Direct numerical simulation of nonisothermal turbulent wall jets

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(Received 20 November 2008; accepted 23 December 2008; published online 4 March 2009)

Direct numerical simulations of plane turbulent nonisothermal wall jets are performed and compared to the isothermal case. This study concerns a cold jet in a warm coflow with an ambient to jet density ratio of $\rho_a/\rho_j=0.4$, and a warm jet in a cold coflow with a density ratio of $\rho_a/\rho_j=1.7$. The coflow and wall temperature are equal and a temperature dependent viscosity according to Sutherland’s law is used. The inlet Reynolds and Mach numbers are equal in all these cases. The influence of the varying temperature on the development and jet growth is studied as well as turbulence and scalar statistics. The varying density affects the turbulence structures of the jets. Smaller turbulence scales are present in the warm jet than in the isothermal and cold jet and consequently the scale separation between the inner and outer shear layer is larger. In addition, a cold jet in a warm coflow at a higher inlet Reynolds number was also simulated. Although the domain length is somewhat limited, the growth rate and the turbulence statistics indicate approximate self-similarity in the fully turbulent region. The use of van Driest scaling leads to a collapse of all mean velocity profiles in the near-wall region. Taking into account the varying density by using semilocal scaling of turbulent stresses and fluctuations does not completely eliminate differences, indicating the influence of mean density variations on normalized turbulence statistics. Temperature and passive scalar dissipation rates and time scales have been computed since these are important for combustion models. Except for very near the wall, the dissipation time scales are rather similar in all cases and fairly constant in the outer region. © 2009 American Institute of Physics. [DOI: 10.1063/1.3081554]

I. INTRODUCTION

A plane wall jet is obtained by injecting fluid along a solid wall such that the velocity of the jet supersedes that of the ambient flow. The resulting inner boundary layer and outer free shear layer have different length and time scales, which has implications for mixing and heat transfer. Ahlman et al. studied an isothermal wall jet using direct numerical simulations (DNSs). The inner layer showed similarities to a zero-pressure boundary layer, while the outer layer in the DNS was found to resemble a plane jet shear layer. Approximate collapse of statistics in the near wall and outer region was achieved by applying inner and outer scalings, respectively, thereby revealing the self-similar development of the jet, which was also observed in the experiments by Eriksson et al. In the study a passive scalar was added in the jet inlet to study mixing properties of the wall jet. The scalar fluctuations in outer scaling were found to correspond to values reported for free plane jets. Streamwise and wall-normal scalar fluxes in the outer layer were of comparable magnitude. Outer scaling also led to a collapse of the scalar dissipation rate profiles.

In this paper we extend the work by Ahlman et al. and study a warm jet in a cold surrounding and a cold jet in a warm environment by fully compressible DNS. The study of the evolution and dynamics of a cold and warm jet is of relevance for thin film cooling and combustion applications. Of special interest is how the flow development and mixing are influenced by the varying density.

Structural compressibility effects on turbulence are in general not expected in fluid flow, as long as the density fluctuations are small. This is often referred to as the Morkovin hypothesis. In this case turbulence statistics of compressible flows become similar to incompressible flows by properly accounting for the mean density variations in the scaling. This has been shown in a number of studies of compressible wall-bounded flows.

In simulations of supersonic channel flow and supersonic boundary layers the van Driest transformation leads to a collapse of the mean streamwise velocity profiles both for isothermal and adiabatic boundary conditions. To account for the variation in mean density near the wall in compressible flow, Huang et al. proposed a “semilocal” scaling. The semilocal velocity scale is also consistent with the velocity scale proposed by Morkovin for similarity of the Reynolds stress in compressible flows. In the supersonic boundary layer simulation by Guarini et al. and the channel flow simulation by Coleman et al. and Morinishi et al. the velocity fluctuation intensities and the shear stress in semilocal scaling were reported to compare well with data for incompressible flows. Semilocal scaling was also applied in a compressible channel flow by Foysi et al. They reported the normal stress peak positions to collapse, but not the peak magnitudes.

Scalar mixing is of interest in a range of areas including e.g., combustion and atmospheric pollutant transport. A number of studies have been devoted to mixing in plane and round jet flows (see, e.g., Refs. 9–11). Mixing in a reacting environment has recently been studied by means of DNS in a...
reducing the compressibility effects in relatively high Mach number flows. Studies of shear flows with significant density gradient and low Mach numbers are scarce, and the low-speed compressible effects in these flows are not well understood. To our knowledge, no numerical investigation concerning turbulent nonisothermal wall jets has been published. In the present study, we therefore analyze the development and statistics of plane nonisothermal turbulent wall jets with low Mach numbers, by means of three-dimensional DNSs. Cold and warm jets are simulated and the temperature differences between the ambient and jet fluid at the inlet are about 400 and 200 K, respectively. The aim of this investigation is to study how the wall-jet development is influenced by the varying density. Properties of the nonisothermal jets are compared to results obtained in an isothermal jet. Proper scaling approaches in the respective inner and outer layers are investigated. The influence of the varying density on the mixing and transport of scalars is also studied, and the self-similarity of the velocity and scalar fields is evaluated.

