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Deriving fluid-particle correlation closures for Eulerian two-fluid models through use of Langevin equations

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ABSTRACT

The correlation between the fluctuating particle and gas velocity in isotropic turbulence is studied with a set of stochastic differential equations taking into account both particle–particle collisions and the particle feedback on the turbulence. The principal aim of this work is to use the Langevin equations to formulate closures for two-fluid gas–particle flow models. Using Itô calculus we derived solutions for the turbulent kinetic energy of the particle phase and the particle-gas velocity correlations. If particle–particle collisions and particle feedback on the turbulence are neglected the new relations approach the ones derived by Tchen and Hinze but if these effects are included additional terms in the relations appear. In this study we only use a very simple model for the particle–particle collisions. The new relation and the classical relation of Tchen and Hinze for the particle turbulent kinetic energy as well as a relation based on the kinetic theory of granular flows have been implemented in a two-fluid model for turbulent gas–particle flow in a channel in order to make comparison for different particle Stokes numbers. Results show that while the two-fluid model using Hinze's relations only gives good results for small Stokes numbers, the new relation yields significant improvements for a large range of Stokes numbers.

1. Introduction

One approach to predict particle-laden turbulent flows is to use a two-fluid model with, for example, $K_g - \epsilon$ and $K_g - \omega$ models for the gas-phase turbulence and additional equations for the particle-phase [1]. This approach requires relations for the particle kinetic energy, K_p, which in general is a function of the particle response time, τ_p, and the particle volume fraction.

Turbulent gas-particle flows can be divided into three regimes. At very low particle volume fractions, i.e. $<10^{-6}$, particles with small response times, τ_p, will be governed by the turbulence and the ratio between the particle and the gas phase mean turbulent kinetic energy can be modelled according to

$$\kappa = \frac{K_p}{K_g} = \frac{1}{1 + \frac{\tau_p}{T_L}} \tag{1}$$

where $\tau_p = \frac{\rho_p D_p^2}{18 \rho_g \nu_g}$ is the particle response time, ρ_p, D_p and ν_g are the particle density, particle diameter and the kinematic viscosity of the fluid, respectively, and T_L is the Lagrangian macro time scale of the turbulence. When τ_p approaches zero the particle kinetic energy will approach the gas phase kinetic energy, K_g. Eq. (1) was derived by Hinze [2] and gives good predictions of K_p for small τ_p and small particle volume fractions [3]. The model has been used by for example [4–6]. As particle volume fraction increases to 10^{-4} particle–particle collisions will start to influence the flow field [3,7]. However, this is not taken into account in the expression above. For particle volume fractions larger than 10^{-3} particle–particle collisions will dominate the flow. The common approach to model K_p for larger particle volume fractions is to solve a transport equation for K_p based on the kinetic theory of granular flows [8].

The purpose of this work is to derive an analytic expression for the ratio κ that is valid for particle volume fractions where particle–particle collisions have an impact on the flow field. The new expression for κ is derived, starting from a simple set of stochastic differential equations describing turbulent particle-laden flows. The momentum transfer from the particles to the fluid, i.e. two-way coupling, and diffusion due to particle–particle collisions are incorporated in the present model. These two effects were not included in the analysis by Hinze [2].

The developed model here as well as Hinze's model and the kinetic theory of granular flows [9] are implemented into a two-fluid model. Model simulations of turbulent gas-particle channel flow with the three models are compared with experiments for different Stokes numbers. It is found that the present model can significantly extend the range of Stokes numbers where K_p can be predicted without solving a transport equation for K_p.

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2. Stochastic modelling of particle kinetic energy

Stochastic differential equations can be used to study different flow phenomena. Originally the Langevin model was used as a stochastic model to describe the motion of microscopic particles (Brownian motion), but it is also a good model to describe the motion of a fluid particle in turbulence. Several studies have previously used this model in different ways, for example the so-called generalised Langevin model has a corresponding Reynolds stress model [10,11] and can thus be used for turbulent flow predictions, the difference being that in the former approach there is no need for closures of third- and other higher-order velocity correlations since they are specified exactly. Stochastic equations have furthermore been used to model subgrid scale motions [12], to study atmospheric dispersion [13–15] and to develop a three dimensional model for the motion of particle pairs in order to study the particle concentration variance in isotropic turbulence [16].

