21st Century Mechanics
Part 3: Dynamics of Bodies

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Chapter 1

Angular Momentum of Particle Systems

This chapter presents the principles of linear and angular momentum in the form they take for a system of particles. The elimination of the internal forces and moments is discussed. The forms that the principle of angular momentum takes for some special cases are derived and applied.

1.1 Linear and Angular Momentum of Particle Systems

In particle mechanics we have come across the concepts: (linear) momentum, $\mathbf{p} = m\mathbf{v}$, and angular momentum, $\mathbf{L}_{\mathcal{A}} = \overline{\mathcal{AP}} \times \mathbf{p}$, of a single particle of mass m and velocity $\mathbf{v}$ at $\mathcal{P}$. In terms of these concepts one finds that the equations of motion lead to

$$\dot{\mathbf{p}} = \mathbf{F} \tag{1.1}$$
$$\dot{\mathbf{L}}_{\mathcal{A}} = \mathbf{M}_{\mathcal{A}} \tag{1.2}$$

and these equations are referred to as the *principles of linear momentum* and *angular momentum* respectively. In the second of these equations it is assumed that the base point $\mathcal{A}$ with respect to which the moments are taken is at rest. Moving base points require special treatment and are discussed in a separate section. The purpose of this section is to generalize these principles to a system of particles.

1.1.1 Centre of Mass Relations

A system of particles is simply a set of particles, at points $\mathcal{P}_j(t)$ at time t, with masses m_j where $j = 1, \ldots, N$. Any mechanical system can be thought of as a system of particles, so this is a very important concept. To start with we define the (linear) *momentum* of the system to be the vector sum of the individual momenta

$$\mathbf{p} = \sum_{j=1}^{N} \mathbf{p}_j = \sum_{j=1}^{N} m_j \mathbf{v}_j. \tag{1.3}$$

We denote the total mass m so that

$$m = \sum_{j=1}^{N} m_j. \tag{1.4}$$

If we multiply and divide with this in the left hand side of the previous equation we get, using the centre of mass definition

$$\mathbf{r}_{\mathcal{G}} = \frac{\sum_{j=1}^{N} m_j \mathbf{r}_j}{\sum_{j=1}^{N} m_j}, \tag{1.5}$$

the relationship

$$\mathbf{p} = m\frac{\sum_{j=1}^{N} m_j \mathbf{v}_j}{\sum_{j=1}^{N} m_j} = m\mathbf{v}_{\mathcal{G}}. \tag{1.6}$$

So, for a system of particles we find that the momentum can be expressed as total mass times velocity of centre of mass:

$$\mathbf{p} = m\mathbf{v}_{\mathcal{G}}. \tag{1.7}$$

The *angular momentum* of a system is also defined as the vector sum of the the corresponding quantities for the individual particles

$$\mathbf{L}_{\mathcal{A}} = \sum_{j=1}^{N} \mathbf{L}_{\mathcal{A}j} = \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times m_j \mathbf{v}_j. \tag{1.8}$$

Here we can again attempt to introduce the centre of mass but things will not become as simple as for linear momentum. We first put $\overline{\mathcal{AP}}_j = \overline{\mathcal{AG}} + \overline{\mathcal{GP}}_j$ and, since $\mathcal{A}$ is assumed to be at rest, we get

$$\mathbf{v}_j = \frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{AP}}_j = \frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathcal{AG}} + \overline{\mathcal{GP}}_j) = \frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathcal{AG}} + \mathbf{r}'_j) = \mathbf{v}_{\mathcal{G}} + \mathbf{v}'_j. \tag{1.9}$$

Here $\mathbf{r}'_j = \overline{\mathcal{GP}}_j$ is the position vector of particle j with respect to the centre of mass and $\mathbf{v}'_j$ is the velocity of particle j with respect to the centre of mass or, equivalently, the velocity of particle j in the 'centre of mass system'. According to their definition these quantities obey

$$\sum_{j=1}^{N} m_j \mathbf{r}'_j = \sum_{j=1}^{N} m_j \mathbf{v}'_j = \mathbf{0}, \tag{1.10}$$

see figure 1.1. We now rewrite the angular momentum using this:

$$\mathbf{L}_{\mathcal{A}} = \sum_{j=1}^{N} (\overline{\mathcal{AG}} + \mathbf{r}'_j) \times m_j(\mathbf{v}_{\mathcal{G}} + \mathbf{v}'_j). \tag{1.11}$$

the two parts of the angular momentum

When the parentheses are expanded we get four terms but two of these are seen to give zero because of the relationships 1.10 so finally

$$\mathbf{L}_{\mathcal{A}} = \overline{\mathcal{AG}} \times m\mathbf{v}_{\mathcal{G}} \; + \; \sum_{j=1}^{N} \mathbf{r}'_j \times m_j \mathbf{v}'_j$$
$$\Longleftrightarrow$$
$$\mathbf{L}_{\mathcal{A}} = \overline{\mathcal{AG}} \times \mathbf{p} \; + \; \mathbf{L}'_{\mathcal{G}}. \tag{1.12}$$

The total angular momentum of a system is thus the sum of two parts: one, external, which only refers to the centre of mass of the system and one, internal, $\mathbf{L}'_{\mathcal{G}}$, which depends only on the positions and velocities of the particles with respect to the centre of mass system. When the system reduces to a single particle this second internal part vanishes.

The angular momentum with respect to the centre of mass, $\mathbf{L}_{\mathcal{G}}$, is by definition

$$\mathbf{L}_{\mathcal{G}} = \sum_{j=1}^{N} \overline{\mathcal{GP}}_j \times m_j \mathbf{v}_j \tag{1.13}$$

so if we use that $\overline{\mathcal{GP}}_j = \mathbf{r}'_j$ and insert the result 1.9 we get

$$\mathbf{L}_{\mathcal{G}} = \sum_{j=1}^{N} \mathbf{r}'_j \times m_j(\mathbf{v}_{\mathcal{G}} + \mathbf{v}'_j) = (\sum_{j=1}^{N} m_j \mathbf{r}'_j) \times \mathbf{v}_{\mathcal{G}} + \mathbf{L}'_{\mathcal{G}}. \tag{1.14}$$

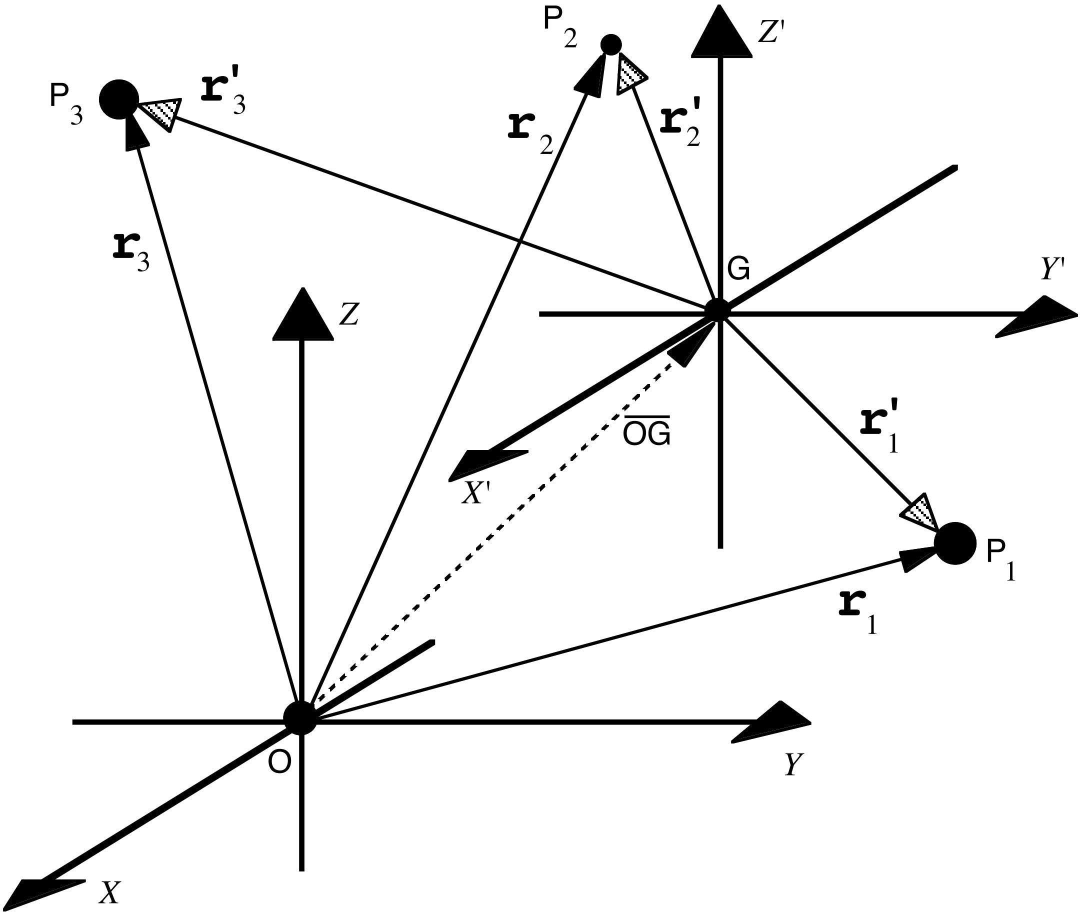


Figure 1.1: This figure shows the definition of the centre of mass system $(\mathcal{G}, X', Y', Z')$ of a three particle system. The position vectors of the particles in the fixed system, $\overline{\mathcal{OP}}_i = \mathbf{r}_i$ $(i = 1, 2, 3)$, are shown with filled black heads. The position vectors of the particles in the centre of mass system, $\overline{\mathcal{GP}}_i = \mathbf{r}'_i$, are shown with dashed heads. By definition we then have that $\sum m_i \mathbf{r}_i = m\overline{\mathcal{OG}}$ while $\sum m_i \mathbf{r}'_i = m\overline{\mathcal{GG}} = \mathbf{0}$.

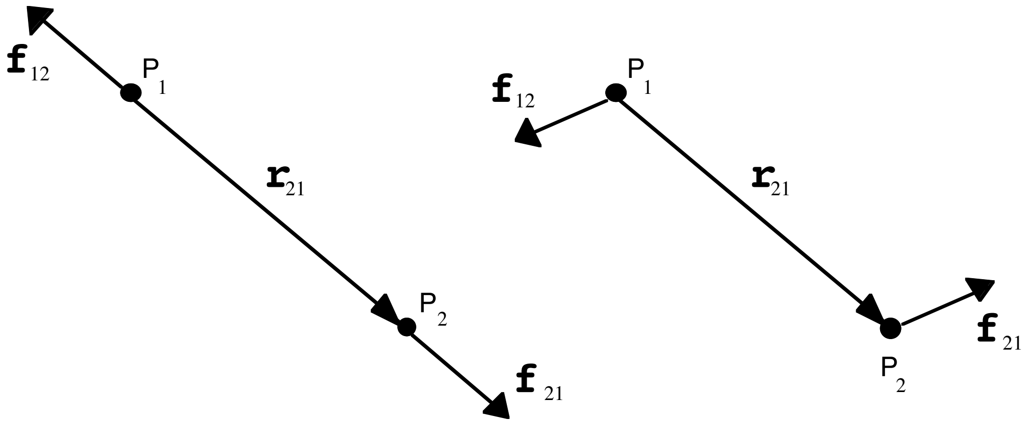


Figure 1.2: Two examples of internal (inter particle) forces that both obey Newton's third law $\mathbf{f}_{12} = -\mathbf{f}_{21}$. In the example on the left the forces are parallel to the vector between the particles, $\mathbf{f}_{21} \parallel \mathbf{r}_{21}$. In the example on the right this is not the case. Such cases must be excluded if the angular momentum principle for systems is to hold in its usual form.

Because of 1.10 we now find

$$\mathbf{L}_{\mathcal{G}} = \mathbf{L}'_{\mathcal{G}}. \tag{1.15}$$

It should be noted that this equality is non-trivial since the symbol $\mathbf{L}'_{\mathcal{G}}$ not only means that the base point is taken to be $\mathcal{G}$. The prime on the symbol indicates that the velocities of the particles are to be taken relative to the centre of mass system, a system which may be accelerated and thus *not* necessarily an inertial system.

1.1.2 Eliminating the Internal Forces

The force on one of the particles of a system can always be thought of as a sum of two contributions: an *internal* force from the other particles in the system and an *external* force from outside the system. The force on particle number k is then

$$\mathbf{F}_k = \mathbf{F}_k^{\mathrm{i}} + \mathbf{F}_k^{\mathrm{e}} \tag{1.16}$$

where the superscript 'i' stands for internal and 'e' stands for external. The internal force on a given particle can always be thought of as arising from the other particles of the system and it can therefore be expressed as the vector sum of of contributions from the rest of the system. If we denote the force from particle j on particle k by

$$\text{Force on } k \text{ from } j = \mathbf{f}_{kj} \tag{1.17}$$

we can thus write

$$\mathbf{F}_k^{\mathrm{i}} = \sum_{j=1}^{N} \mathbf{f}_{kj}. \tag{1.18}$$

It is natural to define $\mathbf{f}_{kk} = \mathbf{0}$ so that the term for $j = k$ does not have to be excluded from the sum. This is also a natural consequence of Newton's third law,

$$\mathbf{f}_{jk} = -\mathbf{f}_{kj}, \tag{1.19}$$

should we extend it to the case $j = k$, see figure 1.2

Let us now use this knowledge of the forces to study the time derivative of the momentum for the particle system. We get

$$\dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{N} m_j \mathbf{v}_j = \sum_{j=1}^{N} m_j \ddot{\mathbf{r}}_j = \tag{1.20}$$

$$= \sum_{j=1}^{N} \mathbf{F}_j = \sum_{j=1}^{N} (\mathbf{F}^{\mathrm{i}}_j + \mathbf{F}^{\mathrm{e}}_j) = \tag{1.21}$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbf{f}_{jk} \ + \ \mathbf{F}^{\mathrm{e}} \ = \ \mathbf{F}^{\mathrm{i}} + \mathbf{F}^{\mathrm{e}}. \tag{1.22}$$

Here $\mathbf{F}^{\mathrm{e}}$ denotes the force sum of the external forces. The double sum over the internal forces gives $\mathbf{0}$ because of the relations 1.19 so that

$$\mathbf{F}^{\mathrm{i}} = \mathbf{0}, \tag{1.23}$$

and consequently we now have

$$\dot{\mathbf{p}} = \mathbf{F}^{\mathrm{e}}. \tag{1.24}$$

This equation is called the *principle of* (linear) *momentum* for a system, or simply the 'momentum principle'.