II. GOVERNING EQUATIONS

The governing equations in all jet simulations are the fully compressible Navier–Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0,$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j},$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho E u_i}{\partial x_i} = \frac{\partial}{\partial x_j} \left( \frac{\partial T}{\partial x_j} \lambda + \frac{\partial \left( \frac{u_i}{\tau_{ij}} - p \delta_{ij} \right)}{\partial x_j} \right),$$

where $\rho$ is the mass density, $u_i$ is the velocity vector, $p$ is the pressure, and $E = e + \frac{1}{2} \rho u_i u_i$ is the total energy, being the sum of the internal energy $e$ and kinetic energy. Fourier’s law, where $\lambda$ is the coefficient of thermal conductivity, is used to approximate the energy fluxes. The stress tensor is defined as

$$\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_k}{\partial x_k},$$

where $\mu$ is the dynamic viscosity.

The fluid is assumed to be calorically perfect and to obey the ideal gas law

$$e = c_v T,$$

$$p = \rho RT,$$

and a ratio of specific heats of $\gamma = c_p/c_v = 1.4$ is used. To account for the substantial variations in density and temperature a temperature dependent viscosity is used in the nonisothermal cases. The viscosity is determined through Sutherland’s law

$$\frac{\mu}{\mu_j} = \left( \frac{T}{T_j} \right)^{3/2} \frac{T_j + S_0}{T + S_0},$$

where $T$ is the local temperature and $T_j$ is the jet center temperature at the inlet. The reference coefficient is $S_0 = 110.4K$ which is valid for air at moderate temperatures and pressures.

A transport equation for a passive scalar

$$\frac{\partial \rho \theta}{\partial t} + \frac{\partial (\rho u_i \theta)}{\partial x_i} = \frac{\partial}{\partial x_j} \left( \rho D \frac{\partial \theta}{\partial x_j} \right),$$

where $D$ is the scalar diffusion coefficient is solved to investigate the mixing. For the energy equation a constant Prandtl number Pr = $\mu c_p / \lambda = 0.72$ is assumed and for the passive scalar a constant Schmidt number $Sc = \mu / \rho D = 1$ is used. This implies that the heat conductivity $\lambda$ and the scalar diffusion $\rho D$ have the same temperature dependence as the dynamic viscosity $\mu$ (Sutherland’s law) since we also assume constant $c_p$.

III. NUMERICAL METHOD

The simulations are performed employing a sixth-order compact finite difference scheme$^{14}$ for the spatial discretization, and a third-order low-storage Runge–Kutta scheme for the temporal integration.$^{15}$ To minimize reflections at in- and outlets, boundary zones as described by Freund$^{16}$ are applied. Within the boundary zones the solution is smoothly forced toward target profiles and spurious fluctuations are damped. The outlet target functions must be constructed with care, especially in the nonisothermal cases. Smaller runs were performed to compute the half widths and maximum velocity positions at the outlet to obtain target functions for the final simulations.

The goal of the investigation is to study turbulent wall jets, hence high magnitude disturbances, $u'_{max} = 0.065U_j$ where $U_j$ is the inlet jet center velocity, are applied at the inlet to facilitate a fast and efficient transition to turbulence. Three types of disturbances are used; random but correlated in time and space using the method of Klein et al.$^{17}$ stream-wise vortices in the upper shear layer and harmonic stream-wise disturbances. The disturbances are superimposed at the inlet and added at every time step. For the correlated disturbances a correlation length of $h/3$ was used in all three directions, except in the streamwise direction in the cold and warm jet simulations, where the correlation length was about 40% lower. The simulation method is presented in more detail in Ahlman et al.$^1$

IV. WALL-JET SIMULATION

An isothermal planar wall jet was investigated by Ahlman et al.$^1$ and data from their DNS are used in this study for comparison. We have carried out DNS of three nonisothermal wall jets. To generate a cold jet in a warm surrounding and a warm jet in a cold environment, the inlet energy and density profiles are varied appropriately. The respective flow cases are characterized by the density ratio of the ambient to jet fluid at the center of the inlet $\alpha_p = \rho_a / \rho_j$. In the cold jet cases a ratio of $\alpha_p = 0.4$ is used and in the warm $\alpha_p = 1.7$.
The simulation domain is a rectangular box with a no-slip wall at the bottom. Periodic boundary conditions are used in the spanwise direction. The streamwise, wall-normal, and spanwise directions are denoted by \( x, y, \) and \( z \), respectively. Above the jet a slight coflow of \( U_{\text{top}} = 0.10 U_j \) is applied for computational reasons. In particular, at startup large scale vortices may develop above the jet. The coflow advects these large persistent vortices out of the domain. The temperature at the wall is constant and equal to the ambient condition in all three cases. For the passive scalar a no-flux boundary condition is imposed, \( \partial \theta / \partial y \biggr|_{y=0} = 0 \). At the top of the domain an inflow velocity \( U_{\text{top}} \) of 0.026\( U_j \) is applied for cases I2, C2, and W2 while \( U_{\text{top}} = 0.065 U_j \) for C7 because the entrainment is larger in this case. The inlet profiles for the velocity and passive scalar, plotted in Fig. 4, are similar to the ones used in the previous investigation of the isothermal jet.\(^1\) The jet height \( h \) is defined as the velocity half width at the inlet, i.e., \( h = y_{1/2}(x=0) \). The velocity half width is the distance from the wall where the mean velocity is equal to half the mean excess value, i.e., \( \bar{U} [x, y_{1/2}(x)] = \frac{1}{2} (\bar{U}_j(x) - \bar{U}_c) \). The inlet density profile is defined, using a thin near-wall layer \( f_{y,w} \), as

