashed-dotted line; filtered DNS, dotted line.

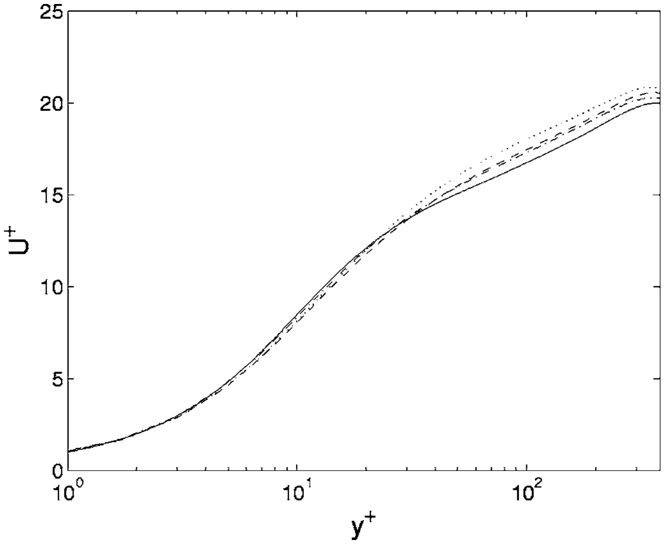

FIG. 9. Mean velocity profile U/u_τ. Stochastic model, dashed line; Smagorinsky model, dotted line; dynamic model, dashed-dotted line; filtered DNS, solid line.

the intermittency of Π according to the clipped dynamic model and the stochastic model with $C=0.8$ [Fig. 8(a)] is much too large. Hence, by choosing $C=0.2$ the stochastic model provides for backscatter in the form of "negative viscosity" without being too intermittent. The time scale of the backscatter is an important parameter considering numerical instabilities that can result from large values of F. A large time scale allows the instabilities to grow in time. Both the dynamic model and the stochastic model allow locally negative total viscosity. The present results indicate that the reason for the result of the dynamic model to be much more intermittent is the time scale of the backscatter rather than the negative viscosity itself.

The PDF of Π and Q in Sec. V B indicated that Q is more intermittent than Π. From Fig. 8(b) we see that this is the case. Apparently, the resolved scalar gradient is more intermittent than the resolved strain rate. According to the filtered DNS data, using a Gaussian filter, the ratio of the flatness factor of Q to that of Π varies from 2.5 to 1.6 within the present range of filter scales. The ratio predicted by the Smagorinsky model is somewhat smaller and the result of the stochastic model is slightly larger. The flatness ratio obtained using the spectral cutoff filter is very similar to that using the Gaussian filter.

VI. TURBULENT CHANNEL FLOW

A. Mean velocity and Reynolds stresses

Figure 9 shows the mean velocity profile for turbulent channel flow computed with different SGS models. Both the stochastic model and the Smagorinsky model have a log-layer slope but overpredict the magnitude of the mean velocity in comparison with the DNS, as could be expected from a model based on van Driest damping, see for example Hughes *et al.*[26] However, the inclusion of stochastic noise decreases slightly the magnitude of the mean velocity profile in the log-layer which yields a better agreement with the DNS. The mean velocity according to the dynamic model, with averaging of the model constant along the two homogeneous directions, is similar to that of the stochastic model within the log-layer.

The diagonal Reynolds stresses are also improved by the inclusion of the stochastic term [Figs. 10(a)–10(c)]. The stochastic model decreases the peak values of streamwise fluctuations and increases the peak values of spanwise and wall normal fluctuations, which yields a significantly better agreement with the filtered DNS data. Another positive feature resulting from the stochastic backscatter is that the spanwise and wall normal fluctuations are larger in the near-wall region. Such increased near wall activity due to stochastic backscatter was also found by Mason and Thomson[12] in their LES of a turbulent boundary layer.

The dynamic model also improves the diagonal stresses compared to the Smagorinsky model by increasing the spanwise and wall normal fluctuations in the near-wall region. This is not due to any stochastic backscatter but to a reduced near-wall eddy viscosity compared to a van Driest damped constant. It is interesting to note that the stochastic model decreases the level of anisotropy of the fluctuations compared to the Smagorinsky model. Figure 10(d) shows that the improvement in the isotropy measure $I=\langle ww\rangle/\langle uu\rangle$ is considerable. The fluctuations predicted by the Smagorinsky model are significantly too anisotropic.