We now try to do the same thing for angular momentum. If we take the time derivative we get

$$\dot{\mathbf{L}}_{\mathcal{A}} = \frac{d}{dt} \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times m_j \mathbf{v}_j = \sum_{j=1}^{N} (\mathbf{v}_j \times m_j \mathbf{v}_j + \overline{\mathcal{AP}}_j \times m_j \ddot{\mathbf{r}}_j) = \tag{1.25}$$

$$= \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times (\mathbf{F}^{\mathrm{i}}_j + \mathbf{F}^{\mathrm{e}}_j) = \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times \mathbf{F}^{\mathrm{i}}_j + \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times \mathbf{F}^{\mathrm{e}}_j = \tag{1.26}$$

$$= \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times \mathbf{F}^{\mathrm{i}}_j \ + \ \mathbf{M}^{\mathrm{e}}_{\mathcal{A}} \ = \ \mathbf{M}^{\mathrm{i}}_{\mathcal{A}} + \mathbf{M}^{\mathrm{e}}_{\mathcal{A}}. \tag{1.27}$$

Here $\mathbf{M}^{\mathrm{e}}_{\mathcal{A}}$ stands for the moment of the external forces with respect to $\mathcal{A}$. The sum of the moments of the internal forces corresponds to the sum that became zero when we derived the momentum principle. We can rewrite it as follows:

$$\mathbf{M}^{\mathrm{i}}_{\mathcal{A}} = \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times \mathbf{F}^{\mathrm{i}}_j = \sum_{j=1}^{N} (\overline{\mathcal{AP}}_j \times \sum_{k=1}^{N} \mathbf{f}_{jk}) = \sum_{k,j} \overline{\mathcal{AP}}_j \times \mathbf{f}_{jk} = \tag{1.28}$$

$$= \frac{1}{2} \left(\sum_{k,j} \overline{\mathcal{AP}}_j \times \mathbf{f}_{jk} + \sum_{k,j} \overline{\mathcal{AP}}_k \times \mathbf{f}_{kj} \right) = \tag{1.29}$$

$$= \frac{1}{2} \sum_{k,j} \left(\overline{\mathcal{AP}}_j \times \mathbf{f}_{jk} + \overline{\mathcal{AP}}_k \times \mathbf{f}_{kj} \right) = \frac{1}{2} \sum_{k,j} \left(\overline{\mathcal{AP}}_j - \overline{\mathcal{AP}}_k \right) \times \mathbf{f}_{jk} = \tag{1.30}$$

$$= \frac{1}{2} \sum_{k,j} \overline{\mathcal{P}_k \mathcal{P}}_j \times \mathbf{f}_{jk} \ = \ \frac{1}{2} \sum_{k,j} \mathbf{r}_{jk} \times \mathbf{f}_{jk} = \mathbf{M}^{\mathrm{i}}, \tag{1.31}$$

where $\mathbf{r}_{jk} \equiv \mathbf{r}_j - \mathbf{r}_k$. So Newton's third law makes the internal moment independent of base point, but it does not make it zero. This expression shows that the sum will become zero provided that the force from particle k to particle j is parallel to the vector from particle k to particle j, i.e. if $\mathbf{f}_{jk} \parallel \mathbf{r}_{jk}$. This behavior of inter-particle forces agree with those of the gravitational and electrostatic interactions. When magnetic interactions are taken into account, however, it may be violated. Since magnetic forces are many orders of magnitude weaker than the corresponding Coulomb forces between charges it seems as if it might at least be a good approximation to neglect this sum. We thus assume that

$$\mathbf{M}^{\mathrm{i}}_{\mathcal{A}} = \mathbf{0}, \tag{1.32}$$

and obtain the *principle of angular momentum* for a system, on the form

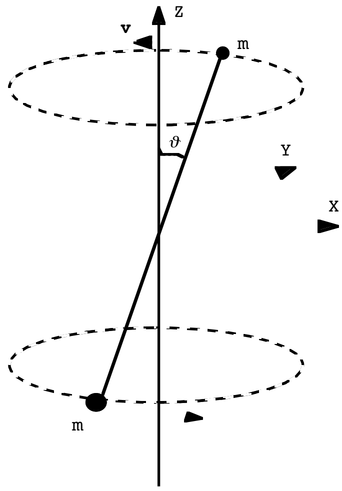


Figure 1.3: This figure shows the skew rotating two particle system discussed in example 1.1.

the principle of angular momentum

$$\dot{\mathbf{L}}_{\mathcal{A}} = \mathbf{M}^{\mathrm{e}}_{\mathcal{A}}. \tag{1.33}$$

That is, the time derivative of the angular momentum vector is equal to the total moment of the external forces. This equation, or something equivalent which amounts to the fact that the internal moments sum to zero, is often postulated in classical mechanics. There are magnetic phenomena in physics where it seems to be violated but this only means that the electro-magnetic field can carry an angular momentum which is not taken into account by classical mechanics. In conclusion, one can safely use equation 1.33, the angular momentum principle, when solving problems in classical mechanics.

Example 1.1 At each end of a light rod of length $2R$ there is a small, heavy ball of mass m. At its midpoint the rod is fixed to a rotating axis with which it makes an angle ϑ. The axis rotates, with fixed direction, with constant angular velocity ω. Calculate the momentum, angular momentum, external force and moment, with the midpoint of the rod as base point, for this two particle system.
Solution: We choose the coordinate system as in figure 1.3, so that the rotation axis is along the Z-axis. For the particle with positive z-coordinate we then have

$$\mathbf{r}_1(t) = R\sin\vartheta(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y) + R\cos\vartheta\,\mathbf{e}_z, \tag{1.34}$$
$$\mathbf{v}_1(t) = R\sin\vartheta\,\omega(-\sin\omega t\,\mathbf{e}_x + \cos\omega t\,\mathbf{e}_y) \tag{1.35}$$

while the other particle has $\mathbf{r}_2 = -\mathbf{r}_1$ and $\mathbf{v}_2 = -\mathbf{v}_1$. This means that the linear momentum is

$$\mathbf{p} = m\mathbf{v}_1 + m\mathbf{v}_2 = m(\mathbf{v}_1 - \mathbf{v}_1) = \mathbf{0}. \tag{1.36}$$

The force is thus also zero: $\mathbf{F}^{\mathrm{e}} = \dot{\mathbf{p}} = \mathbf{0}$. The angular momentum is

$$\mathbf{L}_{\mathcal{O}} = \mathbf{r}_1 \times m\mathbf{v}_1 + \mathbf{r}_2 \times m\mathbf{v}_2 = 2mR^2\sin\vartheta\,\omega[-\cos\vartheta(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y) + \sin\vartheta\,\mathbf{e}_z] \tag{1.37}$$

Note that $L_{\mathcal{O}z} = 2mR^2\sin^2\vartheta\,\omega$ =const. but that $\mathbf{L}_{\mathcal{O}}(t)$ as a whole is not. By taking the time derivative we find

$$\mathbf{M}^{\mathrm{e}}_{\mathcal{O}} = \dot{\mathbf{L}}_{\mathcal{O}} = 2mR^2\omega^2\sin\vartheta\cos\vartheta(\sin\omega t\,\mathbf{e}_x - \cos\omega t\,\mathbf{e}_y) \tag{1.38}$$

The rotation axis must thus act with this moment on the light rod at its midpoint. □

1.1.3 Summary

To sum up this section we have found that the two principles of linear and angular momentum for a particle also hold, in the same form, for systems of particles provided the total force and moment acting on the particle are replaced by the sum of the *external* forces and their moments, respectively:

$$\dot{\mathbf{p}} = \mathbf{F}^{\mathrm{e}}, \tag{1.39}$$
$$\dot{\mathbf{L}}_{\mathcal{A}} = \mathbf{M}^{\mathrm{e}}_{\mathcal{A}}. \tag{1.40}$$

The superscript 'e' on the right hand side quantities appear because we have needed to carefully distinguish between internal and external forces in this section. In what follows we will assume that it has been understood that only the external forces contribute and skip the superscript. The momentum and angular momentum appearing here can be expressed as follows

$$\mathbf{p} = m\mathbf{v}_{\mathcal{G}}, \tag{1.41}$$
$$\mathbf{L}_{\mathcal{A}} = \overline{\mathcal{AG}} \times \mathbf{p} \;+\; \mathbf{L}'_{\mathcal{G}}. \tag{1.42}$$

The centre of mass motion of a particle system or body is thus just like the motion of a particle, but the angular momentum will behave differently because of the presence of the internal part $\mathbf{L}'_{\mathcal{G}}$. We will investigate this further below.

1.2 Moving the Base Point

We first investigate what happens to the angular momentum $\mathbf{L}_{\mathcal{A}}$ of system of particles when the base point is changed from $\mathcal{A}$ to another point $\mathcal{B}$. According to the definition 1.8 we get

$$\mathbf{L}_{\mathcal{B}} = \sum_{j=1}^{N} \mathbf{L}_{\mathcal{B}j} = \sum_{j=1}^{N} \overline{\mathcal{BP}}_j \times m_j \mathbf{v}_j = \tag{1.43}$$
$$= \sum_{j=1}^{N} (\overline{\mathcal{BA}} + \overline{\mathcal{AP}}_j) \times m_j \mathbf{v}_j = \overline{\mathcal{BA}} \times \sum_{j=1}^{N} m_j \mathbf{v}_j \;+\; \sum_{j=1}^{N} \overline{\mathcal{AP}}_j \times m_j \mathbf{v}_j \tag{1.44}$$
$$= \overline{\mathcal{BA}} \times \mathbf{p} + \mathbf{L}_{\mathcal{A}}. \tag{1.45}$$

We have thus derived the connection formula for angular momentum

connection formula for angular momentum

$$\mathbf{L}_{\mathcal{B}} = \mathbf{L}_{\mathcal{A}} + \overline{\mathcal{BA}} \times \mathbf{p}. \tag{1.46}$$

This formula is of the same form and is derived in the same way as the corresponding formula for the moment of a force system: $\mathbf{M}_{\mathcal{B}} = \mathbf{M}_{\mathcal{A}} + \overline{\mathcal{BA}} \times \mathbf{F}$.

1.2.1 The Angular Momentum Principle for a Moving Base Point

When the angular momentum principle in the form of equation 1.33 was derived it was assumed that the base point $\mathcal{A}$ was at rest. We will now investigate how this formula changes if we allow the base point to move.

Consider the connection formula 1.46 and assume that the point $\mathcal{A}$ is fixed but that the point $\mathcal{B}$ is moving, and take the time derivative of both sides of the equation. This gives

$$\frac{\mathrm{d}\mathbf{L}_{\mathcal{B}}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{L}_{\mathcal{A}}}{\mathrm{d}t} + \frac{\mathrm{d}(\overline{\mathcal{BA}})}{\mathrm{d}t} \times \mathbf{p} + \overline{\mathcal{BA}} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{1.47}$$

Since $\frac{\mathrm{d}(\overline{\mathcal{BA}})}{\mathrm{d}t} = -\frac{\mathrm{d}(\overline{\mathcal{AB}})}{\mathrm{d}t} = -\mathbf{v}_\mathcal{B}$ and we can rearrange this into the form

$$\dot{\mathbf{L}}_\mathcal{A} = \dot{\mathbf{L}}_\mathcal{B} + \mathbf{v}_\mathcal{B} \times \mathbf{p} + \overline{\mathcal{AB}} \times \dot{\mathbf{p}}. \tag{1.48}$$

We now put this expression into $\dot{\mathbf{L}}_\mathcal{A} = \mathbf{M}_\mathcal{A}$ and use the fact that $\dot{\mathbf{p}} = \mathbf{F}$:

$$\dot{\mathbf{L}}_\mathcal{B} + \mathbf{v}_\mathcal{B} \times \mathbf{p} + \overline{\mathcal{AB}} \times \mathbf{F} = \mathbf{M}_\mathcal{A}. \tag{1.49}$$

angular momentum principle for moving base point

When the last term on the left hand side is moved to the right and the connection formula for moments is taken into account we end up with the result

$$\dot{\mathbf{L}}_\mathcal{B} + \mathbf{v}_\mathcal{B} \times \mathbf{p} = \mathbf{M}_\mathcal{B}. \tag{1.50}$$

The angular momentum principle must thus have the additional term $\mathbf{v}_\mathcal{B} \times \mathbf{p}$ on the left hand side when the base point $\mathcal{B}$ moves with velocity $\mathbf{v}_\mathcal{B}$. When this velocity is zero this formula correctly reduces to the old result.

The use of a moving base point is particularly convenient when the extra term vanishes. There are two (non-trivial) cases when this happens. Firstly is clearly zero if $\mathbf{p} = m\mathbf{v}_\mathcal{G} = \mathbf{0}$, i.e. if the centre of mass of the body is at rest. Secondly it is zero if the the two vectors of the vector product are parallel, that is, if $\mathbf{v}_\mathcal{G} \parallel \mathbf{v}_\mathcal{B}$. This happens if the base point moves so that its velocity is parallel to that of the centre of mass.

1.2.2 Centre of Mass as Base Point

angular momentum principle for centre of mass as base point

If we put $\mathcal{B} = \mathcal{G}$ the above formula 1.50, use of $\mathbf{p} = m\mathbf{v}_\mathcal{G}$, gives the result

$$\dot{\mathbf{L}}_\mathcal{G} = \mathbf{M}_\mathcal{G}. \tag{1.51}$$

When the centre of mass is used as base point no extra terms arise independently of how $\mathcal{G}$ moves. If we recall the result of equation 1.15 we also have

$$\dot{\mathbf{L}}'_\mathcal{G} = \mathbf{M}_\mathcal{G} \tag{1.52}$$

which means that one may calculate the angular momentum using quantities in the centre of mass system.

With the help of the connection formula for moments, $\mathbf{M}_\mathcal{B} = \mathbf{M}_\mathcal{A} + \overline{\mathcal{BA}} \times \mathbf{F}$, and $\dot{\mathbf{p}} = \mathbf{F}$, one can rewrite the above formulae so that one gets, for example,

$$\dot{\mathbf{L}}'_\mathcal{G} + \overline{\mathcal{AG}} \times \dot{\mathbf{p}} = \mathbf{M}_\mathcal{A}. \tag{1.53}$$

This form is valid independently of the motion of the point $\mathcal{A}$.