\[
\frac{\rho_{y}(y)}{\rho_j} = \begin{cases} 
\alpha_p + (1 - \alpha_p) \tanh(G_{w} y), & 0 \leq y \leq \frac{h}{5} \\
1 - (1 - \alpha_p) \frac{1}{2} [1 + \tanh(G(y - h))], & \frac{h}{5} \leq y \leq L_y,
\end{cases}
\]

where \( G_{w} = 50 \) and \( G = 15.6 \) define the density gradient in the near wall and outer layer, respectively. The outer layer gradient coefficient is the same for the velocity, the passive scalar, and the density. The temperature profile is then determined by the ideal gas law with constant pressure. We have performed simulations of a warm and a cold jet with an inlet Reynolds number of \( \text{Re} = U_j h/\nu_j = 2000 \), where \( \nu_j \) is the kinematic viscosity at the inlet jet center, and a corresponding Mach number of \( M = U_j/c = 0.5 \). These values correspond to those used in our previous isothermal jet simulation.\(^1\) Since the resulting friction Reynolds number in the C2 case is low (see Sec. V A), it is complemented with a cold jet simulation with \( \text{Re} = U_j h/\nu_j = 7000 \) and \( M = U_j/c = 0.5 \). That Re is chosen so that the resulting friction Reynolds number becomes similar to that for the isothermal case.

The inlet density ratios, initial temperatures, box dimensions, and resolutions are presented in Table I. The table also includes the reference name for each case. As will be discussed later, the cold and warm jet flow differ significantly in terms of the range of scales present, and hence different resolutions and box sizes are used in the four cases. The computational grid is stretched in the wall-normal direction using a combination of a hyperbolic tangent and logarithmic function to obtain a clustering of nodes near the wall and keeping a sufficient number of nodes in the outer layer. In the streamwise direction the grid is slightly stretched with the highest resolution in the transition region and a gradually reduced resolution downstream. The smallest scales in the jet are found close to the wall, which is natural since the energy dissipation attains its maximum at the wall. Wall units are therefore used in Table II to quantify the numerical resolution in the four cases. Values in the region where the flow is fully turbulent, \( x/h > 15 \), are presented. The streamwise stretching approximately follows the flow development. The resolution in the isothermal and warm jets is comparable to resolutions used in DNS of channel flow simulations (see, e.g., del Álamo et al.\(^18\)). The resolution used in the cold jet simulations is comparable or significantly better.

Wall-jet statistics are computed by applying ensemble averaging over time and the periodic spanwise direction. The

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Jet} & \text{Case} & \text{Re} & \rho_j/\rho_i & T_a & T_j & L_x \times L_y \times L_z & N_x \times N_y \times N_z & \text{Line style} \\
\hline
\text{Isothermal} & \text{I2} & 2000 & 1.0 & 293.0 & 293.0 & 47 \times 18 \times 9.6 & 384 \times 192 \times 128 & \quad \\
\text{Cold} & \text{C2} & 2000 & 0.4 & 732.5 & 293.0 & 35 \times 17 \times 7.2 & 384 \times 192 \times 160 & \quad \\
\text{Warm} & \text{W2} & 2000 & 1.7 & 293.0 & 498.1 & 28 \times 14 \times 7.2 & 448 \times 256 \times 160 & \quad \\
\text{Cold} & \text{C7} & 7000 & 0.4 & 732.5 & 293.0 & 30 \times 16 \times 8.0 & 256 \times 192 \times 128 & \quad \\
\hline
\end{array}
\]

<table>
<thead>
<tr>
<th>Case</th>
<th>( \Delta x^+ )</th>
<th>( \Delta y^+ )</th>
<th>( \Delta z^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I2</td>
<td>10.7–11.8</td>
<td>0.865–1.30</td>
<td>5.49–8.26</td>
</tr>
<tr>
<td>C2</td>
<td>3.05–3.10</td>
<td>0.285–0.373</td>
<td>1.28–1.68</td>
</tr>
<tr>
<td>W2</td>
<td>12.5–12.9</td>
<td>1.17–1.45</td>
<td>7.61–9.46</td>
</tr>
<tr>
<td>C7</td>
<td>11.6–12.7</td>
<td>1.05–1.12</td>
<td>7.48–7.99</td>
</tr>
</tbody>
</table>
TABLE III. Time scales sampled in terms of the inlet time scale, \( t_j = h/U_j \), and an outer time scale \( t_s = y_{1/2}/U_m \), defined at a downstream distance of \( x/h = 25 \) and \( x/h = 22 \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Start ([t_j])</th>
<th>Separation ([t_s])</th>
<th>Sampled ([t_j])</th>
<th>Sampled ([t_s])</th>
</tr>
</thead>
<tbody>
<tr>
<td>I2</td>
<td>185</td>
<td>0.500</td>
<td>309</td>
<td>70.4 ([t_{25}])</td>
</tr>
<tr>
<td>C2</td>
<td>152</td>
<td>0.568</td>
<td>366</td>
<td>93.7 ([t_{25}])</td>
</tr>
<tr>
<td>W2</td>
<td>153</td>
<td>0.600</td>
<td>367</td>
<td>85.5 ([t_{25}])</td>
</tr>
<tr>
<td>C7</td>
<td>242</td>
<td>0.647</td>
<td>363</td>
<td>101.0 ([t_{25}])</td>
</tr>
</tbody>
</table>

start time for the sampling after the startup of the simulations, the time separation between samplings and the total time over which averaging is carried out are presented in Table III in terms of the inlet time scale \( t_j = h/U_j \) and an outer time scale \( t_s = y_{1/2}/U_m \), defined in the downstream region where turbulent statistics are acquired.