B. Energy transfer statistics

The mean SGS dissipation $\langle\Pi\rangle$, the variance $V[\Pi]$, the length scale $L_x[\Pi]$, and the mean backscatter according to the LES and filtered DNS are shown in Fig. 11. The mean SGS dissipation is predicted correctly by all three SGS models, except for the peak near the wall, but the variance $V[\Pi]$ is not. The filtered DNS data attain very large values of $\sqrt{V[\Pi]}$, up to 10 times of that of the standard and dynamic Smagorinsky models, indicating an intermittent behavior of the real energy transfer. The variance of the energy transfer becomes more realistic with the stochastic model. It is twice as large as for the standard and dynamic Smagorinsky models which, hence, yields a somewhat better agreement with the DNS.

The length scale of the subgrid dissipation is computed from the correlation

$$L_x[\Pi] = \int_0^{1/2 l_x} \frac{\langle \Pi'(x_0)\Pi'(x_0+x)\rangle}{\langle \Pi'^2\rangle} dx, \tag{13}$$

where l_x is the box length in the streamwise direction, and $\Pi'=\Pi-\langle\Pi\rangle$ is the fluctuating part of the subgrid dissipation. According to the DNS data, $L_x[\Pi]$, is smaller than the mean filter scale $\Delta=(\Delta x\langle\Delta y\rangle\Delta z)^{1/3}$, indicating a large amount of spatially decorrelated noise in the SGS stress. The standard and dynamic Smagorinsky models predict $L_x[\Pi]$ to be larger than 5Δ away from the wall. This deficiency is presumably associated with the large influence of low wave number information on the SGS stress in these models. $L_x[\Pi_{\rm smag}]$ is, in fact, comparable with the integral length scale of the streamwise velocity in the middle of the channel, see Fig. 11(c). The inclusion of stochastic noise into the SGS model rem-