1.3 Time Integrals and Conservation laws

If we time integrate the principle of momentum $\dot{\mathbf{p}} = \mathbf{F}$ from $t = t_1$ to $t = t_2$ we find

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t. \tag{1.54}$$

The time integral of the external force is a vector quantity, the *impulse* of the force and we write

$$\mathbf{I} \equiv \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t. \tag{1.55}$$

If a force acts continuously the impulse will depend on the two integration limits. Mostly, however, one speaks of the impulse of a force that is non-zero only for some finite time. The quantity $\mathbf{I}$ is then independent of the integration interval as long it contains the time-interval when the force is non-zero. The integrated form of the momentum principle can also be written

$$\Delta\mathbf{p} = \mathbf{I} \tag{1.56}$$

which says that the change in momentum for a system, or a body, is given by the impulse of the (external) force on the system. If there is no external force there will consequently be no change in the momentum, i.e.

$$\mathbf{p} = \text{const.} \qquad \text{when} \;\; \mathbf{F} = \mathbf{0}. \tag{1.57}$$

This is sometimes called the conservation law for the momentum.

Results and definitions analogous to those above can also be found for the angular momentum principle $\dot{\mathbf{L}}_{\mathcal{A}} = \mathbf{M}_{\mathcal{A}}$. The time integrated form is

$$\mathbf{L}_{\mathcal{A}}(t_2) - \mathbf{L}_{\mathcal{A}}(t_1) = \int_{t_1}^{t_2} \mathbf{M}_{\mathcal{A}}\,\mathrm{d}t \tag{1.58}$$

so if one defines the *angular impulse* as follows

$$\mathbf{H}_{\mathcal{A}} \equiv \int_{t_1}^{t_2} \mathbf{M}_{\mathcal{A}}\,\mathrm{d}t \tag{1.59}$$

one finds that the change in the angular momentum vector $\Delta\mathbf{L}_{\mathcal{A}}$ over some time interval, is given by the angular impulse delivered by the external moment during this time interval

$$\Delta\mathbf{L}_{\mathcal{A}} = \mathbf{H}_{\mathcal{A}}. \tag{1.60}$$

Should the (external) moment on the system be zero the angular momentum will be constant and we have the conservation law for angular momentum:

$$\mathbf{L}_{\mathcal{A}} = \text{const.} \qquad \text{when} \;\; \mathbf{M}_{\mathcal{A}} = \mathbf{0}. \tag{1.61}$$

The conservation laws for momentum and angular momentum are often useful in problem solving. Note that these equations are vector equations and that they thus really correspond to three real equations each. With suitable choice of basis vector directions one can often find that the conservation laws apply, if not in full, at least for one or more of the component equations.

The ideas of impulse and angular impulse are of particular importance when large forces act during short times, i.e. when one has *impact*. We will return to this later.

1.4 Continuous Mass Distributions

Even if all kinds of matter, in principle, may be thought of as systems of particles it is often more convenient to to describe a body with the help of the continuous scalar field, the *mass density* $\varrho_m(\mathbf{r})$. This field has the property that the amount of mass, in the small volume element $\mathrm{d}V$ at $\mathbf{r}$, is $\mathrm{d}m = \varrho_m\,\mathrm{d}V$. With the help of this one finds the expression

$$\mathbf{r}_{\mathcal{G}} = \frac{\int_\Omega \mathbf{r}\varrho_m(\mathbf{r})\,\mathrm{d}V}{\int_\Omega \varrho_m(\mathbf{r})\,\mathrm{d}V} = \frac{1}{m}\int_\Omega \mathbf{r}\varrho_m(\mathbf{r})\,\mathrm{d}V \tag{1.62}$$

for the centre of mass of the body.

If the body moves there one also has a velocity field $\mathbf{v}(\mathbf{r})$ which gives the velocity of the element of mass $\mathrm{d}m(\mathbf{r}) = \varrho_m\,\mathrm{d}V$ at the point with position vector $\mathbf{r}$. The momentum of a body can now be written

$$\mathbf{p} = \int_\Omega \mathbf{v}(\mathbf{r})\,\mathrm{d}m = \int_\Omega \mathbf{v}(\mathbf{r})\varrho_m(\mathbf{r})\,\mathrm{d}V \tag{1.63}$$

and the angular momentum

$$\mathbf{L}_{\mathcal{A}} = \int_{\Omega} [(\mathbf{r} - \mathbf{r}_{\mathcal{A}}) \times \mathbf{v}(\mathbf{r})] \, \mathrm{d}m. \tag{1.64}$$

Many of the derivations that we have made above become more intricate when we must consider integrals over the bodies but these difficulties are mainly of a technical nature and it is often taken as a postulate that the same laws hold for for continuously distributed matter as for systems of particles. We shall adhere to this view in this text. We will, however, mostly present the general definitions and derivations assuming the particle system description and pass to the integrals over continuous distributions only when it is convenient.

1.5 Projection on a Fixed Direction

All vector equations derived so far will give rise to scalar component equations if we take the scalar product of the equations with a unit vector in the desired direction. This can be done for any direction, or unit vector $\mathbf{e}$, but we will, as a matter of convention, choose this direction as the z-direction. The angular momentum principle thus gives

$$\dot{\mathbf{L}}_{\mathcal{A}} \cdot \mathbf{e}_z = \mathbf{M}_{\mathcal{A}} \cdot \mathbf{e}_z \implies \dot{L}_{\mathcal{A}z} = M_{\mathcal{A}z} \tag{1.65}$$

Recall that if the point $\mathcal{A}$ is on the Z-axis then $M_{\mathcal{A}z} = M_z$ is the moment with respect to the Z-axis.

Let us now calculate the z-component of the angular momentum. To simplify the notation we put the origin at the base point $\mathcal{A} = \mathcal{O}$. We now have

$$L_{\mathcal{O}z} = \sum_{j=1}^{N} (\overline{\mathcal{O}\mathcal{P}}_j \times m_j \mathbf{v}_j) \cdot \mathbf{e}_z \tag{1.66}$$

$$= \sum_{j=1}^{N} m_j (\mathbf{r}_j \times \mathbf{v}_j) \cdot \mathbf{e}_z = \sum_{j=1}^{N} m_j (x_j \dot{y}_j - y_j \dot{x}_j). \tag{1.67}$$

We now introduce cylindrical (polar) coordinates ρ, φ, and z. In terms of these we have for the position vectors $\mathbf{r}_j = \rho_j \, \mathbf{e}_\rho + z_j \, \mathbf{e}_z$ and for the velocities $\dot{\mathbf{r}}_j = \dot{\rho}_j \, \mathbf{e}_\rho + \rho_j \dot{\varphi}_j \, \mathbf{e}_\varphi + \dot{z}_j \, \mathbf{e}_z$, see figure 1.4. This gives us

$$(\mathbf{r}_j \times \mathbf{v}_j) \cdot \mathbf{e}_z = [(\rho_j \, \mathbf{e}_\rho + z_j \, \mathbf{e}_z) \times (\dot{\rho}_j \, \mathbf{e}_\rho + \rho_j \dot{\varphi}_j \, \mathbf{e}_\varphi + \dot{z}_j \, \mathbf{e}_z)] \cdot \mathbf{e}_z = \rho_j^2 \dot{\varphi}_j, \tag{1.68}$$

so that

$$L_{\mathcal{O}z}(t) = \sum_{j=1}^{N} m_j \rho_j^2(t) \dot{\varphi}_j(t). \tag{1.69}$$

If we now define the average angular velocity of the body (with respect to the Z-axis) by

$$\omega_{\mathrm{av}}(t) \equiv \frac{\sum_{j=1}^{N} m_j \rho_j^2(t) \dot{\varphi}_j(t)}{\sum_{j=1}^{N} m_j \rho_j^2(t)} \tag{1.70}$$

and the *moment of inertia*, J_z, of the body with respect to the Z-axis by

$$J_z \equiv \sum_{j=1}^{N} m_j \rho_j^2 \tag{1.71}$$

we see that we get

$$L_{\mathcal{O}z} = J_z \omega_{\mathrm{av}}. \tag{1.72}$$

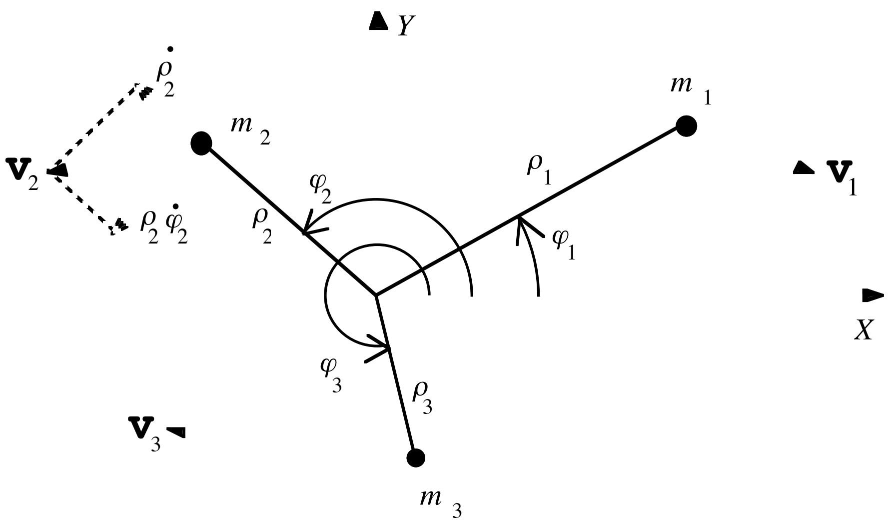


Figure 1.4: A system of three particles viewed along the Z-axis, which points up from the paper. The cylindrical coordinates ρ_i, φ_i $(i = 1, 2, 3)$ of the particles are shown as well as their velocity vectors $\mathbf{v}_i$. For particle number 2 the components of the velocity along the radial ($\mathbf{e}_\rho$) direction and the $\mathbf{e}_\varphi$-direction are indicated.

We can now apply these results to the z-component of the angular momentum principle, equation 1.65, in the form

$$\boxed{\dot{L}_{\mathcal{O}z} = M_{\mathcal{O}z}.} \tag{1.73}$$

We find that

$$\dot{J}_z \omega_{\text{av}} + J_z \dot{\omega}_{\text{av}} = M_{\mathcal{O}z}, \tag{1.74}$$

and this is thus a general expression for the z-component of the angular momentum principle.

If all particles have the same angular velocity $\dot{\varphi}$ then, of course, $\omega_{\text{av}} = \dot{\varphi}$. The most important case for which this happens is when the system is *rigid* and rotates around the Z-axis. In general, however, it will happen whenever the velocities of the particles are such that the $\dot{\varphi}_i$ all are equal while $\dot{\rho}_i$ and $\dot{z}_i$ are arbitrary. For these cases, when the angular velocity is well defined, one can simply write

$$L_{\mathcal{O}z} = J_z \dot{\varphi}. \tag{1.75}$$

If we furthermore assume that all ρ_i are constant, which they will be if the body is rigid and rotates around the Z-axis, then J_z is constant. In this case equation 1.74 reduces to

$$J_z \ddot{\varphi} = M_{\mathcal{O}z}, \tag{1.76}$$

i.e. the angular accelaration is simply proportional to the moment.

If there is no external moment with respect to the Z-axis through $\mathcal{O}$ so that $M_{\mathcal{O}z} = 0$ then the z-component of the angular momentum vector will be conserved, $L_{\mathcal{O}z}$ =const., and this thus implies that

$$J_z \omega_{\text{av}} = \text{const.} \tag{1.77}$$

This equation tells us that a large moment of inertia J_z implies small (average) angular velocity and vice versa, a fact used by springboard divers and figure skaters, see figure 1.5.

Example 1.2 A person walks on a horizontal platform that can rotate freely around a vertical Z-axis. The empty platform has the moment of inertia J_z. The person $\mathcal{P}$, which can be treated

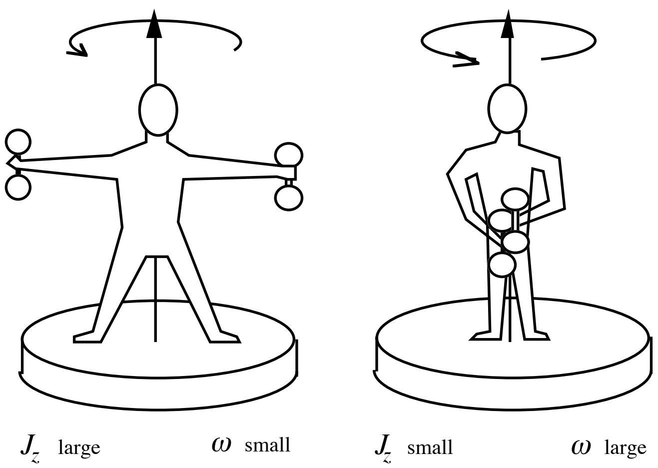


Figure 1.5: This picture shows a person on a platform that can rotate with negligible friction around a fixed vertical axis. Then $J_z(t)\omega_{\text{av}}(t)$ = constant and this means that the angular velocity is larger when the weights in the hands are held close to the body and the body is close to the rotation axis, and vice versa.

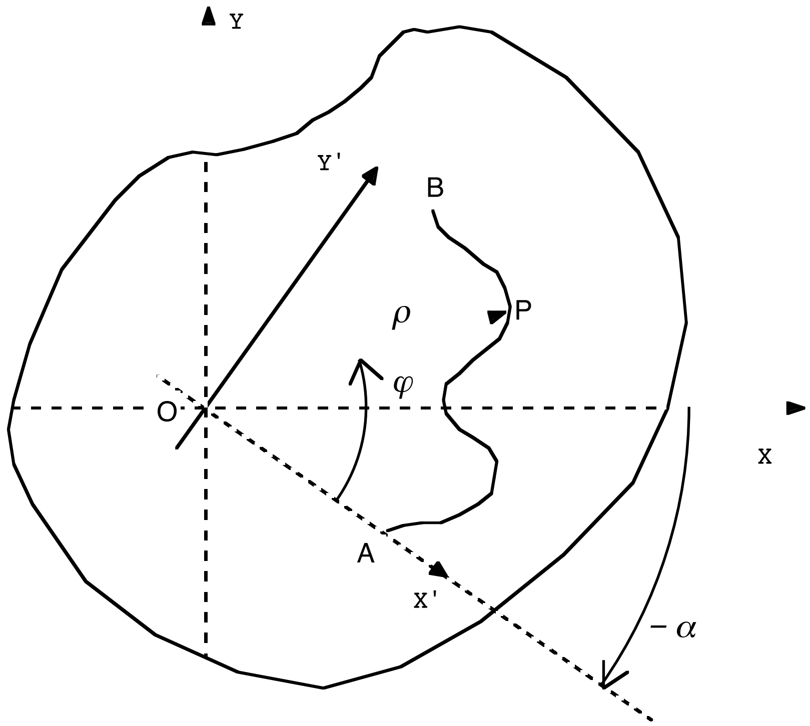


Figure 1.6: This figure shows the platform discussed in example 1.2 as seen from above. It also shows the path that the person walking on the platform has taken between points $\mathcal{A}$ and $\mathcal{B}$ of the platform and the cylindrical coordinates (ρ, φ) of a point $\mathcal{P}$ on this path relative to a coordinate system fixed to the platform. The angle $-\alpha$ is the angle that the platform has rotated relative to a fixed direction (the X-axis) when the person has walked from the initial point $\mathcal{A}$ to $\mathcal{P}$.

as a particle, has mass m. The position of the person relative to a coordinate system $\mathcal{O}X'Y'Z'$ fixed to the platform is measured in terms of cylindrical coordinates ρ and φ, see figure 1.6.
a) Find a relationship between the small change in angle $\mathrm{d}\varphi$ of the person and the corresponding absolute rotation angle $\mathrm{d}\alpha$ of the platform relative to a fixed direction (the X-axis).
b) Use this relationship to calculate the rotation angle $\Delta\alpha$ of the platform when the person walks along a given path $(\rho(t), \varphi(t))$ on the platform from points $\mathcal{A}$, with $\varphi = 0$, to $\mathcal{B}$, with $\varphi = \Delta\varphi$, and show that this angle depends in general on the path.