Stochastic differential equations have found a fruitful application not only in single phase flow modelling but also in the modelling of turbulent two-phase flows, see for example [17–19] who present extensive reviews on the use of Langevin equations to model the motion of particles in the context of polydispersed two-phase turbulent flows. The Lagrangian approach in Langevin models is a natural way to describe the fluctuating particle motions in turbulent flows. Models based on Langevin equations, which are sometimes coupled to Reynolds-stress models for the flow field in a hybrid approach, have been applied for instance to bluff-body gas-particle flows [20] and to study particle deposition in turbulent flows [21–23]. Although Langevin models can be applied to real two-phase flow applications, in some cases it might be more convenient to use Eulerian two-fluid models instead, for example, such models can easily be implemented in a single-phase flow code without having to change the numerical methodology. The basic idea of the present work is to utilise the more complete description of the particle fluctuations in Langevin models in order to find expressions for fluid-particle correlations that are unclosed in two-fluid models of turbulent gas-particle flows.

The simplest stochastic diffusion process giving a statistically stationary solution for $t \to \infty$ is called the Ornstein–Uhlenbeck process and has the corresponding Langevin equation for the fluid particle velocity [11]

$$\mathrm{d}U(t) = -U(t)\frac{\mathrm{d}t}{T_L} + \left(\frac{2\sigma_0^2}{T_L}\right)^{1/2} \mathrm{d}w(t), \tag{2}$$

where $\sigma_0^2 = 2K_g$ and w is a Wiener process. The first term on the right hand side (R.H.S.) represents the deterministic drift and the second term on the R.H.S. represents the diffusion. Due to the irregularity of the noise the process is not differentiable and standard tools of differential calculus cannot be used. Instead an Itô integral is defined from which the Itô calculus follows. The differential of a stochastic process $Y(t) = g(t, U(t))$ where $U(t)$ is given by (2), can be obtained by Taylor expanding $Y(t)$ up to second order and then using the following rules

$$E[\mathrm{d}w \cdot \mathrm{d}w] = \mathrm{d}t, \qquad E[\mathrm{d}t \cdot \mathrm{d}t] = 0, \qquad E[\mathrm{d}t \cdot \mathrm{d}w] = 0 \tag{3}$$

and $E[\cdot]$ denotes the expected value. For more background about Itô calculus see [24].

Assuming isotropy the turbulent kinetic energy can be expressed as $K = \frac{3}{2}\langle u'u'\rangle$ where u' is the fluctuating velocity and $\langle\cdot\rangle$ is the average. The fluctuating part is defined as $u' = U - \langle U\rangle$. The variance of the process $U(t)$ defined in (2) is $\mathrm{var}(U(t)) = E[(U(t) - E[U(t)])^2]$.

In this study we use a Langevin equation to model gas phase turbulent velocity seen by the particle, U_g, and an additional stochastic differential equation for the fluctuating particle velocity, U_p. Basically, we follow the same approach as in many previous studies, e.g [20,19]. A Langevin model in general defines the one-point probability density functions of all statistical moments [17,18]. A major beneficial feature of such a model is therefore that the correlations between the gas-phase and particle velocity are specified by the Lagrangian stochastic equations and do not require additional closures. Instead, we use the Langevin equations to derive expressions for fluid-particle velocity correlations in a Eulerian two-fluid model where they are unclosed. Minier & Peirano [17] already showed how a Langevin model can be used to deduce Tchen's expression for the fluid-particle velocity correlations. In this study we advance this idea.

We assume here statistically stationary and isotropic turbulence and negligible effects of particle inertia on the Lagrangian time scale of the gas-phase seen by the particle. Later we will also take the stationary limit of the particle statistics. The governing equations for the gas-phase fluctuations seen by the particles and the fluctuating particle velocity then read [19]

$$\begin{cases} \mathrm{d}U_g = -\underbrace{\frac{U_g\,\mathrm{d}t}{T_L}}_{\text{Drift}} + \underbrace{\phi\frac{U_p - U_g}{\tau_p}\mathrm{d}t}_{\text{Drag}} + \underbrace{\left(\frac{2\sigma^2}{T_L}\right)^{\frac{1}{2}} \mathrm{d}w_g(t)}_{\text{“noise”}} \\ \mathrm{d}U_p = -\underbrace{\frac{U_p - U_g}{\tau_p}\mathrm{d}t}_{\text{Drag}} + \underbrace{C\,\mathrm{d}w_p(t)}_{\text{“noise”}} \end{cases} \tag{4}$$