V. RESULTS

Statistics of the nonisothermal wall jets are presented below and compared to those of the isothermal jet. The line styles defined in Table I are used throughout this paper to discriminate between statistics in the warm, isothermal and cold jets. If not stated otherwise, wall-normal profiles and correlations presented are acquired at a downstream position of \( x/h = 22 \) in all cases.

Density-weighted or Favre decomposed statistics are often used in combustion models since the averaged equations take the same form as the incompressible Reynolds averaged ones. The familiar incompressible modeling approaches can then be applied also for the compressible case. In the present study statistics using Reynolds and Favre decomposition, according to

\[
\bar{f} = \bar{f} + f',
\]

\[
f = \bar{f} + f'' = \frac{\rho f}{\bar{\rho}} + f'',
\]

will be presented. Reynolds averages and fluctuations are denoted by over bars and single accents while Favre averages and the corresponding fluctuations are denoted by tildes and double accents.

A. Structures and mean flow development

To provide an overview of the turbulence structures in the different jets, snapshots of the passive scalar concentration are shown in Fig. 1. The turbulence structures are distinctly different in the three \( Re=2000 \) cases. The warm jet contains the largest range of scales and has significantly more small scale energy than the other two cases. Correspondingly the isothermal case contains a larger range of scales and more small scale energy than the cold case. The same phenomenon is also seen in snapshots of the fluctuating velocity components. The observed difference in turbulence structure will have a profound influence on, for example, the heat transfer to the wall in cooling and combustion applications.

The friction Reynolds number

\[
Re_\tau = \frac{\delta}{\nu_w} = \frac{\delta}{\nu_w} \sqrt{\frac{dU}{dy}} |_{y=0}
\]

is examined in Fig. 2. When an appropriate outer length scale \( \delta \) is used, the friction Reynolds number provides an estimate of the outer to inner layer length scale ratio. In the fully turbulent region, \( Re_\tau \) in the warm jet is about 4.5 times higher than in the cold jet at \( Re=2000 \) which explains the presence of smaller structures in the warm jet. The friction Reynolds number can therefore be considered to be the effective Reynolds number of nonisothermal wall jets, rather than the inlet Reynolds number. The difference in \( Re_\tau \) is related to the temperature dependence of the viscosity at wall temperature. In the warm jet \( \nu_w \) is low near the relatively cold wall, whereas in the cold jet \( \nu_w \) is high near the relatively warm wall, which results in high and low \( Re_\tau \), respec-

FIG. 1. (Color online) Snapshots of the passive scalar concentration \( \theta/\Theta_0 \), in the cold jet at \( Re=2000 \) (a) and \( Re=7000 \) (b), isothermal jet (c), and warm jet (d).

FIG. 2. Downstream development of the friction Reynolds number \( Re_\tau = u_{w} y_{1/2}/\nu_w \).
The higher inlet Reynolds number in case C7 leads to a Reynolds number about equal to that in case I2.

The resulting heat transfer to the wall in the warm and cold jets is shown in Fig. 3 in terms of the Nusselt number defined for the inner shear layer

$$\text{Nu}_i = q_w \frac{y_m}{H} \frac{T_m - T_w}{T_m - T_w},$$

where $y_m$ is the jet center position and $T_m$ denotes the maximum mean temperature in the warm jet and the minimum mean temperature in the cold. Due to the different Reynolds numbers the Nusselt numbers differ in cases C2 and C7, but in both the cold cases $\text{Nu}_i$ is significantly smaller than in the warm jet.

Cross stream profiles of the velocity, temperature, and passive scalar concentration are shown in Fig. 4. The streamwise velocity and temperature profiles, normalized by the inlet conditions, show that the warm jet has the fastest streamwise decay, followed by the isothermal and the cold jets, respectively. The passive scalar concentration development has a different character. The decay of the maximum concentration at the wall in the warm and isothermal jets are practically the same, while the concentration at the wall in the cold jets decays significantly slower. The region near the wall, where the concentration gradient develops, is also visibly larger in case C2. This is likely caused by the low friction Reynolds number in this case.

Figure 5 shows $xz$-plane snapshots of the streamwise velocity fluctuations $u''/u_1$ in the cold, isothermal and warm jet at $y^+ = 7$. Elongated streamwise streaks, typically present in the viscous sublayer of boundary layers, are seen in all four jets. The width of the streaks is however influenced by the varying density and the streaks with the smallest physical width are seen in the warm jet. Note, however, that the opposite is true in viscous scaling. To determine the size of the turbulence structures in the near-wall layer, two-point velocity correlations in the spanwise direction at $y^+ = 7$ are shown in Fig. 6(a). The streamwise correlations contain minima corresponding to half the spanwise streak spacing. Using this measure the streak spacings are approximately 46, 90, 94, and 140 wall units in the cold jet cases C2 and C7, isothermal and warm jet, respectively. In terms of wall units the warm jet contains the widest streaks. The streak spacing does, therefore, not depend solely on the viscous length scale. The streak spacing found in the isothermal wall jet and case C7 is similar to what is found in incompressible turbulent boundary layers and channel flows, whereas in case C2 it is significantly smaller. Two-point correlations of the temperature and passive scalar acquired at the half-width position are shown in Fig. 6(b). In the outer layer the temperature and passive scalar correlations are practically identical, except for case C7. This indicates that temperature mixing and transport can be considered passive in this region. In the inner region the correlations differ due to the different boundary conditions.