Solution:

a) There is no external moment with respect to the Z-axis on the system of particles defined by the platform and the person on it. Therefore L_z = constant. There are two contributions to L_z, that of the platform, which is $J_z\dot{\alpha}$, and that of the person which is given by $m\rho^2(\dot{\alpha}+\dot{\varphi})$. Note that the angular velocity of the person with respect to the fixed system is equal to the angular velocity of the platform, $\dot{\alpha}$, plus the angular velocity of the person with respect to the platform, which is $\dot{\varphi}$. We thus have

$$L_z = J_z\dot{\alpha} + m\rho^2(\dot{\alpha} + \dot{\varphi}). \tag{1.78}$$

If we have initial conditions such that $\dot{\alpha}(0) = \dot{\varphi}(0) = 0$ we get $L_z = 0$ and the above equation can be written

$$0 = J_z\frac{\mathrm{d}\alpha}{\mathrm{d}t} + m\rho^2(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\mathrm{d}\varphi}{\mathrm{d}t}). \tag{1.79}$$

This gives us the relation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{m\rho^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}}{J_z + m\rho^2} \tag{1.80}$$

between the two angular velocities.If we multiply by $\mathrm{d}t$ we get the differential relation

$$\mathrm{d}\alpha = -\frac{m\rho^2\mathrm{d}\varphi}{J_z + m\rho^2} \tag{1.81}$$

for small angles.

b) If we integrate the relation 1.80 between the time $t = 0$, when the person is at $\mathcal{A}$, and the time $t = T$, when the person is at $\mathcal{B}$, we get

$$\Delta\alpha = \int_0^T \dot{\alpha}\,\mathrm{d}t = -\int_0^T \frac{m\rho^2(t)\dot{\varphi}(t)}{J_z + m\rho^2(t)}\mathrm{d}t = -\int_0^{\Delta\varphi} \frac{m\rho^2(\varphi)\mathrm{d}\varphi}{J_z + m\rho^2(\varphi)}. \tag{1.82}$$

This integral will obviously depend on e.g. the function $\rho(\varphi)$. If the radius ρ is kept constant, $\rho(\varphi) = \rho_0$, during the walk, the integral can be evaluated and one finds the relation

$$\Delta\alpha = -\frac{m\rho_0^2}{J_z + m\rho_0^2}\Delta\varphi \tag{1.83}$$

between the two angles. Note that the angle of rotation of the platform is of opposite sign to that of $\Delta\varphi$ as indicated in figure 1.6. □

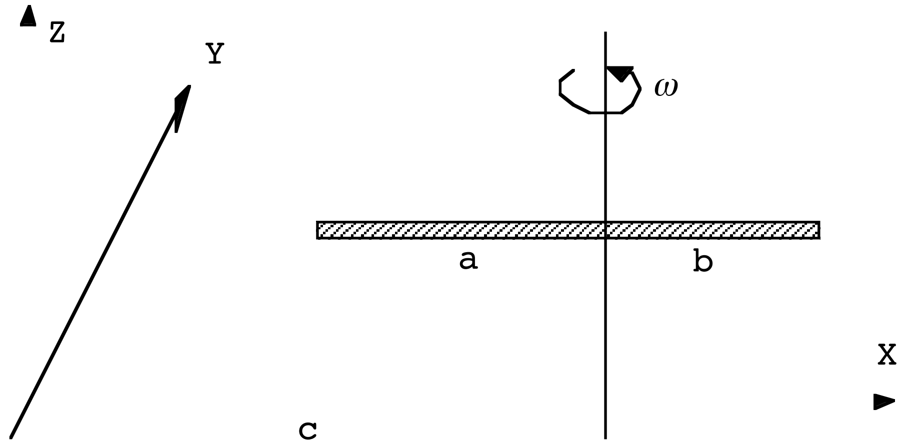


Figure 1.7: This picture refers to problem 1.3 and shows the rod, its rotation axis, and the placement in the coordinate system.

1.6 Problems

Problem 1.1 A particle $\mathcal{P}$ of mass m has constant speed v in a circular trajectory of radius r in the xy-plane with centre at the origin of the coordinate system.
a) Calculate the angular momentum $\mathbf{L}_{\mathcal{A}}$ with respect to a point $\mathcal{A}$ with Cartesian coordinates (a, b, c).
b) Then calculate the time derivative of this quantity to get $\mathbf{M}_{\mathcal{A}}$ and investigate whether the point $\mathcal{A}$ can be chosen so that $\mathbf{M}_{\mathcal{A}} = \mathbf{0}$.
c) Use the result of b) to find the direction of the force acting on the particle.

Problem 1.2 A three particle system consists of particles with masses $3m$, m, and $5m$. Their position vectors have Cartesian components $(t, -2, 3t^2)$, $(t-1, t^3, 5)$, and $(2-t^2, t, t^3)$ respectively. Calculate as functions of time
a) the total force acting on the system,
b) the total moment with respect to the origin.

Problem 1.3 A thin straight homogeneous rod of length $a + b$ rotates in a plane parallel to the xy-plane around an axis which is parallel to the Z-axis and which lies in the xz-plane. The axis goes through the rod at a point which is at the distance a from one endpoint and the distance b from the other. The axis is at the distance c from the Z-axis along the positive X-axis, see figure 1.7. Determine the distance c so that the angular momentum of the rod with respect to the Z-axis is zero when the rod is in the position shown in the figure i.e. when the rod is parallel to the X-axis and the end of length a points towards the Z-axis. Use the following two methods to get L_z:
a) Integration along the rod.
b) Use of the connection formula 1.46.

Problem 1.4 In order to measure the moment of inertia J_L of the complicated rotor of an electric motor one mounts the rotor on bearings of negligible friction so that it can rotate freely around a horizontal axis L. A thin flexible string is then wound around the axis of the rotor. This axis has radius r. A mass M is then hung in the string and it is found that, when starting from rest, a length x of the string becomes unwound in time T. What is J_L? (See figure 1.13 in the hints and answer section if necessary.)

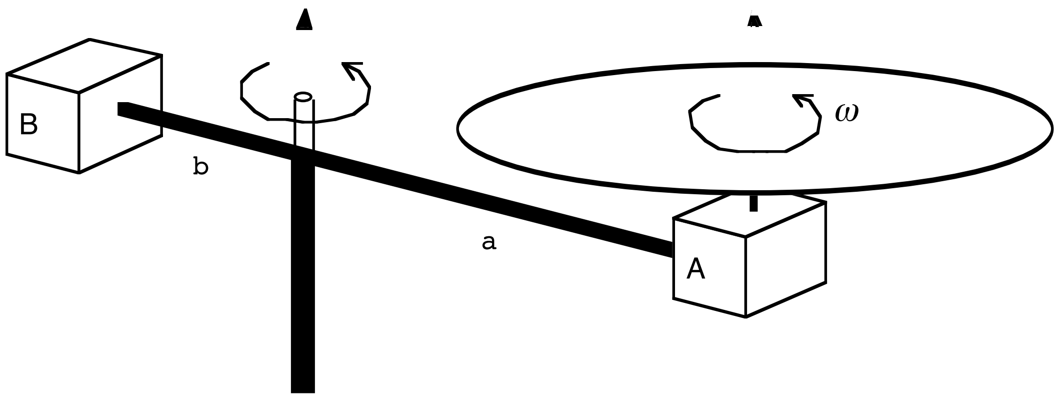


Figure 1.8: This picture refers to problem 1.6 and shows the disc, the lever arm $\mathcal{AB}$, the counterweight at $\mathcal{B}$. It also indicates the vertical rotation axis of the disk and the fixed vertical rotation axis of the horizontal lever arm.

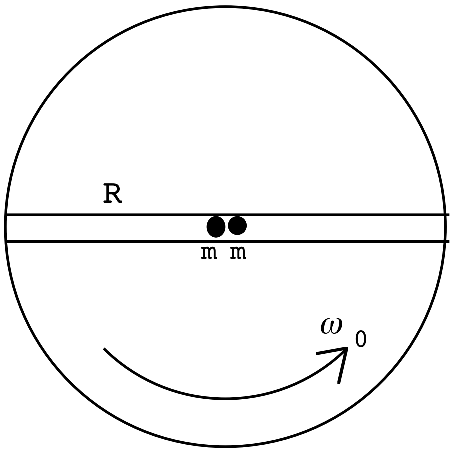
Problem 1.5 A person stands at rest on a horizontal platform that can rotate freely around a fixed vertical axis. By turning the upper body relative to the lower the person tries to look backwards. When the system is in this position the upper body has been turned an angle α while the lower part of the body together with the platform has turned an angle β, both with respect to the same fixed direction. The moment of inertia of the upper body is J_1 while that of the lower body plus platform is J_2. Calculate α as a function of the relative 'twist' angle $\alpha - \beta$.

Problem 1.6 A homogeneous circular disc of mass M and radius R is being rotated by a small electric motor at $\mathcal{A}$, around a vertical axis, see figure 1.8. The electric motor has mass m and is mounted at one end of a horizontal lever arm $\mathcal{AB}$. At the other end $\mathcal{B}$ of the lever arm, which can rotate freely around a fixed vertical axis at the distance a from $\mathcal{A}$, there is a counterweight. The distance of the counterweight from the fixed axis is b and it balances the mass at the other end of the arm. The power supply of the motor is suddenly cut off and the disc starts to slow down due induction effects in the electric motor. Assume that the angular velocity of the disc initially was ω while the lever arm was at rest. Calculate the final angular velocity of the disc.

Problem 1.7 A homogeneous solid sphere of mass M and radius R is rotating freely with angular velocity ω_0 around a vertical axis through its centre of mass. Along a horizontal diameter a smooth narrow channel has been drilled through the sphere. Two small balls each of mass m are initially at rest in the middle of the channel, see figure 1.9. A small charge between the balls suddenly explodes and gives them initial speeds v in opposite directions. When the balls leave the sphere the angular velocity has changed. Calculate the change in angular velocity and explain how it came about.

Problem 1.8 A horizontal homogeneous circular platform of mass m and radius R if is mounted on a bearing so that it can rotate around a vertical axis with negligible friction. A person of mass M walks on the platform along the path shown in figure 1.10. First 90° in a positive sense in a circular path at radius R, then radially inwards from R to $R/2$, then 90° in a negative sense along a circular arc of radius $R/2$ and finally back to the starting point along a radial path from $R/2$ to R. The 90° angles are measured with respect to directions fixed on the platform. Use the result of example 1.2 to calculate the net absolute rotation of the disc after completion of the closed path.

Problem 1.9 A straight homogeneous bar $\mathcal{AB}$ of length ℓ and mass m is suspended horizontally in two vertical strings from each of its two endpoints $\mathcal{A}$ and $\mathcal{B}$. A match



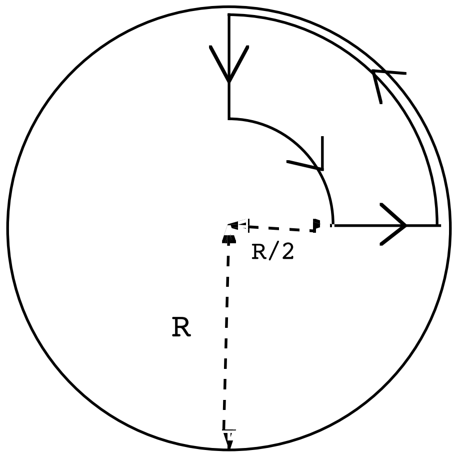
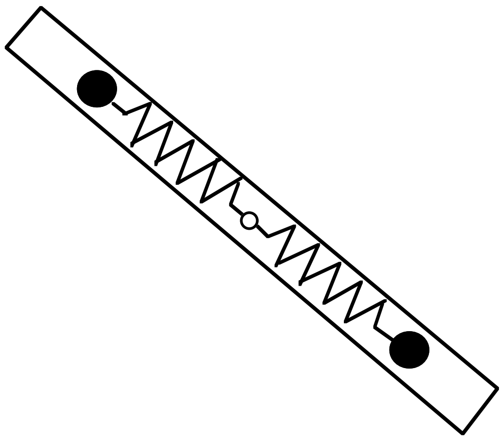


Figure 1.9: The figure on the left refers to problem 1.7. Two small balls are initially in a drilled channel in the middle of the rotating sphere.

Figure 1.10: The figure on the right refers to problem 1.8 and shows, from above, the closed path followed by the person on the freely rotating platform.



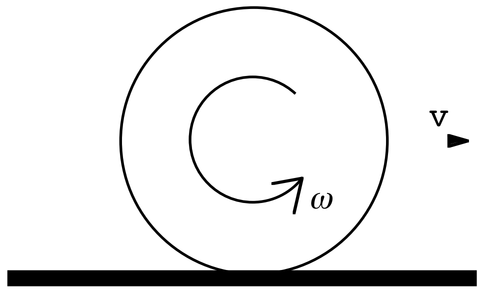


Figure 1.11: The figure on the left refers to problem 1.10. Two particles are attached to two springs symmetrically placed in a smooth pipe which can move on a smooth horizontal plane.

Figure 1.12: The figure on the right refers to problem 1.11. The cylinder starts with the velocity and the angular velocity shown in the figure. It moves on a rough horizontal plane.

is held to the string at $\mathcal{B}$ so that it burns off. Calculate the ratio of the tensions in the other string before and just after the burning.