which is appropriate when the turbulence is homogeneous and its statistics Gaussian [16]. The second term on the R.H.S. of the gas-phase equation represents the particle feedback caused by the Stokes drag where $\phi = \Phi/(1-\Phi)$ and Φ is the particle volume fraction. Using perturbation analysis it can be shown that $\sigma = \sigma_0 + O(\phi^2)$. Since ϕ is rather small we will from here on use $\sigma = \sigma_0$. If there is no preferential concentration of particles and therefore no statistically biased sampling we can in addition assume that $\sigma_0^2 = 2K_g$. For the particle phase the R.H.S is represented by the drag term and the diffusion due to particle–particle collisions is represented by a Wiener process w_p that is independent of the Wiener process describing the fluid turbulence w_g, i.e. $E[w_g w_p] = 0$. In this study, we use a rather simple model for the parameter C, i.e.

$$C = \frac{\nu_{tp}^{1/2}}{\tau_c}, \tag{5}$$

where ν_{tp} is the particle kinematic viscosity and τ_c is a collision time scale. The Langevin model presented here is the isotropic limit of the one presented by Peirano et al. [19] but extended with a particle–particle collision term.

In matrix form (4) can be expressed as

$$\mathrm{d}X = AX\mathrm{d}t + K\mathrm{d}W_t \tag{6}$$

where

$$\mathrm{d}X = \begin{pmatrix} \mathrm{d}U_g \\ \mathrm{d}U_p \end{pmatrix}, \qquad X = \begin{pmatrix} U_g \\ U_p \end{pmatrix}, \qquad A = \begin{pmatrix} \left(-\frac{1}{T_L} - \frac{\phi}{\tau_p}\right) & \frac{\phi}{\tau_p} \\ \frac{1}{\tau_p} & -\frac{1}{\tau_p} \end{pmatrix}$$

$$K = \begin{pmatrix} \left(\frac{2\sigma_0^2}{T_L}\right)^{\frac{1}{2}} & 0 \\ 0 & C \end{pmatrix} \qquad \mathrm{d}W = \begin{pmatrix} \mathrm{d}w_g \\ \mathrm{d}w_p \end{pmatrix}.$$

The following solution for X is obtained by solving (6) according to the Itô calculus

$$X_t = \mathrm{e}^{At}X_0 + \mathrm{e}^{At}\int_0^t \mathrm{e}^{-As}K\mathrm{d}W(s). \tag{7}$$

In order to get the solution to (7) an expression for A must be found. To obtain this we solve a similar but non-stochastic system of differential equations (DE) that gives the same value for A as (4), i.e.

$$\begin{cases} dU_g' = -\frac{U_g'}{T_L}dt + \phi\frac{U_p' - U_g'}{\tau_p}dt \\ dU_p' = -\frac{U_p' - U_g'}{\tau_p}dt. \end{cases} \tag{8}$$

In matrix form this can be expressed as

$$dX' = AX'dt, \tag{9}$$

where dX' and X' are defined analogous as above. The solution is formally given by

$$X_t' = e^{At}X_0', \tag{10}$$

where X_0' is the initial condition of X_t'. Its solution is written as a linear function of the initial conditions

$$U_g' = \left(\frac{\tau_p(\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t})}{\tau_p(\lambda_2-\lambda_1)} + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)}\right)U_{g0}' + \Bigg((1+\tau_p\lambda_1) \times\left(e^{\lambda_1 t} + \frac{\tau_p(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)} + \frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)}\right)\Bigg)U_{p0}' \tag{11}$$

$$U_p' = \left(\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)}\right)U_{g0}' + \left(e^{\lambda_1 t} + \frac{1+\tau_p\lambda_1}{\tau_p(\lambda_2-\lambda_1)}(e^{\lambda_1 t} - e^{\lambda_2 t})\right)U_{p0}' \tag{12}$$

where λ_1 and λ_2 are given by Eq. (13), given in Box I. Since we, in the expression for σ_0^2, neglect terms of order ϕ^2, it is suitable here to expand the above expression, resulting in:

$$\lambda_1 = -\frac{1}{\tau_p} - \frac{\phi T_L}{\tau_p(T_L-\tau_p)} + O(\phi^2) \tag{14}$$

$$\lambda_2 = -\frac{1}{T_L} - \frac{\phi}{T_L-\tau_p} + O(\phi^2). \tag{15}$$

Now (11) and (12) are put into the matrix form (10) which gives the following expression

$$e^{At} = \begin{pmatrix} \frac{\tau_p(\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}) + e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)} & (1+\tau_p\lambda_1)\times\left(e^{\lambda_1 t} + \frac{\tau_p(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}) + (e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)}\right) \\ \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)} & e^{\lambda_1 t} + (1+\tau_p\lambda_1)\frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)} \end{pmatrix}. \tag{16}$$