B. Mean density effects

According to the hypothesis by Morkovin (also described in Refs. 21–23) the effects of density on the turbu-
lence structure are small as long as the density fluctuation intensity compared to the absolute density is small. A common measure is $\rho' / \bar{\rho} < 0.1$. Following this assumption, the turbulent statistics of compressible flows are similar to statistics of incompressible flows when a proper scaling, accounting for the mean density variation, is used. Although Morkovin’s investigation concerned boundary layers, it is often applied to other types of shear flows as well. Turbulence statistics in compressible boundary layers with free stream Mach numbers of $M < 5$ and compressible jets with Mach numbers of $M < 1.5$ generally compare well with the statistics of their incompressible counterparts, provided that a proper scaling is applied. However, ratios of turbulent quantities to mean flow values may still be greatly affected by density since Morkovin’s hypothesis does not include the effects of mean density gradients.

Normalized density fluctuation intensities in the nonisothermal jets are shown in Fig. 7. When scaled by the mean density, the fluctuation intensities are approximately at the level where compressible effects could become noticeable, according to the Morkovin criteria above, but can still be considered small. The higher intensity in the cold jets is presumably due to the higher relative difference in $\rho_u$ and $\rho_j$. Coleman et al. observed, in a compressible channel flow with isothermal walls, values up to $\rho' / \bar{\rho} = 0.13$ and concluded that Morkovin’s hypothesis holds. In the warm jet the position of the outer fluctuation maximum is further away from the wall than in the cold jets. The pressure fluctuation intensity, scaled by the density-weighted turbulent kinetic energy at the half-width position, is plotted in Fig. 8. The scaled fluctuations are all of the order of one, but the fluctuation intensity is affected by the mean density differences and $\text{Re}_\tau$. Both in the warm and cold jet case C2 the fluctuation levels are higher than in the isothermal case, whereas in case C7 it is about equal. Comparing the full profiles the fluctuation intensity is higher in the outer layer, reflecting the higher sensitivity of free shear layers to density effects compared to boundary layers.

Mean density effects can also be studied through comparison of statistics using Favre and Reynolds decomposition. The differences between Reynolds and Favre averaged mean velocities and turbulent stresses can be written as

$$\bar{U}_i - \bar{\tilde{U}}_i = \bar{\rho} \tilde{u}_i = -\rho' \tilde{u}_i \bar{\rho} = -\rho' \tilde{u}_i \bar{\rho},$$

$$\bar{\rho}' \tilde{u}_i \rho' \tilde{u}_j = \bar{\tilde{u}}_i \bar{\tilde{u}}_j - \rho' \tilde{u}_i \rho' \tilde{u}_j,$$

i.e., differences are caused by correlations of the density and velocity fluctuations and by ensemble averages of the Favre fluctuations. Mean velocities using Favre and Reynolds av-
The visualizations and the Re-plot have shown that the jets are fully turbulent for \( x/h \geq 15 \). Here we present statistics obtained in the turbulent region.

Mean streamwise velocity profiles at downstream positions \( x/h = 20 \) and \( x/h = 25 \) for cases C2, I2, and W2, and at \( x/h = 17 \) and \( x/h = 22 \) for case C7 are shown in Fig. 10. In case C7 different outflow boundary conditions had to be used than in the other cases due to the higher Reynolds number. These appear to affect the statistics beyond \( x/h = 22 \) in the C7 case whereas in the other cases the statistics at \( x/h = 25 \) are unaffected. In Ahlman et al.\textsuperscript{14} the near-wall layer of the isothermal wall jet was shown to closely resemble a zero-pressure gradient boundary layer. Mean profiles are therefore shown in Fig. 10 using three types of inner scaling. In Fig. 10(a) conventional boundary layer scaling is used. Here \( y^+ = y/l^+ = yu_\tau /\nu_w \) and \( U^+ = U/u_\tau \), where the friction velocity is defined as \( u_\tau = \sqrt{\tau_w/\rho_w} \) and the subscript \( w \) denotes conditions at the wall. The marked differences between the velocity profiles are consistent with the different Re- in the simulations.

Using wall variables all profiles collapse in the viscous sublayer, which also has the same approximate width in wall units, \( y^+ \leq 5 \), in the four cases. The physical size on the other hand varies since the viscous length scale \( l^+ \) varies with temperature, implying that the near-wall region with significant viscosity effects is largest in the cold jet case C2. Further away from the wall, an inertial sublayer in correspondence to a boundary layer has been found in experiments and large-eddy simulations at sufficiently high Reynolds numbers (see, Fig. 10).
e.g., Refs. 2, 25, and 26). No such region is found in the present study, presumably due to the moderate Reynolds numbers. As a result of the varying density near the wall, the mean profiles in conventional wall variable scaling do not collapse outside the viscous sublayer. This has also been observed to occur in compressible boundary layers or channel flows with significant temperature variations near the wall. Alternative scaling approaches have been developed which take into account the mean density variation. One of these is the van Driest transformation \( u'\bar{\tau}_w / u'_w = \int_0^{U^*} (\bar{\rho} \bar{\rho}_w)^{1/2} dU^*. \) (16)

Using the van Driest transformation, the mean velocity profiles of compressible flows and incompressible flows are similar (see, e.g., Refs. 4–6). However, as concluded by Huang and Coleman, the transformation does not lead to simultaneous collapse in the sublayer and logarithmic layer. Van Driest transformed wall-jet profiles are shown in Fig. 10(b). As a result of the transformation the profiles become slightly more similar, but they still do not collapse outside the buffer layer. In the warm jet, which has the highest \( \Re_w \) a logarithmic region in accordance with the incompressible inertial sublayer starts to appear.