Problem 1.10 A homogeneous straight smooth pipe of mass M and length a can move on a smooth horizontal plane. In the middle of the pipe the ends of two identical springs, each of unloaded length L and stiffness k, are attached. At the other ends of the springs there are particles of mass m attached, see figure 1.11. At time $t = 0$ the two particles are pulled out a distance b from their symmetric equilibrium positions inside the pipe and are then released from rest. The pipe is at the same time given an angular velocity ω_0. Describe the qualitative motion of the pipe.

Problem 1.11 A homogeneous circular cylinder of mass m and radius r is given a speed v and an angular velocity ω on a rough horizontal floor with coefficient of (kinetic) friction f. Determine the value of ω so that the cylinder returns to the starting point after a tour along the floor. See figure 1.12.

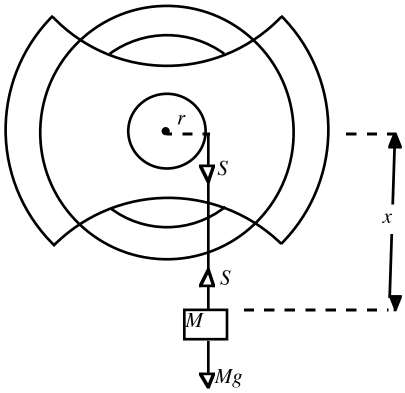


Figure 1.13: This picture refers to answer 1.4 and is a view of the rotor along its axis (of radius r). It also shows the string, wound round the axis, in which the mass M hangs down. The forces acting on M are indicated.

1.7 Hints and Answers

Answer 1.1
a) $\mathbf{L}_\mathcal{A} = mv\left(c\cos(\frac{v}{r}t), c\sin(\frac{v}{r}t), r - a\cos(\frac{v}{r}t) - b\sin(\frac{v}{r}t)\right)$
b) $\dot{\mathbf{L}}_\mathcal{A} = \mathbf{M}_\mathcal{A} = \mathbf{0}$ only for $a = b = c = 0$.
c) $\mathbf{M}_\mathcal{O} = \mathbf{0}$ requires that $\mathbf{F}$ is parallel to $\overline{\mathcal{OP}}$.

Answer 1.2
a) $\mathbf{F} = \sum m_k \ddot{\mathbf{r}}_k = 2m(-5, 3t, 9 + 15t)$,
b) $\mathbf{M}_\mathcal{O} = \sum \mathbf{r}_k \times m_k \ddot{\mathbf{r}}_k = 2m(-18 - 15t + 15t^2, -39t + 10t^3, 2t + 3t^2)$.

Answer 1.3
a) Elements of the rod of length dx have mass $\mathrm{d}m = m\,\mathrm{d}x/(a+b)$ and the velocities of these elements are

$$v(x) = \dot{y}(x) = (x - c)\omega$$

so that they are negative for $x < c$ and positive otherwise. Now

$$\mathrm{d}L_z = (\mathbf{r} \times \mathrm{d}m\,\dot{\mathbf{r}}) \cdot \mathbf{e}_z = x\,\mathrm{d}m\,\dot{y}(x)$$

. Integration thus gives

$$L_z = \int_{c-a}^{c+b} \mathrm{d}L_z(x) = \int_{c-a}^{c+b} x\frac{m\,\mathrm{d}x}{a+b}(x-c)\omega = \frac{m\omega}{a+b}\left[c\frac{1}{2}(b^2 - a^2) + \frac{1}{3}(a^3 + b^3)\right].$$

From this one finds that $L_z = 0$ when

$$c = \frac{2}{3}\frac{a^2 + b^2 - ab}{a - b}$$

which thus is the answer. One notes that when $a \to b$ it is not possible to make $L_z = 0$ except in the limit $c \to \infty$.

Answer 1.4
The equations of motion can be written

$$J_L\ddot{\varphi} = rS,$$

$$M\ddot{x} = Mg - S.$$

See figure 1.13 for notation. The length of unwound string x must obey

$$x = r\varphi \;\Rightarrow\; \ddot{x} = r\ddot{\varphi}.$$

This system of equations is easily solved and then integrated with respect to time. Finally solving for J_L gives

$$J_L = \frac{1}{2}\frac{Mgr^2}{x}T^2 - Mr^2$$

for the moment of inertia of the rotor.

Answer 1.5
The answer is $\alpha = J_2(\alpha - \beta)/(J_1 + J_2)$. Note the limits $J_2 \to 0$ and $J_2 \to \infty$.

Answer 1.6
Use the fact that L_z is conserved since there is no external moment with respect to the fixed vertical axis through the lever arm. Also note that the disc will slow down until it has the same angular velocity as the motor and lever arm. The mass m' of the counterweight is calculated from the relation $b\,m' = a(M{+}m)$. The relation $J_0\omega = J_1\omega'$ with $J_0 = \frac{1}{2}MR^2$ and $J_1 = (J_0 + Ma^2) + ma^2 + m'b^2$ then gives

$$\omega' = \frac{\omega}{1 + 2\frac{(M+m)}{M}\frac{a(a+b)}{R^2}}$$

and this is the answer.

Answer 1.7
Use the fact that L_z is conserved for the system as a whole. The two balls contribute the amount $2mr[r\omega(r)]$ to L_z when at radius r. Here $\omega(r)$ is the angular velocity of the sphere when the ball are at radius r. The initial angular velocity is thus, with this notation, $\omega_0 = \omega(0)$. The moment of inertia of the sphere itself is $J_{\rm s} = \frac{2}{5}MR^2$, the balls contribute the amount $J_{\rm b} = 2mr^2$, when at radius r. The final angular velocity is thus $\omega(R) = \omega(0)M/(M + 5m)$ and the this gives

$$\Delta\omega = \omega(0) - \omega(R) = \omega_0\frac{5m}{M + 5m}$$

for the decrease.

Answer 1.8
The angle is given by

$$\Delta\alpha = -\int_0^{\pi/2}\frac{MR^2\,\mathrm{d}\varphi}{\frac{1}{2}mR^2 + MR^2} - \int_{\pi/2}^{0}\frac{M(R/2)^2\,\mathrm{d}\varphi}{\frac{1}{2}mR^2 + M(R/2)^2} = -\frac{\pi}{2}\frac{3\frac{m}{M}}{(2+\frac{m}{M})(1+2\frac{m}{M})}.$$

Note that it goes to zero when $m/M \to \infty$ i.e. when the platform becomes heavy. It also becomes zero when the platform has negligible mass ($m = 0$). If the numerator and denominator of the formula are both multiplied by M^2/m^2 the simplified result is the same formula with m/M replaced by M/m.

Answer 1.9
Before the burning the tension in each string must be $S' = mg/2$. Immediately after the burning the bar is still horizontal. With a vertical Y-axis and with φ the angle that the bar makes with the horizontal the principles of linear and angular momentum give the equations of motion:

$$\begin{aligned} m\ddot{y}_{\mathcal{G}} &= S - mg, \\ J_{\mathcal{G}}\ddot{\varphi} &= S\ell/2, \end{aligned}$$

where S is the tension in the string at $\mathcal{A}$ and $J_{\mathcal{G}} = \frac{1}{12}m\ell^2$ is the moment of inertia of the bar for an axis perpendicular to the bar through it centre of mass (mid point). We also have that for small φ (immediately after the burning)

$$y_{\mathcal{G}} = -\frac{\ell}{2}\sin\varphi \;\Rightarrow\; \dot{y}_{\mathcal{G}} = -\frac{\ell}{2}\dot{\varphi}\cos\varphi \;\Rightarrow\; \ddot{y}_{\mathcal{G}} = \frac{\ell}{2}(\dot{\varphi}^2\sin\varphi - \ddot{\varphi}\cos\varphi)$$

The initial conditions $\varphi(0) = \dot{\varphi}(0) = 0$ thus give

$$\ddot{y}_{\mathcal{G}} = -\frac{\ell}{2}\ddot{\varphi}.$$

This equation together with the two equations of motion give us a system of three equations for three unknowns $(S, \ddot{y}_{\mathcal{G}}, \ddot{\varphi})$. Solution of this system of equations gives us the result $S = mg/4$. The ratio of the two tensions is thus

$$S'/S = (mg/2)/(mg/4) = 2$$

and this is the answer.

Answer 1.10

Since there are no external forces on the pipe its centre of mass will move with constant velocity; if this velocity is initially zero the centre of mass (the middle of the pipe) will remain at rest. The two particles will oscillate radially in the pipe and this means that the moment of inertia of the pipe will vary and therefore the angular velocity of the pipe will also vary.

Answer 1.11

The velocity of the centre of mass of the cylinder must have changed direction when rolling without slipping occurs. This gives the condition $\omega > 2v/r$.

Chapter 2

Kinematics of Rigid Bodies

This chapter presents rigid body kinematics. This means that methods for describing the position, the orientation, and the velocity state of the rigid body are presented.

2.1 Position and Orientation of a Rigid Body

A rigid body is a piece of matter for which it is possible to find a reference frame with respect to which the mass distribution of the body is constant and all mass of the body at rest. This reference frame will be called the *body fixed* reference frame. If one considers the mass distribution to be a system of particles, $\{\{m_k, \mathbf{r}_k\};\ k = 1, \ldots, N\}$, the *rigid body* is characterized by the fact that the distance between any pair of particles is fixed and constant:

$$|\mathbf{r}_i - \mathbf{r}_j| = c_{ij} = \text{constants for all } i, j = 1, \ldots, N. \tag{2.1}$$

There are two types of *displacement* of a body, or system of particles, which conserve the the distances within the body: *translations* and *rotations* (see figure 2.1).

A translation is characterized by a translation vector, $\mathbf{d}$, and all points of a body move in the same way under a translation

$$\mathbf{r}'_i = \mathbf{r}_i + \mathbf{d} \text{ for all } i = 1, \ldots, N. \tag{2.2}$$

Here $\mathbf{r}'_i$ are the position vectors of the new positions of the particles that where at $\mathbf{r}_i$ before the translation.

A rotation can be defined to be a rigid displacement of space which leaves one point fixed. We will study rotations in more detail below. It turns out that a rotation in three-dimensional space leaves not just one point fixed but a line of points which define an axis of rotation. A rotation can thus always be thought of as a turn some angle ϕ around some axis parallel to $\mathbf{e}_a$ through some point $\mathcal{A}$ of space.

In order to specify completely the position of the particles of a given rigid body, A, with respect to a reference position in the observer reference frame O, we must, first of all, know the position of some given point $\mathcal{A}$ of the body or rigidly connected to the body. To give the position of the point $\mathcal{A}$ all we need is its position vector,

$$\overline{\mathcal{O}\mathcal{A}} = \mathbf{r}_{\mathcal{A}} = x_{\mathcal{A}}\,\mathbf{e}^{\mathrm{O}}_1 + y_{\mathcal{A}}\,\mathbf{e}^{\mathrm{O}}_2 + z_{\mathcal{A}}\,\mathbf{e}^{\mathrm{O}}_3, \tag{2.3}$$

from the origin $\mathcal{O}$ of the observer fixed coordinates system with components $x_{\mathcal{A}}, y_{\mathcal{A}}, z_{\mathcal{A}}$ in the observer fixed basis $\mathbf{e}^{\mathrm{O}}_1, \mathbf{e}^{\mathrm{O}}_2, \mathbf{e}^{\mathrm{O}}_3$. This thus requires three coordinates. Secondly we must know how the body has rotated around an axis through $\mathcal{A}$. This requires the knowledge of the direction of the rotation axis $\mathbf{e}_a$, i.e. two parameters, plus the angle ϕ of rotation around the axis, one more parameter. Sometimes it is convenient to formally combine these three parameters into the 'rotation vector'

$$\boldsymbol{\phi} = \phi\,\mathbf{e}_a, \tag{2.4}$$

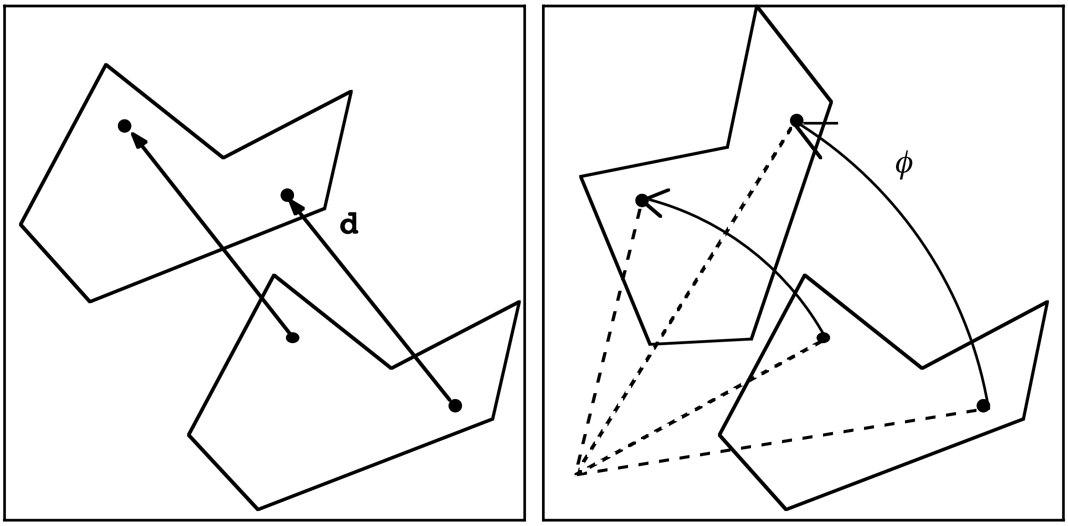


Figure 2.1: This figure illustrates translation of a rigid body (on the left). A translation is a displacement of the body such that all points of the body undergo the same translation. On the right a rotation is illustrated. A rotation of a rigid body is a displacement of the body such that one point, of the body or rigidly connected to the body, remains fixed.

but one must keep in mind that finite rotations do *not* add vectorially. The symbol $\boldsymbol{\phi}$ is, in this sense, only a convenient way of writing the three parameters specifying the rotation. We will see later, however, that infinitesimal rotations do add, so that it is meaningful to write for example $\delta\boldsymbol{\phi} = \delta\boldsymbol{\phi}_1 + \delta\boldsymbol{\phi}_2$. Rotations around a fixed axis direction also are additive in this way.