Then

$$e^{A(t-s)} = \begin{pmatrix} \frac{\tau_p(\lambda_2 e^{\lambda_2(t-s)} - \lambda_1 e^{\lambda_1(t-s)}) + e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)} & (1+\tau_p\lambda_1)\left(e^{\lambda_1(t-s)} + \frac{\tau_p(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}) + (e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\right) \\ \frac{e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)} & e^{\lambda_1(t-s)} + (1+\tau_p\lambda_1)\frac{(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)} \end{pmatrix}.$$

We can now go back to our original problem (4) and write its solution (7) as

$$\begin{aligned} U_g = {} & \left(\frac{\tau_p(\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}) + e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)}\right)U_{g0} + (1+\tau_p\lambda_1) \\ & \times\left(e^{\lambda_1 t} + \frac{\tau_p(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}) + (e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)}\right)U_{p0} \\ & + \int_{s=0}^{t}\left(\frac{2\sigma_0^2}{T_L}\right)^{\frac{1}{2}} \\ & \times\frac{\tau_p(\lambda_2 e^{\lambda_2(t-s)} - \lambda_1 e^{\lambda_1(t-s)}) + e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)}dw_g(s) \\ & + \int_{s=0}^{t} C(1+\tau_p\lambda_1)\Bigg(e^{\lambda_1(t-s)} \\ & + \frac{\tau_p(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}) + (e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\Bigg)dw_p(s) \end{aligned} \tag{17}$$

$$\begin{aligned} U_p = {} & \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)}U_0 + \left(e^{\lambda_1 t} + (1+\tau_p\lambda_1)\frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)}\right)U_{p0} \\ & + \left(\frac{2\sigma_0^2}{T_L}\right)^{\frac{1}{2}}\int_{s=0}^{t}\frac{e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)}dw_g(s) \\ & + \int_{s=0}^{t} C\left(e^{\lambda_1(t-s)} + (1+\tau_p\lambda_1)\frac{(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\right)dw_p(s). \end{aligned} \tag{18}$$

An expression for $K_p = \langle u_{pi}' u_{pi}' \rangle$, where $u_{pi}' = u_{pi} - U_{pi}$, can now be calculated using

$$\mathrm{var}(U_p) = E[(U_p - E[U_p])^2]. \tag{19}$$

The expectation value for U_p is

$$E[U_p] = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\tau_p(\lambda_2-\lambda_1)}U_0 + \left(e^{\lambda_1 t} + (1+\tau_p\lambda_1)\frac{(e^{\lambda_1 t} - e^{\lambda_2 t})}{\tau_p(\lambda_2-\lambda_1)}\right)U_{p0} \tag{20}$$

and the variance then becomes

$$\begin{aligned} \mathrm{var}(U_p) = E\Bigg[& \frac{2\sigma_0^2}{T_L}\int_{s=0}^{t}\frac{e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)}dw_g(s) \\ & \times\int_{s=0}^{t}\frac{e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)}dw_g(s) \\ & + \left(\frac{2\sigma_0^2}{T_L}\right)^{\frac{1}{2}}\int_{s=0}^{t}\frac{e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}}{\tau_p(\lambda_2-\lambda_1)}dw_g(s) \\ & \times\int_{s=0}^{t} C\Bigg(e^{\lambda_1(t-s)} + (1+\tau_p\lambda_1) \\ & \times\frac{(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\Bigg)dw_p(s) \\ & + \Bigg(\int_{s=0}^{t} C\big(e^{\lambda_1(t-s)} + (1+\tau_p\lambda_1) \\ & \times\frac{(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\big)dw_p(s)\Bigg)^2\Bigg]. \end{aligned} \tag{21}$$

We can now use a basic property of Itô integrals [25]. Suppose that $f, g : [0, T] \to \mathbb{R}$ are Itô integrable. Then

$$\lambda_{1,2} = -\frac{1}{2}\frac{T_L + \tau_p + \phi T_L \pm \sqrt{T_L^2 - 2T_L\tau_p + 2\phi T_L^2 + \tau_p^2 + 2\phi\tau_p T_L + \phi^2 T_L^2}}{T_L\tau_p}. \tag{13}$$