Another approach to account for the varying mean density, the semilocal scaling proposed by Huang et al.\(^7\) is used in Fig. 10(c). In this scaling the wall variables are based on the local mean density and viscosity, and their relation to the conventional wall units becomes, as a result,

\[
U^*_w = \frac{\bar{\tau}_w}{\bar{\rho}} = \frac{1}{\bar{\rho}} u'_w
\]

\[
\bar{l}^* = (\bar{\rho} / \bar{\rho}_w)^{1/2} l^*,
\]

where \( u'_w = \sqrt{\bar{\tau}_w / \bar{\rho}_w} \) and \( l^* = \bar{\rho}_w / u'_w \) are the conventional friction velocity and length scales, respectively, and wall conditions are denoted by a subscript \( w \). In the results presented properties scaled by semilocal quantities are denoted by a superscript star, hence \( U^* = \bar{U} / u'_w \) and \( y^* = y / l^* \). The semilocal scaling provides the best scaling of both the position and magnitude of the jet center in the four jets. The warm, isothermal and cold jet case C7 collapse out to the beginning of the logarithmic layer, while in the cold jet case C2 the jet profile deviates closer to the wall due to its very low \( \Re_w \).

The Reynolds shear stress in the four jets is plotted in Fig. 11 using inner scaling. When the conventional boundary layer scaling is applied, the differences between the profiles are significant, which was also found for the mean velocity and kinetic energy profiles. Both the inner minimum and the outer maximum have higher magnitudes and extend to higher \( y^* \) values for the warm jet. This development is expected on the basis of the variation of \( \Re_w \). Figure 11(b) shows shear stress profiles in semilocal scaling. The sum of the viscous and shear stress must be equal to \( u'_w^2 \) in the near-wall region. However, in case C7 it stays large up to relatively large values of \( y^* \) whereas in the other cases the sum of the viscous and shear stress decays more rapidly away from the wall (results not shown here). Correspondingly, in the inner layer the scaled shear stress is significantly more negative in case C7 than in the other cases, as seen in Fig. 11(b). The correlation coefficient of the shear stress also reaches larger negative values in the cold jets than in the isothermal and warm jet.

The streamwise fluctuation intensity and the kinetic energy in all four cases are shown in Fig. 12. The semilocal scaling improves the collapse but notable differences still exist, also in the inner region. This is the case also for the spanwise fluctuations. In channel flow the scaled streamwise fluctuation intensity increases with increasing \( \Re_w \). We see the same trend when the cases C2 and C7 are compared, but in our simulations the increase is much larger for an equivalent span of \( \Re_w \) also when using semilocal scales. In case C7 the scaled streamwise fluctuation intensity and the kinetic energy are higher than in the isothermal jet, although \( \Re_w \) is about equal in both cases. The differences can thus not solely be attributed to Reynolds number effects. The turbulent intensity is also affected by the mean density gradient.

The rate of viscous dissipation of turbulent kinetic energy, \( \epsilon = (\bar{\rho} / \bar{\rho}_w) \int_0^{l^*} (\bar{\rho}_w / \bar{\rho}) \frac{\partial \bar{U}^2}{\partial x} d\bar{x} \), is shown in Fig. 13. Similar to turbulent channel flows, slight kinks are present in the inner region approximately at the position of maximum production of kinetic energy.\(^29\) In all four cases the scaled viscous dissipation at the wall is higher than the values 0.16–0.17 attained on an isothermal wall in the incompressible and compressible channel flows simulated by Morinishi et al.\(^6\) In all four jet cases the dissipation rate magnitude in the outer layer, in terms of the outer scaling \( \overline{U^*} \) (not
shown), is lower than the value 0.015 found at the half-width position in the plane jet simulation of Stanley et al. 10

**D. Self-similarity**

Self-similarity in the simulated wall jets is assessed by studying the downstream development and the application of scaling to collapse statistics in the inner and outer layers. In Fig. 14 the wall-normal growth and the streamwise decay in the jets are evaluated. The wall-normal growth is characterized by the development of the density-weighted half width $y_1^{o}$, which is the position in the outer shear layer where the density-weighted velocity is equal to half its maximum excess value, i.e., $\frac{1}{2}(\rho \bar{U}_m - \rho \bar{U}_c)$. Incompressible wall jets are known to display a linear half-width growth, similar to plane jets.2,25,26,30 Linear growth was previously observed in the isothermal simulation and in Fig. 14(a), this is also found to hold for the nonisothermal jets. The streamwise decay is evaluated in Fig. 14(b) which shows the development of the maximum momentum excess

$$M_e = \frac{(\rho \bar{U})_m - \rho \bar{U}_c}{\rho U_f - \rho \bar{U}_c}.$$  

(19)

The streamwise momentum decay in the warm and cold jets is also seen to be of the same type as in the isothermal wall jet. The results show that the generality of the outer layer evolution of isothermal wall jets also applies in the nonisothermal cases. The varying density wall jets can hence be considered self-similar, and the characteristic scales of the outer layer are similar to those of the isothermal wall jet.