To summarize we see that we need six numbers, three coordinates and three angles, to specify the position and orientation of a rigid body. We can take the reference state of the body to be one for which the point $\mathcal{A}$ coincides with the origin and for which it has some given orientation. To bring it to any other position one then translates it with $\mathbf{d} = \mathbf{r}_{\mathcal{A}}$ without rotation. One then rotates it around an axis through $\mathcal{A}$ with rotation vector $\boldsymbol{\phi}$ to bring it to the desired orientation. One says that the rigid body has *six degrees of freedom* and the six coordinates can be taken as the six components of the vectors $\mathbf{r}_{\mathcal{A}}, \boldsymbol{\phi}$.

The translational degrees of freedom of the body are similar to those of a particle which we already know about. The rotational degrees of freedom are, however, quite different and in order to understand the kinematics of the rigid body we must now study the properties of rotations in greater detail.

2.2 Rotation Matrices

To describe the orientation of a rigid body A quantitatively one fixes an orthonormal triad of basis vectors in the body. We shall denote these body fixed basis vectors by $\mathbf{e}_1^{\mathrm{A}}, \mathbf{e}_2^{\mathrm{A}}, \mathbf{e}_3^{\mathrm{A}}$, where the superscript A indicates that these are fixed in the reference frame defined by the body A. To specify the orientation of the body one can now give the directions of these basis vectors in terms of basis vectors $\mathbf{e}_1^{\mathrm{O}}, \mathbf{e}_2^{\mathrm{O}}, \mathbf{e}_3^{\mathrm{O}}$, fixed in the reference frame O of the observer. We then have

$$\mathbf{e}_1^{\mathrm{A}} = (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}})\,\mathbf{e}_1^{\mathrm{O}} + (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}})\,\mathbf{e}_2^{\mathrm{O}} + (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}})\,\mathbf{e}_3^{\mathrm{O}} \tag{2.5}$$

$$\mathbf{e}_2^{\mathrm{A}} = (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}})\,\mathbf{e}_1^{\mathrm{O}} + (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}})\,\mathbf{e}_2^{\mathrm{O}} + (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}})\,\mathbf{e}_3^{\mathrm{O}} \tag{2.6}$$

$$\mathbf{e}_3^{\mathrm{A}} = (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}})\,\mathbf{e}_1^{\mathrm{O}} + (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}})\,\mathbf{e}_2^{\mathrm{O}} + (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}})\,\mathbf{e}_3^{\mathrm{O}} \tag{2.7}$$

This cumbersome system of equations can be written much more concisely if we introduce matrix notation. We first introduce the three by one (3×1) column matrices of

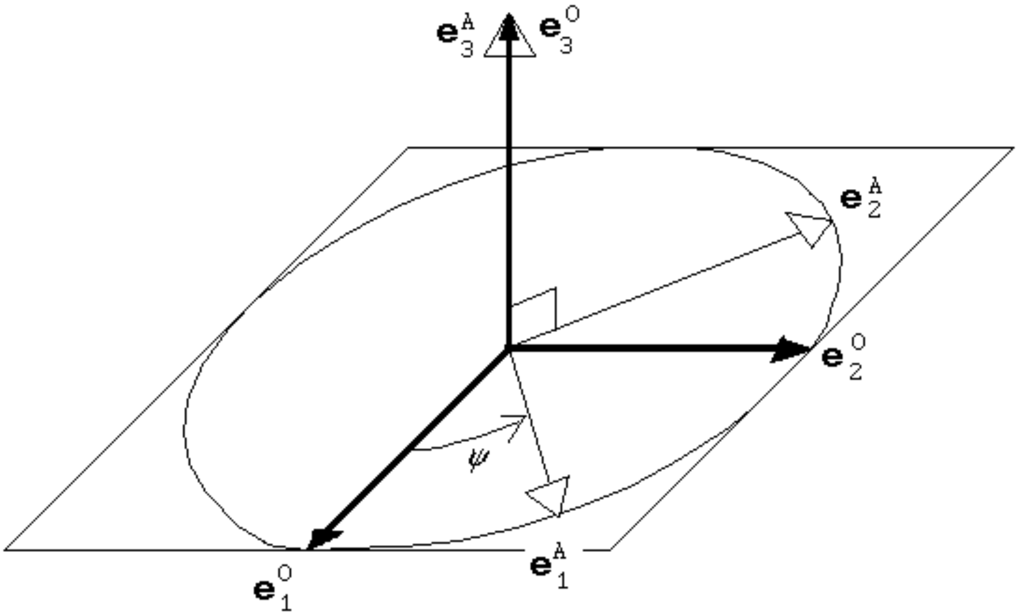


Figure 2.2: In example 2.1 the rotation matrix connecting the two bases in this figure according to formula 2.10 is given.

basis vectors

$$\mathbf{E}^{\mathrm{O}} = \begin{pmatrix} \mathbf{e}_1^{\mathrm{O}} \\ \mathbf{e}_2^{\mathrm{O}} \\ \mathbf{e}_3^{\mathrm{O}} \end{pmatrix} \quad \text{and} \quad \mathbf{E}^{\mathrm{A}} = \begin{pmatrix} \mathbf{e}_1^{\mathrm{A}} \\ \mathbf{e}_2^{\mathrm{A}} \\ \mathbf{e}_3^{\mathrm{A}} \end{pmatrix}, \tag{2.8}$$

and then the three by three (3×3) matrix

$${}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}} = \begin{pmatrix} (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}}) & (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}}) & (\mathbf{e}_1^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}}) \\ (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}}) & (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}}) & (\mathbf{e}_2^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}}) \\ (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_1^{\mathrm{O}}) & (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_2^{\mathrm{O}}) & (\mathbf{e}_3^{\mathrm{A}} \cdot \mathbf{e}_3^{\mathrm{O}}) \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{pmatrix} \tag{2.9}$$

of direction cosines of the body-fixed basis vectors with respect to the basis vectors fixed in the observer reference frame. The equation connecting the two bases can now be written on the form

$$\mathbf{E}^{\mathrm{A}} = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\,\mathbf{E}^{\mathrm{O}} \tag{2.10}$$

where matrix multiplication is implied.

Example 2.1 Calculate the rotation matrix 2.9 explicitly for the case when $\mathbf{E}^{\mathrm{A}}$ is obtained by rotating the basis $\mathbf{E}^{\mathrm{O}}$ the angle ψ around $\mathbf{e}_3^{\mathrm{O}}$ in the positive sense, see figure 2.2.
Solution: The figure shows that $\mathbf{e}_1^{\mathrm{A}} = \cos\psi\,\mathbf{e}_1^{\mathrm{O}} + \sin\psi\,\mathbf{e}_2^{\mathrm{O}}$ and that $\mathbf{e}_2^{\mathrm{A}} = -\sin\psi\,\mathbf{e}_1^{\mathrm{O}} + \cos\psi\,\mathbf{e}_2^{\mathrm{O}}$. This means that the rotation matrix becomes

$${}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}} = \mathbf{R}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.11}$$

and, as indicated, we will denote this matrix by $\mathbf{R}_3(\psi)$ which thus stands for a matrix that rotates an angle ψ around the third basis vector. $\square$

The *rotation matrix* ${}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}$, being a 3×3 matrix, has nine elements but these cannot all be independent since the rows of the matrix are the components of orthonormal basis vectors in an orthonormal basis. Like any orthonormal basis the body-fixed basis is characterized by the six relations

$$\mathbf{e}_i^{\mathrm{A}} \cdot \mathbf{e}_j^{\mathrm{A}} = \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.12}$$

between the scalar products. When these are written in terms of the components they give the six conditions

$$\begin{array}{ll} \cos\alpha_i \cdot \cos\alpha_i + \cos\beta_i \cdot \cos\beta_i + \cos\gamma_i \cdot \cos\gamma_i = 1 & \text{for } i = 1,2,3 \\ \cos\alpha_i \cdot \cos\alpha_j + \cos\beta_i \cdot \cos\beta_j + \cos\gamma_i \cdot \cos\gamma_j = 0 & \text{for } i,j = 1,2;1,3;2,3 \end{array} \tag{2.13}$$

which the nine elements of the of the rotation matrix 2.9 must obey. The matrix can thus depend only on $(9 - 6 = 3)$ three independent parameters. A rotation is thus completely specified by three parameters and this is in agreement with the statement of the previous section where we explained how a rotation can be specified by three components of the rotation vector $\boldsymbol{\phi}$.

The matrix equation 2.10 can be 'solved' for the basis E^{O} in terms of E^{A} by multiplying both sides with the inverse of ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$. We get

$$({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{-1}\,\mathsf{E}^{\mathrm{A}} = ({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{-1}\,{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathbf{1}\,\mathsf{E}^{\mathrm{O}} = \mathsf{E}^{\mathrm{O}}. \tag{2.14}$$

In order for our notation to be consistent we should have $\mathsf{E}^{\mathrm{O}} = {}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}$ so that $({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{-1} = {}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}$. But if we interchange A and O in the first matrix of formula 2.9, to get ${}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}$, then we get a new matrix with rows which are equal to the columns of the old and vice versa. A matrix obtained by interchanging rows and columns in another matrix is called the *transpose* of the old and is denoted by a superscript T. We have now shown that

$$({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{-1} = ({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{T}. \tag{2.15}$$

A matrix which has this property, that the inverse is equal to the transpose, is called an *orthogonal matrix* and the rotation matrices are thus orthogonal matrices. This fact is closely related to the fact that the elements in the rows of such matrices are the components of orthonormal basis vectors in an orthonormal basis. This is best seen by the fact that the relations 2.13 (using 2.9) are equivalent to the matrix equations

$$\begin{array}{ccc} {}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,{}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}} = {}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{T} = {}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})^{-1} & = & \mathbf{1} \\ & \Longleftrightarrow & \\ \begin{pmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{pmatrix} \begin{pmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{pmatrix} & = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array} \tag{2.16}$$

2.3 Rotation of Vectors

Now that we know how to rotate a set of basis vectors we can study the effects of rotation on any vector $\mathbf{r}$. In order to do this we introduce a new way of writing a vector which often is useful. If we denote the 1×3 row matrix of the components of the vector in some basis as follows

$$\mathsf{r} = (x_1\ x_2\ x_3) \tag{2.17}$$

we can express the vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, using the notation introduced in formula 2.8, as the matrix product

$$\mathbf{r} = \mathsf{r}\,\mathsf{E} = (x_1\ x_2\ x_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{2.18}$$

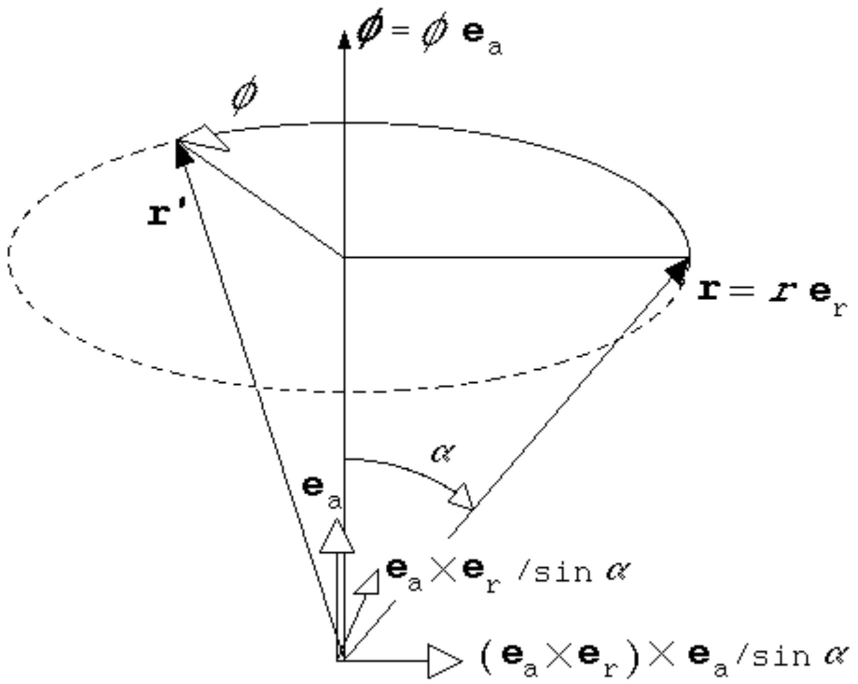


Figure 2.3: This figure illustrates the effect of the rotation implied by the rotation vector $\boldsymbol{\phi}$ on an arbitrary vector $\mathbf{r}$. Note that $|\mathbf{e}_a \times \mathbf{e}_r| = \sin\alpha$ so that the three vectors with open triangular heads make up an orthonormal triad of basis vectors. When the rotated vector $\mathbf{r}'$ is expressed in terms of these one obtains formula 2.23.

If more than one basis is involved in the problem at hand we will indicate this with superscripts on the relevant matrices, components, etc, so that we write

$$\mathbf{r} = \mathsf{r}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = (x_1^{\mathrm{O}}\; x_2^{\mathrm{O}}\; x_3^{\mathrm{O}}) \begin{pmatrix} \mathbf{e}_1^{\mathrm{O}} \\ \mathbf{e}_2^{\mathrm{O}} \\ \mathbf{e}_3^{\mathrm{O}} \end{pmatrix} = \mathsf{r}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}. \tag{2.19}$$

Note that the vector $\mathbf{r}$ itself is independent of the basis so that $\mathbf{r} = \mathsf{r}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathsf{r}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}$. Use of this and 2.10 and 2.16 shows that the components of a vector transforms according to

$$\mathsf{r}^{\mathrm{A}} = \mathsf{r}^{\mathrm{O}}\;{}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}. \tag{2.20}$$

This equation essentially says that when we rotate the basis we must make a compensating ‘rotation’ of the components in order to keep the vector the same. In order to study the rotation of rigid bodies we, however, need to know how to actually rotate vectors. One way of doing this is to rotate the basis without the compensating rotation of the components. The vector then simply follows the basis in its rotation. To every rotation matrix ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$ there therefore corresponds a *rotation operator* $\hat{R}_{\mathrm{O}\to\mathrm{A}}$ defined by

$$\mathbf{r}' = \hat{R}_{\mathrm{O}\to\mathrm{A}}\,\mathbf{r} = \mathsf{r}^{\mathrm{O}}\;{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathsf{r}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{A}}. \tag{2.21}$$

This can also be expressed as follows: the original vector is $\mathbf{r} = x_1^{\mathrm{O}}\mathbf{e}_1^{\mathrm{O}} + x_2^{\mathrm{O}}\mathbf{e}_2^{\mathrm{O}} + x_3^{\mathrm{O}}\mathbf{e}_3^{\mathrm{O}}$ while the rotated vector is $\hat{R}\mathbf{r} = \mathbf{r}' = x_1^{\mathrm{O}}\mathbf{e}_1^{\mathrm{A}} + x_2^{\mathrm{O}}\mathbf{e}_2^{\mathrm{A}} + x_3^{\mathrm{O}}\mathbf{e}_3^{\mathrm{A}}$.