Box I.

$$E\left[\left(\int_0^T f(s,\cdot)\mathrm{d}W(s)\right)\left(\int_0^T g(s,\cdot)\mathrm{d}W(s)\right)\right] = \int_0^T E\big[f(s,\cdot)g(s,\cdot)\big]\mathrm{d}s. \tag{22}$$

This is used in (21) together with (3). The second term on the right hand side is zero because $\mathrm{d}w_g(s)$ and $\mathrm{d}w_p(s)$ are independent of each other. $\mathrm{var}(U_p)$ now becomes

$$\mathrm{var}(U_p) = \frac{2\sigma_0^2}{T_L}\int_{s=0}^{t} E\left[\frac{(\mathrm{e}^{\lambda_2(t-s)} - \mathrm{e}^{\lambda_1(t-s)})^2}{\tau_p^2(\lambda_2-\lambda_1)^2}\right]\mathrm{d}s + \int_{s=0}^{t} C^2\left(\mathrm{e}^{\lambda_1(t-s)} + (1+\tau_p\lambda_1)\frac{(\mathrm{e}^{\lambda_1(t-s)} - \mathrm{e}^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\right) \times \left(\mathrm{e}^{\lambda_1(t-s)} + (1+\tau_p\lambda_1)\frac{(\mathrm{e}^{\lambda_1(t-s)} - \mathrm{e}^{\lambda_2(t-s)})}{\tau_p(\lambda_2-\lambda_1)}\right)\mathrm{d}s. \tag{23}$$

Solving the integrals and letting $t \to \infty$, since the stationary case is considered, $\mathrm{var}(U_p)$ finally becomes

$$\lim_{t\to\infty}\mathrm{var}(U_p) = -\frac{\sigma_0^2}{T_L}\frac{1}{\tau_p^2\lambda_1\lambda_2(\lambda_2+\lambda_1)} - C^2\frac{1}{2}\frac{\tau_p^2\lambda_1^2 + 2\tau_p\lambda_1 + 3\tau_p^2\lambda_1\lambda_2 + 1 + \tau_p^2\lambda_2^2 + 2\tau_p\lambda_2}{\tau_p^2\lambda_1\lambda_2(\lambda_2+\lambda_1)}. \tag{24}$$

Replacing λ_1 and λ_2 with (13), given in Box I, (24) gives

$$\mathrm{var}(U_p) = \frac{\sigma_0^2}{1+\frac{\tau_p}{T_L}+\phi} + C^2\tau_p\frac{1+\frac{\tau_p}{T_L}+2\phi+\phi^2\frac{T_L}{\tau_p}}{2\left(1+\frac{\tau_p}{T_L}+\phi\right)}. \tag{25}$$

An expression for the mean turbulent kinetic energy for the particle phase, K_p has now been derived. To get an expression for κ an expression for K_g is needed. This is derived in the same manner as described above. With the expression for U_g from (17), $\mathrm{var}(U_g)$ becomes

$$\mathrm{var}(U_g) = \frac{\sigma_0^2\left(1+\frac{\tau_p}{T_L}\right)}{1+\frac{\tau_p}{T_L}+\phi} - \frac{C^2}{2\tau_p^2\lambda_2\lambda_1(\lambda_1+\lambda_2)} \times \left(\tau_p^2\lambda_2^2 + 2\tau_p^3\lambda_2^2\lambda_1 + \tau^4\lambda_2^2\lambda_1^2 + 2\tau_p\lambda_2 + 4\tau_p^2\lambda_2\lambda_1 + 2\tau_p^3\lambda_2\lambda_1^2 + 1 + 2\tau_p\lambda_1 + \tau_p^2\lambda_1^2\right). \tag{26}$$

Replacing λ_1 and λ_2 with Eq. (13), given in Box I, $\mathrm{var}(U_g)$ gives

$$\mathrm{var}(U_g) = \frac{\sigma_0^2\left(1+\frac{\tau_p}{T_L}\right)}{1+\frac{\tau_p}{T_L}+\phi} + C^2\frac{\phi^2 T_L}{2\left(1+\frac{\tau_p}{T_L}+\phi\right)}. \tag{27}$$