Despite the similarity in development to isothermal conditions, the imposed density variation slightly influences the wall-normal growth and streamwise decay. The largest differences occur in the transition phase after which the growth rate is similar. The same variation is also seen in the standard half widths that are not density weighted. This effect is probably mainly due to the variation in Re$_e$. Differences in the entrainment process could also play a role, but how this was influenced by the varying density is not clear. In wall-jet experiments both the wall-normal growth and the streamwise decay appear to be Reynolds number dependent.25,31,32 For increasing Reynolds numbers the growth rates decrease slightly, but this trend cannot be observed when cases C2 and C7 are compared.

Density differences influence the growth rate of turbulent free shear layers.33,34 This can also play a role in our simulations. In the growth rate we cannot discern a clear effect of the density differences, but the momentum decay rates in the two cold cases are somewhat higher than in the isothermal and warm jet for $x/h > 12$. The variation in growth and decay rates observed are, however, small despite the significant density differences.

In the isothermal jet, statistics in the near-wall region were found to be self-similar using conventional wall variables.1 The extent of the collapsed region, however, varies between different statistics. Furthermore, the inner layers in the four cases differ due to mean density variation. Accounting for the mean density by semilocal scaling leads to a collapse of the mean velocity profiles in the inner layer, as seen in Fig. 10(c). However, this scaling does not lead to a collapse of the Reynolds stress profiles of the four jets, as
seen in Figs. 11 and 12, which indicates that the mean density variation affects the near-wall turbulence.

To evaluate self-similarity in the outer region, profiles of the four jets are shown in Fig. 15 in terms of the isothermal outer scaling. The outer scaling leads to a collapse of the streamwise velocity profiles of cases I2, W2, and C7, whereas the profiles of case C2 differ probably because of the low Re. The streamwise velocity fluctuation profiles of the warm and isothermal jet are quite similar and the same holds for the wall normal and spanwise components. In contrast, the maximum scaled streamwise fluctuation intensity of the two cold jets in the outer layer is higher than in the isothermal and warm jet which shows the relative high turbulence intensity in the two former cases. On the other hand, the scaled wall-normal scalar flux is lower in case C7 than in the isothermal and warm jet.

E. Temperature and passive scalar statistics

In Fig. 16, the fluctuation intensity and wall-normal fluxes of the temperature and passive scalar are presented. The fluctuations are normalized by the maximum temperature and scalar difference, $\Delta \Theta_m = \Theta_w = \Theta_{\infty}$, and $\Delta T_m = \max((\overline{T_w} - \bar{T}))$. The temperature flux is shown in semilocal scaling, where the inner temperature scale is defined as

$$T' = \frac{q_w}{\bar{\rho}_w c_p \mu_f} = \frac{\bar{\mu}_w}{\bar{\rho}_w \text{Pr} \mu_f} \left( \frac{\partial \overline{T}}{\partial y} \right)_{y=0},$$

where $q_w$ is the wall heat flux. The passive scalar flux is scaled by the wall concentration since the passive scalar, as a result of the no-flux condition at the wall, lacks a natural inner scale. The different boundary conditions are evident in both the fluctuation intensities and the fluxes. Inner and outer peaks are present in the temperature but not in the passive scalar flux profiles.

Passive scalar fluctuations are in the cold jet at Re=7000 near the wall larger than in the outer layer where the fluctuations further decrease in intensity. In contrast, the passive scalar fluctuations are of comparable magnitude in the inner and outer layers in the other jets at Re=2000. In the warm jet the intensity even increases slightly for $y > y_1/2$. Scaled with $\Delta T_m$ and $\Theta_{\infty}$, the temperature and passive scalar fluctuations, respectively, have a similar intensity in the inner region in all simulations, with the exception of case C2 which shows small passive scalar fluctuations there. In the outer layer, the intensity of the passive scalar fluctuations is lower than of the temperature using outer scaling, in particular, in case C7.

The maximum intensity of the temperature fluctuation in the outer layer scales approximately with the maximum temperature difference. Near the wall, the fluctuations show distinct peaks in the two cold jet cases but not in the warm jet. The difference between the inner and outer layer is thus less pronounced in the warm jet than in the cold jets. This is somewhat counterintuitive because an increased scale separation in most cases acts to pronounce differences between the inner and outer regions. The maximum value of the normalized passive scalar flux in the outer layer appears to increase with increasing Re. In the inner layer, the magnitude of the normalized wall-normal temperature flux is larger in case C7 than in the warm jet but in the outer layer it is comparable.

Gradient-diffusion approaches are commonly used to model the Reynolds shear stresses and scalar fluxes. For plane flows the model parameters describing the mixing and heat transfer are usually referred to as the turbulent Schmidt and Prandtl numbers, which are defined as...
respectively, where \( \nu_t \) is the turbulent viscosity, and \( D_\tau \) and \( \alpha_\tau \) are the turbulent passive scalar and heat diffusivities, respectively. In most simple turbulent flows \( \text{Pr}_t \) and \( \text{Sc}_c \) are of the order of one. \( \text{Pr}_t \) and \( \text{Sc}_c \), evaluated \emph{a priori} from the simulations, are shown in Fig. 17. In the near-wall region \( \text{Pr}_t \) and \( \text{Sc}_c \) differ due to the different boundary conditions. The Schmidt number goes to zero due to the vanishing mean gradient, whereas the turbulent Prandtl number increases toward the wall. Outside the near-wall region \( \text{Pr}_t \) and \( \text{Sc}_c \) decrease slightly in all cases. Further out both the shear stress and heat flux change sign. This takes place before the corresponding mean temperature and velocity extremum points, which causes an abrupt decline and negative \( \text{Pr}_t \) and \( \text{Sc}_c \) values over a short distance. Throughout the outer region, the turbulent Schmidt number is approximately constant and around 0.7. Due to the similarity of the heat and passive scalar transport in this layer \( \text{Pr}_t \) is very close to \( \text{Sc}_c \) and therefore not shown. In conclusion, significant variations in \( \text{Pr}_t \) and \( \text{Sc}_c \) exist only in the near-wall region and in the region separating the positions of vanishing turbulent fluxes and vanishing mean gradients. Constant values are good approximations for the whole outer region.