One can also define a rotation operator $\hat{R}(\boldsymbol{\phi})$ by means of the rotation vector $\boldsymbol{\phi} = \phi\,\mathbf{e}_a$. This is done as follows: the rotated vector $\mathbf{r}' = \hat{R}(\boldsymbol{\phi})\mathbf{r}$ is the vector obtained by turning the vector $\mathbf{r}$ with its foot fixed at the origin an angle ϕ around the axis through the origin parallel to the vector $\mathbf{e}_a$, in the positive sense according to the right hand rule. This is illustrated in figure 2.3 which shows that the vector moves on the surface of the cone with axis along $\boldsymbol{\phi}$, vertex at the origin and generators making an angle $\alpha = [\boldsymbol{\phi}, \mathbf{r}]$ with the axis. If $\alpha = 0$ the vectors in the figure become undefined but then $\mathbf{r}$ is parallel to $\mathbf{e}_a$ so the rotation has no effect. It should be clear from the figure that

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{e}_a)\,\mathbf{e}_a + (\mathbf{e}_a\times\mathbf{r})\times\mathbf{e}_a \tag{2.22}$$

and that the rotated vector is given by (note that $\mathbf{r} = r\,\mathbf{e}_r$)

$$\mathbf{r}' = \hat{R}(\boldsymbol{\phi})\,\mathbf{r} = (\mathbf{r}\cdot\mathbf{e}_a)\,\mathbf{e}_a + \cos\phi\,(\mathbf{e}_a\times\mathbf{r})\times\mathbf{e}_a + \sin\phi\,\mathbf{e}_a\times\mathbf{r}. \tag{2.23}$$

The rotation matrix ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$ corresponding to the rotation operator $\hat{R}(\boldsymbol{\phi})$ will now be calculated.

The two equations 2.21 and 2.23 can be used to calculate the elements of the rotation matrix 2.9 explicitly in terms of the quantities of the rotation vector $\boldsymbol{\phi} = \phi\,\mathbf{e}_a$. In order to do this we insert the expressions

$$\mathbf{r} = \mathsf{r}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = x_1^{\mathrm{O}}\,\mathbf{e}_1^{\mathrm{O}} + x_2^{\mathrm{O}}\,\mathbf{e}_2^{\mathrm{O}} + x_3^{\mathrm{O}}\,\mathbf{e}_3^{\mathrm{O}}, \tag{2.24}$$

$$\mathbf{e}_a = \cos\alpha_1\,\mathbf{e}_1^{\mathrm{O}} + \cos\alpha_2\,\mathbf{e}_2^{\mathrm{O}} + \cos\alpha_3\,\mathbf{e}_3^{\mathrm{O}} \tag{2.25}$$

into equation 2.23. Here α_i are the angles between the observer fixed (non-rotated) basis vectors and the direction of the rotation axis $\mathbf{e}_a$. When this has been done it is only a matter of some algebra to rewrite this equation (2.23) on the form

$$\mathbf{r}' = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i^{\mathrm{O}}\mathrm{R}_{ij}\,\mathbf{e}_j^{\mathrm{O}}, \tag{2.26}$$

where R_{ij} stand for algebraic expression in terms of the quantities of the rotation vector. When we compare this with the equation $\mathbf{r}' = \mathsf{r}^{\mathrm{O}}\;{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\;\mathsf{E}^{\mathrm{O}}$ written explicitly as a sum over the matrix elements

$$\mathbf{r}' = \sum_{i=1}^{3}\sum_{j=1}^{3}(\mathsf{r}^{\mathrm{O}})_i\,({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})_{ij}\,(\mathsf{E}^{\mathrm{O}})_j = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i^{\mathrm{O}}\,({}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}})_{ij}\,\mathbf{e}_j^{\mathrm{O}} \tag{2.27}$$

we see that the algebraic expressions R_{ij} are in fact the matrix elements of the rotation matrix ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$. When this program is carried out one finds the result

$${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}(\boldsymbol{\phi}) = \tag{2.28}$$

$$\begin{pmatrix} (1-\cos\phi)\cos^2\alpha_1+\cos\phi & (1-\cos\phi)\cos\alpha_1\,\cos\alpha_2+\sin\phi\,\cos\alpha_3 & (1-\cos\phi)\cos\alpha_1\,\cos\alpha_3-\sin\phi\,\cos\alpha_2 \\ (1-\cos\phi)\cos\alpha_2\,\cos\alpha_1-\sin\phi\,\cos\alpha_3 & (1-\cos\phi)\cos^2\alpha_2+\cos\phi & (1-\cos\phi)\cos\alpha_2\,\cos\alpha_3+\sin\phi\,\cos\alpha_1 \\ (1-\cos\phi)\cos\alpha_3\,\cos\alpha_1+\sin\phi\,\cos\alpha_2 & (1-\cos\phi)\cos\alpha_3\,\cos\alpha_2-\sin\phi\,\cos\alpha_1 & (1-\cos\phi)\cos^2\alpha_3+\cos\phi \end{pmatrix}$$

This expression can be broken up into a sum of somewhat simpler parts

$$\begin{aligned} {}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}(\boldsymbol{\phi}) &= \cos\phi\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin\phi\begin{pmatrix} 0 & \cos\alpha_3 & -\cos\alpha_2 \\ -\cos\alpha_3 & 0 & \cos\alpha_1 \\ \cos\alpha_2 & -\cos\alpha_1 & 0 \end{pmatrix} \\ &+(1-\cos\phi)\begin{pmatrix} \cos^2\alpha_1 & \cos\alpha_1\,\cos\alpha_2 & \cos\alpha_1\,\cos\alpha_3 \\ \cos\alpha_2\,\cos\alpha_1 & \cos^2\alpha_2 & \cos\alpha_2\,\cos\alpha_3 \\ \cos\alpha_3\,\cos\alpha_1 & \cos\alpha_3\,\cos\alpha_2 & \cos^2\alpha_3 \end{pmatrix} \end{aligned} \tag{2.29}$$

where the first and last matrices are seen to be symmetric and the middle one is anti-symmetric. For small rotation angles ϕ the three terms in the sum represent constant+quadratic, linear, and quadratic terms in ϕ, respectively. Note that the $\cos\alpha_i$ are direction cosines of the axis direction so they obey $\cos^2\alpha_1+\cos^2\alpha_2+\cos^2\alpha_3 = 1$ and only two of them are independent. This means that this rotation matrix depends on three independent parameters as we have discussed above. We will now find a different, more direct way of parameterizing a rotation matrix.

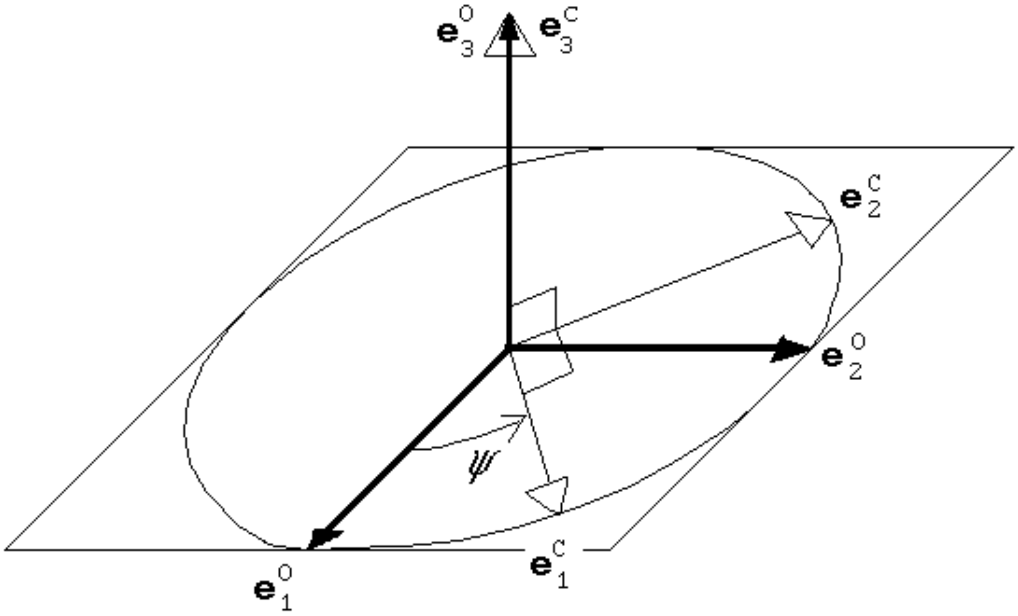


Figure 2.4: This figure shows the effect of the first step of the three necessary to achieve general rotation. The rotation is from the fixed basis triad E^{O} to the intermediate basis E^{C} and corresponds to rotation an angle ψ around the direction given by $\mathbf{e}_3^{\mathrm{O}}$.

2.4 Euler Angles and Non-commutation of Rotations

One way of constructing a general rotation and its matrix is to use three steps of the kind used in example 2.1. The strategy then is to build up the rotation in terms of three simple rotations each around one of the coordinate axes. Rotation matrices that rotate around the respective axes are given by (see example 2.1)

$$\mathsf{R}_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \mathsf{R}_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \mathsf{R}_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.30}$$

In order to make a rotation matrix that rotates from the fixed basis E^{O} to an arbitrarily oriented basis E^{A} we use now use these as follows. We first rotate an angle ψ around $\mathbf{e}_3^{\mathrm{O}}$ and get a new basis E^{C} given by

$$\mathsf{E}^{\mathrm{C}} = {}^{\mathrm{C}}\mathsf{R}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathsf{R}_3(\psi)\,\mathsf{E}^{\mathrm{O}}. \tag{2.31}$$

This step is illustrated in figure 2.4. We now rotate an angle θ around $\mathbf{e}_1^{\mathrm{C}}$ from the basis E^{C} to the basis E^{B} so that

$$\mathsf{E}^{\mathrm{B}} = {}^{\mathrm{B}}\mathsf{R}^{\mathrm{C}}\,\mathsf{E}^{\mathrm{C}} = \mathsf{R}_1(\theta)\,\mathsf{E}^{\mathrm{C}}. \tag{2.32}$$

This step is shown in figure 2.5. Note how the plane spanned by $\mathbf{e}_1^{\mathrm{B}}$ and $\mathbf{e}_2^{\mathrm{B}}$ now is tilted an angle θ. The orientation of the basis E^{B} is still not arbitrary since the basis vector $\mathbf{e}_1^{\mathrm{B}}$ necessarily is in the original 1,2-plane. This is now changed by the third step: a rotation an angle φ around $\mathbf{e}_3^{\mathrm{B}}$. This is shown in figure 2.6 and the formula is

$$\mathsf{E}^{\mathrm{A}} = {}^{\mathrm{A}}\mathsf{R}^{\mathrm{B}}\,\mathsf{E}^{\mathrm{B}} = \mathsf{R}_3(\varphi)\,\mathsf{E}^{\mathrm{B}}. \tag{2.33}$$

Any desired orientation of the basis triad E^{A} can clearly be achieved with this sequence of steps. If we now put it all together we have

$$\mathsf{E}^{\mathrm{A}} = {}^{\mathrm{A}}\mathsf{R}^{\mathrm{B}}\,{}^{\mathrm{B}}\mathsf{R}^{\mathrm{C}}\,{}^{\mathrm{C}}\mathsf{R}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathsf{R}_3(\varphi)\,\mathsf{R}_1(\theta)\,\mathsf{R}_3(\psi)\,\mathsf{E}^{\mathrm{O}} = {}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}(\psi,\theta,\varphi)\,\mathsf{E}^{\mathrm{O}}. \tag{2.34}$$

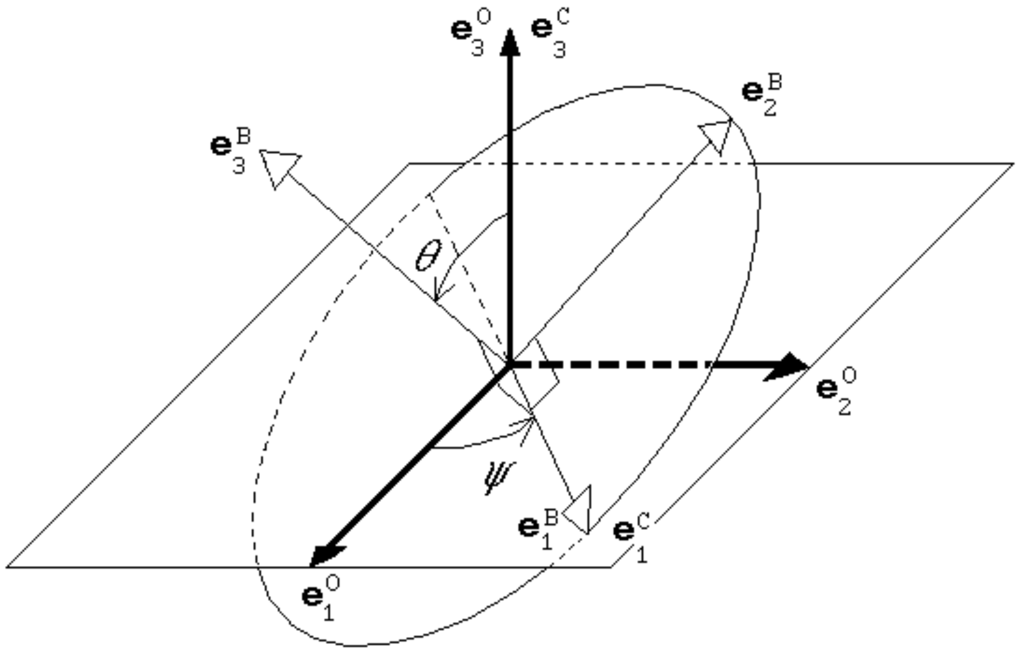


Figure 2.5: This figure shows the effect of the second rotation from the basis triad $\mathbf{E}^{\mathrm{C}}$ to the second intermediate basis $\mathbf{E}^{\mathrm{B}}$. This rotation is an angle θ around $\mathbf{e}_1^{\mathrm{C}}$. The line of intersection, parallel to $\mathbf{e}_1^{\mathrm{C}} = \mathbf{e}_1^{\mathrm{B}}$, between the 1,2-plane of the O-system and the 1,2-plane of the B-system, is sometimes called the line of nodes.