The ratio between particle and gas mean kinetic energy now becomes

$$\kappa = \frac{K_p}{K_g} = \frac{\mathrm{var}(U_p)}{\mathrm{var}(U_g)} = \frac{1+\frac{C^2\tau_p}{2\sigma_0^2}\left(1+\frac{\tau_p}{T_L}+2\phi+\phi^2\frac{T_L}{\tau_p}\right)}{1+\frac{\tau_p}{T_L}+\frac{C^2\tau_p}{2\sigma_0^2}\phi^2\frac{T_L}{\tau_p}}. \tag{28}$$

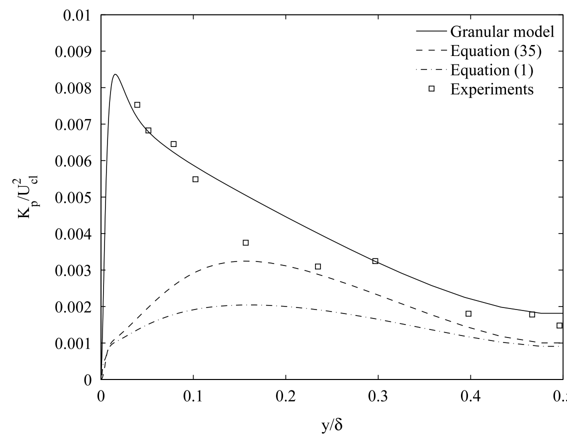

Fig. 1. Particle turbulent kinetic energy normalised with the centre line velocity in the presence of 50 μm glass particles with mass loading 10% shown for the granular model, simulations with Eq. (35), simulations with Eq. (1) and experiments from [31]. $St^+ = 371$.

Since ϕ^2 terms are neglected in the approximation $\sigma^2 = \sigma_0^2$ one should simplify (28) to

$$\kappa = \frac{1}{1+\frac{\tau_p}{T_L}} + \frac{C^2\tau_p}{2\sigma_0^2}\left(1+\frac{2\phi}{1+\frac{\tau_p}{T_L}}\right) + O(\phi^2) \tag{29}$$

if also $\phi^2 T_L/\tau_p \ll 1$. Here, two different dimensionless parameters can be identified i.e., $\frac{\tau_p}{T_L}$ and $\frac{C^2\tau_p}{2\sigma_0^2}$, the latter is a measure of the relative influence of particle collisions. Using (5) we get

$$\frac{C^2\tau_p}{2\sigma_0^2} \sim \frac{\nu_{tp}\tau_p}{\tau_c^2 K_g} \tag{30}$$

where it is seen that collisions increase when the collisional time scale decreases. If the collisional term is neglected i.e., $C = 0$, we obtain Eq. (1).

The correlation between fluctuating fluid and particle velocities $\langle u'_{gi}u'_{pi}\rangle$ is needed in the equation for K_g when modelling particle laden turbulent flows. The correlation $\langle u'_{gi}u'_{pi}\rangle = \mathrm{cov}(U_g U_p)$, where U_g and U_p are given by (17) and (18) respectively. Using the same method as above we get

$$\mathrm{cov}(U_g U_p) = \frac{\sigma_0^2}{1+\tau_p/T_L+\phi} - \frac{C^2}{2\tau_p^2\lambda_1\lambda_2(\lambda_1+\lambda_2)} \times \left(1 + 2\tau_p(\lambda_1+\lambda_2) + \tau_p^2(\lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2) + \tau_p^3\lambda_1\lambda_2(\lambda_1+\lambda_2)\right) \tag{31}$$

where λ_1 and λ_2 are given by (14) and (15). Neglecting terms of higher order in ϕ we get:

$$\mathrm{cov}(U_g U_p) = \frac{\frac{2}{3}K_{g0}}{1+\tau_p/T_L+\phi} + \frac{\phi C^2\tau_p T_L^2(1-\tau_p/T_L)^2}{2(1+\tau_p/T_L)} + O(\phi^2). \tag{32}$$