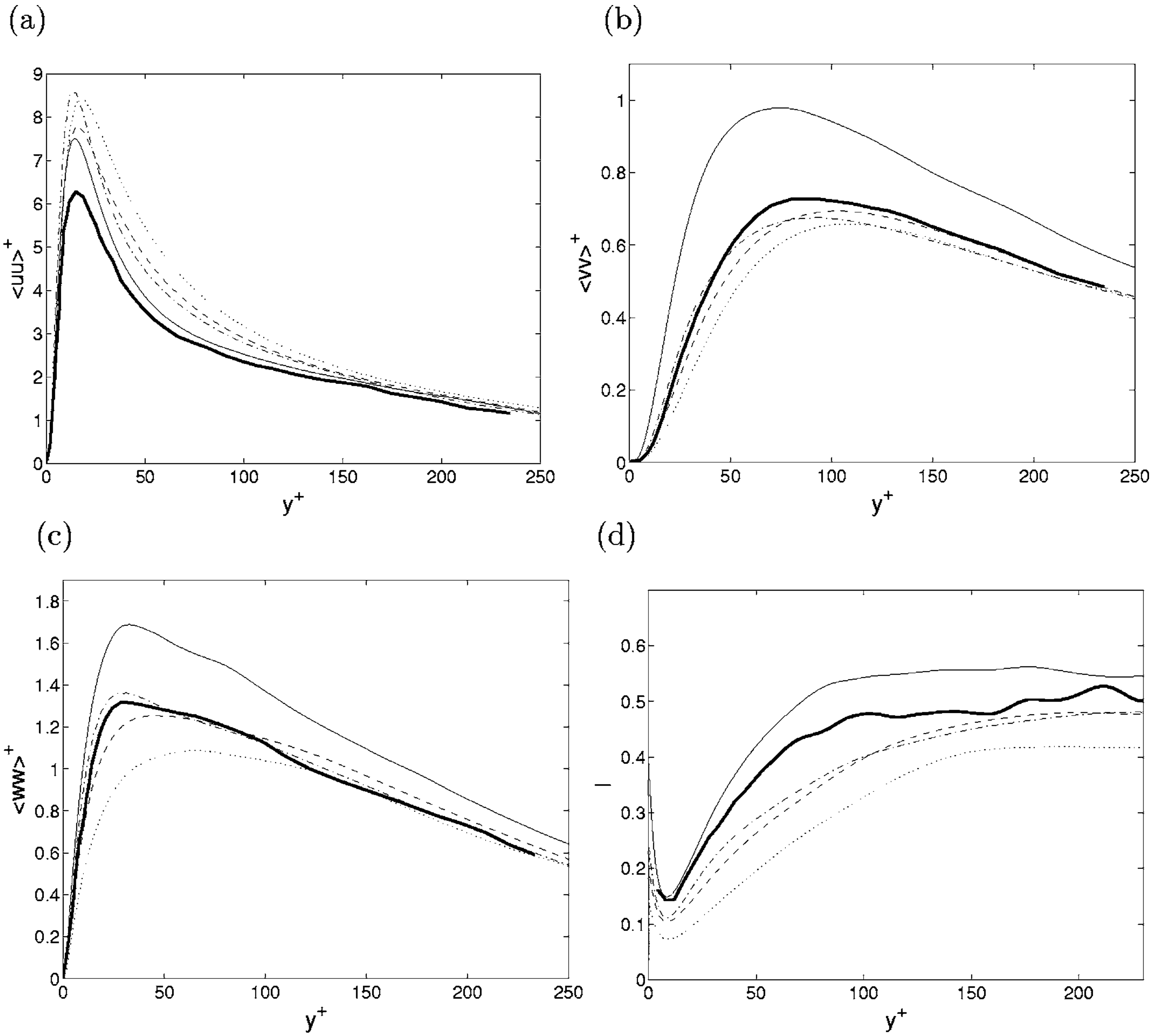

FIG. 10. (a) $\langle uu\rangle^{+}$; (b) $\langle vv\rangle^{+}$; (c) $\langle ww\rangle^{+}$; (d) $\langle ww\rangle/\langle uu\rangle$. Stochastic model, dashed line; Smagorinsky model, dotted line; dynamic model, dashed-dotted line; DNS, solid line; filtered DNS, thick solid line.

edies the problem. $L_x[\Pi]$ is slightly larger than twice the filter scale Δ according to the stochastic model, which is in much better agreement with the filtered DNS data. The filtered DNS suggest $L_\Pi < \Delta$ but this is not achievable in LES because the energy transfer has to be resolved by the grid to avoid truncation errors in the numerical method. The present stochastic model predicts $L_\Pi \approx 2.5\Delta$, i.e., the energy transfer can be considered as resolved.

The backscatter derived from DNS using the spectral cutoff filter is almost as large as the forward scatter with a ratio $-\langle\Pi^-\rangle/\langle\Pi^+\rangle \approx 0.8$ or $-\langle\Pi^-\rangle \approx 5.4\langle\Pi\rangle$. This ratio is approximately the same as that we found in homogeneous shear flow using a spectral cutoff filter (Sec. V B). The stochastic model provides for backscatter with $-\Pi^-/\Pi^+ \approx 0.4$.

VII. CONCLUSIONS

A stochastic eddy viscosity Smagorinsky-type of model based on a partly stochastic velocity scale $\Delta(1+X)|\widetilde{S}_{ij}|$ is proposed. The new model provides for stochastic backscatter with the desired k^4 scaling of the backscatter spectrum and enables control of the length and time scales of the SGS energy transfer. LES of isotropic decaying turbulence, rotating homogeneous shear flow with a passive scalar and fully developed channel flow were performed. Three different subgrid models were used: the standard Smagorinsky model, the dynamic Smagorinsky model, and the newly developed stochastic model.

In isotropic turbulence there was no significant influence on the kinetic energy development due to the stochastic term