The scalar dissipation rate is of interest since it corresponds to the local dissipation of scalar fluctuations and hence describes the small scale mixing. The dissipation rate is therefore an important quantity in many mixing and combustion models. Figure 18 shows the dissipation rate of the passive scalar and the temperature, using an outer scaling for the dissipation rate and the wall distance.

The profiles of the passive scalar dissipation show very clear differences in the two cold jet cases. The effective Reynolds number (\( \text{Re}_x \)) is very low for the C2 case and the dissipation rate curve exhibits a character different from that

\begin{align}
\text{Sc}_c &= \frac{\nu_t}{D_\tau} \frac{u''v''}{\theta' \partial \bar{T}/\partial y}, \\
\text{Pr}_t &= \frac{\nu_t}{\alpha_t} \frac{u''v''}{\theta' \partial \bar{U}/\partial y},
\end{align}

FIG. 16. Scalar fluctuation intensities [(a) and (b)] and wall-normal fluxes [(c) and (d)]. Profiles at \( x/h=20 \) and \( x/h=25 \) for cases C2, I2, and W2. Profiles at \( x/h=17 \) and \( x/h=22 \) for case C7.

FIG. 17. Turbulent Schmidt and Prandtl numbers in the inner layer (a) and in the outer layer (b).
of the other cases, with a maximum in the outer layer. In the C7 case the shape of the dissipation profile has a similar character to that in the isothermal and warm jet, but the dissipation rate in terms of the outer scale is much lower. Also for the temperature, the dissipation rate magnitude in the outer layer is significantly lower using outer scaling, as compared to the other cases.

The ratio of the mechanical to passive scalar time scale and the corresponding temperature to passive scalar time scale ratio are presented in Fig. 19. In the outer layer, no clear effect of the density differences is perceivable in the mechanical to passive scalar time scale ratio. The time scale ratio is relatively constant throughout most of the wall jet, with exception for the inner region. For the passive scalar the time scale ratio shows even less variation in the outer region. The ratio of the temperature and passive scalar time scale is less than one indicating a more intense dissipation of temperature fluctuations than of passive scalar fluctuations.

VI. CONCLUSIONS

DNSs of a warm wall jet in a cold environment and a cold jet in a warm environment have been carried out. The inlet Reynolds and Mach numbers are the same as in a previously performed isothermal wall-jet simulation. In addition, a cold jet at a higher Reynolds number has been simulated. The cold jet cases can be seen as an idealized film cooling configuration and the hot jet case mimics the flow of hot exhaust gases over a cold wall.

Statistics of the jet development, turbulence and mixing are presented and the results are compared to statistics of the isothermal jet. Due to the varying viscosity the friction Reynolds numbers, here based on the half-width positions, are different. Correspondingly, in the warm jet smaller turbulence structures are present and the scale separation between the inner and outer shear layer is larger than in the cold jet at the same inlet Reynolds number. As a result of the density variation, conventional wall scaling fails to collapse the nonisothermal and isothermal jets outside the viscous sublayer. Applying the semilocal scaling leads to a collapse of the mean profiles in the inner layer. Semilocal scaling is not capable of collapsing the inner peak magnitudes of the streamwise and spanwise fluctuations intensities. Also in the outer layer, the profiles of the streamwise velocity fluctuations in outer scaling do not collapse. The normalized turbulence statistics thus appear to be influenced by mean density variations. In the nonisothermal jets the development of the density-weighted growth and streamwise momentum decay rate are similar to the isothermal case.

Streamwise streaks are present in the viscous sublayer in all cases, but their width, in terms of wall units, varies. Mean density effects are seen in the density and temperature fluctuations but the levels are small, in a Morkovian sense, despite the fact that for instance the cold jet inflow density is 2.5 times the ambient coflow density.

The profiles of the mean and fluctuating velocities are significantly different in the four jets, but the differences between Favre and Reynolds averages are small. The compressible effects are thus mainly a result of the mean density variations.

The turbulent Schmidt and Prandtl numbers vary significantly only in the near-wall layer and below the jet center. When comparing the scalar dissipation rates in the inner layer, the temperature and the passive scalar dissipation rates are different due to the different boundary conditions.

The difference in Reynolds number between the two cold jet simulations results in large differences in the passive
scalar dissipation profile. At higher Reynolds number, this profile has a similar shape as in the isothermal and warm jet but normalized with an outer scale the dissipation rate is much lower. In the outer layer, the mechanical to scalar time scale behavior is approximately equal in the four jets. The ratio of the temperature to passive scalar time scale is less than one indicating a smaller temperature time scale and hence a more intense dissipation of temperature fluctuations.

ACKNOWLEDGMENTS

Funding for the present work was provided by The Centre for Combustion Science and Technology (CECOST). The computations were performed at the Center for Parallel Computers at KTH, using time granted by the Swedish National Infrastructure for Computing (SNIC). Professor Bendiks Jan Boersma is thanked for providing the original version of the DNS code.