To zeroth order in ϕ this expression agrees with that of e.g. [26] but disagrees with the often used expression [27]

$$\langle u'_{gi}u'_{pi}\rangle = 2\sqrt{K_g K_p}. \tag{33}$$

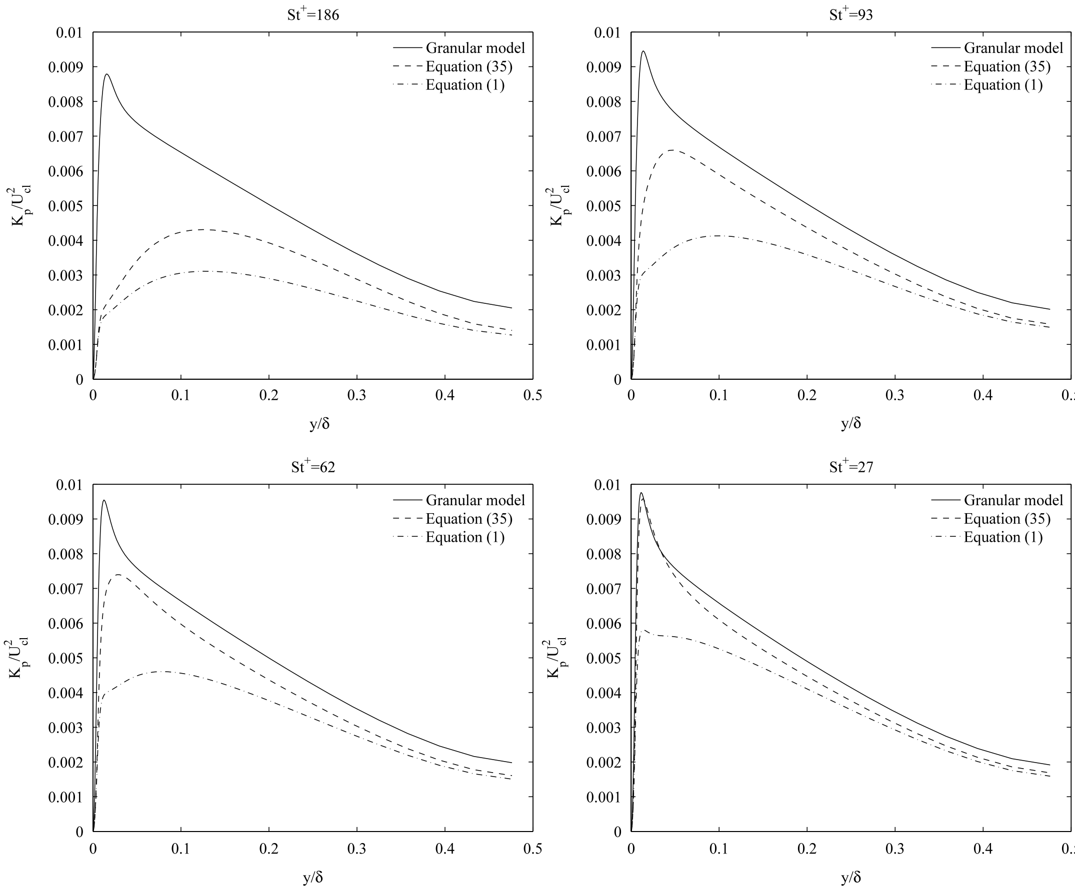

Fig. 2. Particle turbulent kinetic energy normalised with the centre line velocity. $St^+ = 186$ (a); $St^+ = 93$ (b); $St^+ = 62$ (c); $St^+ = 27$ (d).

Here we have used Itô calculus to find expressions for the fluid-particle velocity correlations in the stationary limit but in [17] an alternative approach is discussed.

3. Simulation of particle laden channel flow

We will now investigate the contribution of the second term on the R.H.S. in Eq. (29) in simulations of particle laden turbulent channel flow. Although this equation has been derived assuming isotropic turbulence we apply it to an inhomogeneous flow since we consider this an exploratory step to develop improved expressions for the fluid-particle velocity correlations in two-fluid models. Note that Tchen's expressions are also only strictly valid for isotropic turbulence, but nevertheless they are widely used for wall-bounded flows. In order to examine the influence of the particle–particle collision term, C is modelled according to Eq. (5) and the collisional time scale τ_c in a shear flow is expected to depend on the shear velocity. As a first simple approach it is modelled as

$$\tau_c = \frac{1}{\frac{dU_p}{dy}}. \tag{34}$$

More physics can of course come into the modelling of this time scale, see for example, [28,29]. However, in this study we use this simple expression that only takes into account what is expected to be most dominant for our case. In order to model Eq. (29) we here use $\sigma_0^2 = \frac{2}{3}K_{g0}$ since the model simulation only considers homogeneous isotropic turbulence. Furthermore, $K_{g0} = K_g|_{\phi=0}$, and τ_c and C are modelled according to Eqs. (5) and (34), respectively. Using this the model for κ now becomes (neglecting terms of $O(\phi)$)

$$\kappa = \frac{K_p}{K_g} = \frac{1}{1+\frac{\tau_p}{T_L}} + \frac{3\nu_{tp}\tau_p\left(\frac{dU_p}{dy}\right)^2}{4K_g}. \tag{35}$$