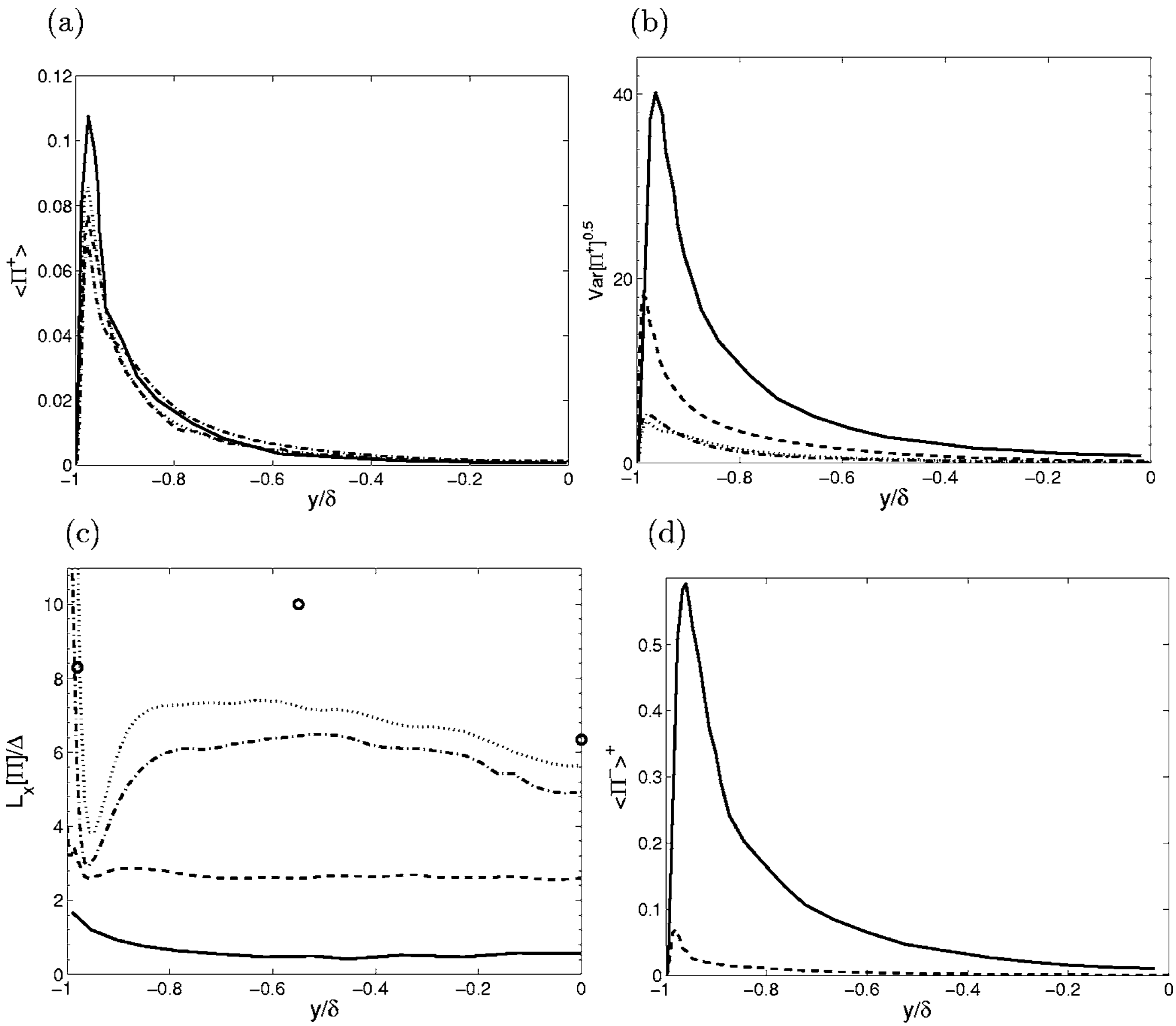

FIG. 11. (a) Mean SGS dissipation $\langle\Pi\rangle$; (b) $\sqrt{\mathrm{Var}[\Pi]}$; (c) $L_x[\Pi]/\Delta$ (the symbols are the velocity integral length scale $L_x[u]/\Delta$ from DNS); (d) averaged backscatter $\langle\Pi^-\rangle$. Stochastic model, dashed line; Smagorinsky model, dotted line; dynamic model, dashed-dotted line; filtered DNS, solid line.

but the new model improved the shape of the enstrophy spectrum at the smallest resolved scales and reduced the time scale of the smallest resolved scales. The latter may be a promising feature for the development of improved subgrid scale models for aeroacoustics because the Smagorinsky model overpredicts the time scale of the smallest resolved scales (He *et al.*[2]). The time scale τ_X of the stochastic process with a constant model parameter $C=0.2$ was shown to have a correct filter scale dependence in the inertial subrange. The correct scaling of τ_X implies that the amount of energy that is transferred to the larger scales during backscatter events decreases with filter scale, despite the constant variance $b=2.3$ of the process. This is not only physically correct but also essential for the stability of the LES. Such scaling of the energy backscatter cannot be guaranteed with a dynamic Smagorinsky model, because the time scale of the backscatter event cannot be controlled.

The choice of the subgrid model has a small influence on the large scale velocity and scalar statistics in rotating homogeneous shear flow, despite significant backscatter. However, it was shown that the intermittency of the SGS dissipation of the stochastic model is much smaller and more realistic than that of the dynamic model with clipping of large negative values of the model constant instead of spatial averaging. The localized dynamic model accounts for local backscatter, but due to long correlation time of the negative eddy viscosity numerical instabilities occurred. The time scale and variance of the backscatter predicted by the proposed model is adjustable. For the suggested model parameters the intermittency of the subgrid dissipation was of the same order of magnitude as for the standard Smagorinsky model and realistic. The local negative eddy viscosity occurring in LES with the stochastic model did not cause numerical instabilities since the correlation time scale of these events is short.

The LES with the dynamical model using spatial averaging did not provide for any backscatter.

In fully developed turbulent channel flow the stochastic model improved the fluctuation magnitudes as well as the anisotropy of the fluctuations, compared to the standard Smagorinsky model with wall damping of C_S. These improvements in the Reynolds stresses were comparable to those obtained with the dynamic model with averaging of the model constants along the homogeneous directions, but the computational cost was much lower than for the dynamic model. Moreover, the stochastic model improved the description of the energy transfer by reducing its length scale and increasing its variance in better agreement with observations. The stochastic backscatter seems to be an important part of the SGS dynamics in wall bounded flows.

Further investigations are needed to account for the advection of the stochastic fluctuations. In the future we also wish to investigate the implications for reacting flows where the small scale scalar statistics are important, see Pitsch.[27] The stochastic model may here offer an interesting approach to this challenging area for LES computations.

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