Table 1
Parameters of the simulated cases. Here $\text{Re} = U_b\delta/\nu_g$ where U_b is the bulk velocity and δ is the channel width, $St^+ = \frac{\tau_p u_\tau^2}{\nu_g}$ is the particle Stokes number based on friction velocity u_τ and viscosity ν_g, Φ_0 is the initial particle volume fraction and m is the mass loading.

Case	Re	St^+	Φ_0	m
1	27 600	371	$5 \cdot 10^{-5}$	0.1
2	27 600	186	$6 \cdot 10^{-5}$	0.03
3	27 600	93	$6 \cdot 10^{-5}$	0.03
4	27 600	62	$6 \cdot 10^{-5}$	0.03
5	27 600	27	$6 \cdot 10^{-5}$	0.03

To validate the two expressions for K_p, Eqs. (1) and (35) have been implemented into a two-fluid model [30] using a two-equation $K-\omega$ turbulence model with an isotropic eddy-viscosity and a corresponding approach for the particle phase. The two-fluid model gives ν_{tp} required in Eq. (35) and is implemented in a finite element code. In Fig. 1 the two model simulations with Eqs. (1) and (35) are compared to fully developed particle laden upward vertical turbulent channel flow experiments by Kulick et al. [31]. Computations with a two-fluid model based on kinetic theory of granular flows for the particle phase [9] have also been carried out. The parameters of the simulations (case 1) are listed in Table 1. Two-fluid model simulations using Eq. (35) give a larger K_p than a simulation using Hinze's model (Eq. (1)), especially around $y/\delta = 0.15$ where the model using Eq. (35) predicts values more than 50%

larger and agrees well with the experiments. However, the peak of K_p in the near wall region is neither captured by simulations using Eq. (1) nor (35). In this region the granular model has the best agreement with experiments while for $y/\delta > 0.15$ both the present model and the granular model agree fairly well with experiments. Simulations using Eq. (1) predict too low values of K_p in the whole region.

To investigate the effect of the three different models of K_p, two-fluid model simulations with the different models are compared in Fig. 2 for four different Stokes numbers. The parameters of the simulations (cases 2–5) are again listed in Table 1. Both model simulations with Eqs. (1) and (35) are seen to increase and approach the value predicted by the granular model with decreasing Stokes numbers. For $St^+ = 186$ simulations with Eqs. (1) and (35) do not predict the peak of K_p in the near wall region. For $St^+ = 93$ and 62 model simulations with Eq. (35) predicts the peak but not its magnitude while both its magnitude and position are captured for $St^+ = 27$. Simulations with Eq. (1) only qualitatively predict the peak of K_p for $St^+ = 27$ but not its magnitude. Strömgren et al. [9] showed that simulations with Eq. (1) reasonably capture both the magnitude and position of the peak of K_p predicted by the granular model for $St^+ < 15$. Thus, using Eq. (35) the model derived by Hinze can be extended to Stokes numbers up to 150 and still give good predictions of K_p without solving a transport equation for K_p.

4. Conclusions

A Langevin model is developed and used in order to formulate an expression for $\kappa = \frac{K_p}{K_g}$ since this ratio is unclosed in two-fluid models of turbulent gas-particle flows. In contrast, it can readily be obtained from the Langevin model which includes particle–particle collisions and two-way coupling. Without the contribution from particle–particle collisions the result coincides with the result from [2] which is only valid for small Stokes numbers and isotropic turbulence.

The expressions for K_p have been implemented in a two-fluid model in order to simulate particle-laden turbulent channel flow. The results show that with the contribution of particle–particle collisions Hinze's model can be extended to larger Stokes numbers. Stochastic models for gas-particle turbulent flows thus have the potential to extend models for K_p.

The expression for the collisional time scale and C could be developed further, since these expressions now only take the shear into account and lack some of the physics such as a dependence on particle concentration. The stochastic model can also be extended to anisotropic flows by taking the direction into account and by modifying the Lagrangian time scale to take crossing-trajectory and inertia effects into account [19].

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