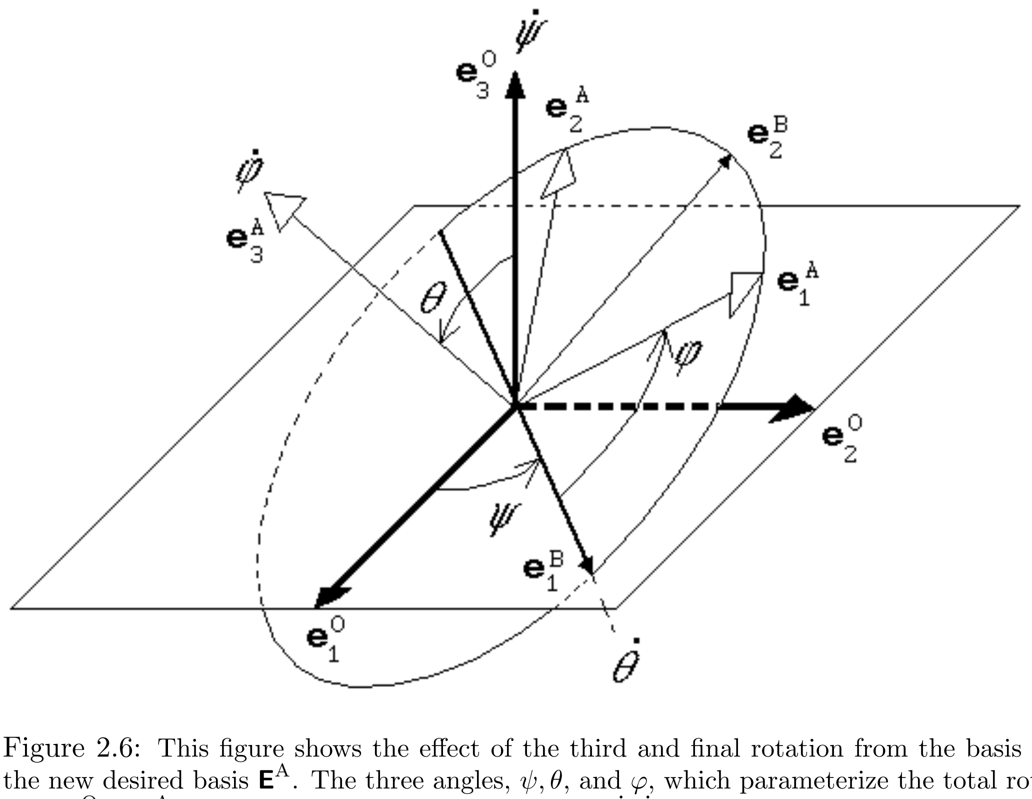


Figure 2.6: This figure shows the effect of the third and final rotation from the basis $\mathbf{E}^{\mathrm{B}}$ to the new desired basis $\mathbf{E}^{\mathrm{A}}$. The three angles, ψ, θ, and φ, which parameterize the total rotation from $\mathbf{E}^{\mathrm{O}}$ to $\mathbf{E}^{\mathrm{A}}$ are called Euler angles. The symbols $\dot{\psi}, \dot{\theta}$, and $\dot{\varphi}$ indicate the axes around which rotation takes place if the corresponding angle changes while the two others are held fixed.

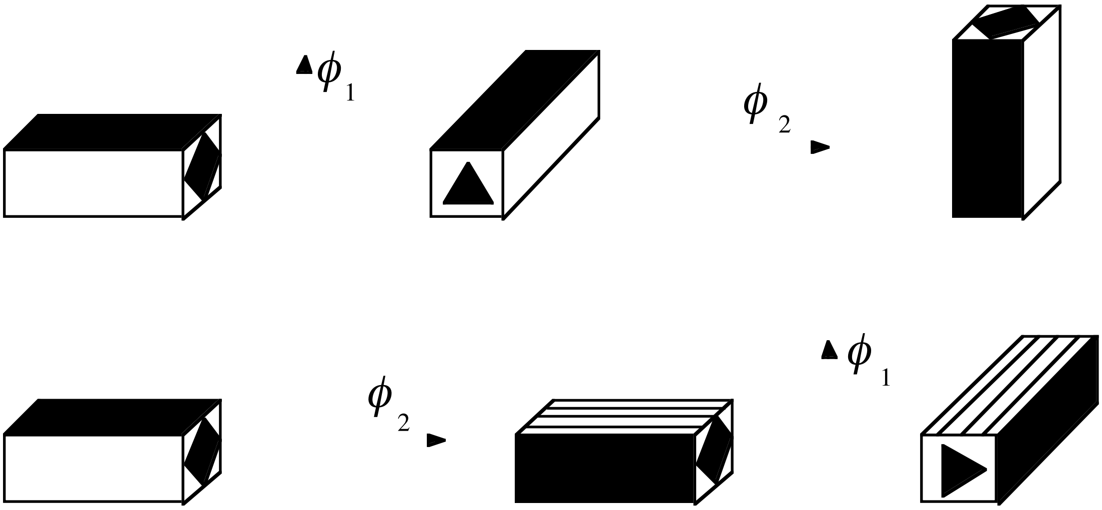


Figure 2.7: This figure illustrates the effect of two consecutive rotations of a rectangular box. In the upper row the box is first rotated 90° around a vertical axis, then 90° around a horizontal axis. In the lower row the same two rotations are performed in the opposite order. The end results are clearly different and this shows that finite rotations do *not* commute; the order in which they are done matters.

The parameters in the parameterization of the rotation matrix constructed in this way are called *Euler angles.* The rotation matrix in terms of Euler angles ψ, θ, φ is thus

$$ {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\psi,\theta,\varphi) = \mathbf{R}_3(\varphi)\,\mathbf{R}_1(\theta)\,\mathbf{R}_3(\psi). \tag{2.35}$$

When the matrix multiplications are carried out explicitly one obtains

$$ {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\psi,\theta,\varphi) = \tag{2.36}$$
$$\begin{pmatrix} \cos\varphi\,\cos\psi - \sin\varphi\,\cos\theta\,\sin\psi & \cos\varphi\,\sin\psi + \sin\varphi\,\cos\theta\,\cos\psi & \sin\varphi\,\sin\theta \\ -\sin\varphi\,\cos\psi - \cos\varphi\,\cos\theta\,\sin\psi & -\sin\varphi\,\sin\psi + \cos\varphi\,\cos\theta\,\cos\psi & \cos\varphi\,\sin\theta \\ \sin\theta\,\sin\psi & -\sin\theta\,\cos\psi & \cos\theta \end{pmatrix},$$

but this form is not particularly useful and is only given here for reference. By equating the antisymmetric part of this matrix with the antisymmetric part in equation 2.29 one can easily find the axis and angle of the net rotation produced by the three steps.

It is important to understand that the first and the last rotations $\mathbf{R}_3$ are not around the same axis; the first is around $\mathbf{e}_3^{\mathrm{O}}$ but the last is around $\mathbf{e}_3^{\mathrm{B}}$. From this one realizes that the order in which rotations are done is important; the end result will change if the order is changed. As long as one rotates about a *fixed axis* this is not true since, clearly,

$$\mathbf{R}_3(\varphi)\,\mathbf{R}_3(\psi) = \mathbf{R}_3(\psi)\,\mathbf{R}_3(\varphi) = \mathbf{R}_3(\varphi+\psi). \tag{2.37}$$

This, however, makes it obvious that

$$\mathbf{R}_3(\varphi)\,\mathbf{R}_1(\theta)\,\mathbf{R}_3(\psi) \neq \mathbf{R}_1(\theta)\,\mathbf{R}_3(\varphi)\,\mathbf{R}_3(\psi). \tag{2.38}$$

Algebraically this depends on the fact the matrix multiplication is not commutative but the reader should also try to visualize the geometric difference between the two rotations implied by the left and the right hand side. The non-commutation of rotations is also illustrated in figure 2.7.

2.5 Infinitesimal Rotations and Angular Velocity

Rotations by very small angles do commute, the condition being that one can ignore quadratic terms. This is most easily seen as follows. For a small rotation vector $\delta\boldsymbol{\phi}$, using $\sin\delta\phi \approx \delta\phi$ and $\cos\delta\phi \approx 1$ in equation 2.29 we get

$$ {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \delta\phi_3 & -\delta\phi_2 \\ -\delta\phi_3 & 0 & \delta\phi_1 \\ \delta\phi_2 & -\delta\phi_1 & 0 \end{pmatrix}. \tag{2.39} $$

Now let $\delta\boldsymbol{\phi}^i$ $(i = 1, 2)$ be two rotation vectors of small magnitude given by

$$ \delta\boldsymbol{\phi}^i = (\delta\phi_1^i, \delta\phi_2^i, \delta\phi_3^i) = \delta\phi_1^i \mathbf{e}_1^{\mathrm{O}} + \delta\phi_2^i \mathbf{e}_2^{\mathrm{O}} + \delta\phi_3^i \mathbf{e}_3^{\mathrm{O}} \tag{2.40} $$

in the basis $\mathbf{E}^{\mathrm{O}}$. Putting these into the matrix formula above we find that

$$ \begin{aligned} &\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \delta\phi_3^1 & -\delta\phi_2^1 \\ -\delta\phi_3^1 & 0 & \delta\phi_1^1 \\ \delta\phi_2^1 & -\delta\phi_1^1 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \delta\phi_3^2 & -\delta\phi_2^2 \\ -\delta\phi_3^2 & 0 & \delta\phi_1^2 \\ \delta\phi_2^2 & -\delta\phi_1^2 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \delta\phi_3^1 + \delta\phi_3^2 & -\delta\phi_2^1 - \delta\phi_2^2 \\ -\delta\phi_3^1 - \delta\phi_3^2 & 0 & \delta\phi_1^1 + \delta\phi_1^2 \\ \delta\phi_2^1 + \delta\phi_2^2 & -\delta\phi_1^1 - \delta\phi_1^2 & 0 \end{pmatrix} + \mathrm{O}(\delta\phi^2). \end{aligned} \tag{2.41} $$

From this it is easy to show that

$$ {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}^1)\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}^2) = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}^2)\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}^1) = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi}^1 + \delta\boldsymbol{\phi}^2) \tag{2.42} $$

provided quadratic terms can be neglected.

For a small rotation we now have according to 2.10 that

$$ \begin{aligned} \mathbf{E}^{\mathrm{A}} = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}(\delta\boldsymbol{\phi})\,\mathbf{E}^{\mathrm{O}} &= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \delta\phi_3 & -\delta\phi_2 \\ -\delta\phi_3 & 0 & \delta\phi_1 \\ \delta\phi_2 & -\delta\phi_1 & 0 \end{pmatrix}\right]\begin{pmatrix} \mathbf{e}_1^{\mathrm{O}} \\ \mathbf{e}_2^{\mathrm{O}} \\ \mathbf{e}_3^{\mathrm{O}} \end{pmatrix} \\ &= \mathbf{E}^{\mathrm{O}} + \delta\mathbf{E}^{\mathrm{O}} = \mathbf{E}^{\mathrm{O}} + \begin{pmatrix} \delta\boldsymbol{\phi}\times\mathbf{e}_1^{\mathrm{O}} \\ \delta\boldsymbol{\phi}\times\mathbf{e}_2^{\mathrm{O}} \\ \delta\boldsymbol{\phi}\times\mathbf{e}_3^{\mathrm{O}} \end{pmatrix} = \mathbf{E}^{\mathrm{O}} + \delta\boldsymbol{\phi}\times\mathbf{E}^{\mathrm{O}}. \end{aligned} \tag{2.43} $$

The effect of the antisymmetric matrix on the basis is thus equivalent to a vector product. In the last term here the vector product in front of the column matrix $\mathbf{E}^{\mathrm{O}}$ of basis vectors means that each vector in the matrix is to be multiplied vectorially by $\delta\boldsymbol{\phi} = \delta\phi_1\mathbf{e}_1^{\mathrm{O}} + \delta\phi_2\mathbf{e}_2^{\mathrm{O}} + \delta\phi_3\mathbf{e}_3^{\mathrm{O}}$. The change in the basis vectors for a small rotation can thus be written

$$ \delta\mathbf{E}^{\mathrm{O}} = \delta\boldsymbol{\phi}\times\mathbf{E}^{\mathrm{O}} \tag{2.44} $$

by means of the vector product with the infinitesimal rotation vector.

We'll now assume that the parameters of the rotation matrix are functions of time so that the matrix describes the rotation of a rigid body A to which the basis $\mathbf{E}^{\mathrm{A}}$ is attached with respect to the observer frame O. We wish to find the time derivatives of the basis vectors of $\mathbf{E}^{\mathrm{A}}$. The time derivative of a vector will depend on which reference frame is considered as fixed. When it is the frame O which is fixed we denote this by writing the time derivative $\frac{{}^{\mathrm{O}}d}{dt}$. We can now define a matrix which gives this time derivative as follows:

$$ \frac{{}^{\mathrm{O}}d}{dt}\mathbf{E}^{\mathrm{A}} = \dot{\mathbf{E}}^{\mathrm{A}} = \begin{pmatrix} \dot{\mathbf{e}}_1^{\mathrm{A}} \\ \dot{\mathbf{e}}_2^{\mathrm{A}} \\ \dot{\mathbf{e}}_3^{\mathrm{A}} \end{pmatrix} = \frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\,\mathbf{E}^{\mathrm{O}} = \left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\,\mathbf{E}^{\mathrm{A}} = {}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}}\,\mathbf{E}^{\mathrm{A}} \tag{2.45} $$

Here we have introduced the *angular velocity matrix* for the rotation of the basis $\mathbf{E}^{\mathrm{A}}$ with respect to the fixed basis $\mathbf{E}^{\mathrm{O}}$

$$ {}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}} \equiv \left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}. \tag{2.46} $$

We now show that this is an anti-symmetric matrix. We denote the three by three unit matrix (with ones on the diagonal) by $\mathbf{1}$ and the three by three null matrix, with all elements zero, by $\mathbf{0}$. We can now write, using equation 2.16

$$\mathbf{0} = \frac{{}^{\mathrm{O}}d}{dt}\mathbf{1} = \frac{{}^{\mathrm{O}}d}{dt}({}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}) = \left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}} + {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\right) \tag{2.47}$$

so that we have shown

$$\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}} = -\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\right). \tag{2.48}$$

But if we use the rule $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ for the transpose of a matrix product, we find that

$$\left[\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\right]^T = (\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}})^T\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)^T = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\right). \tag{2.49}$$

We thus find that

$$\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}} = -\left[\left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}}\right]^T \tag{2.50}$$

or, equivalently,

$${}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}} = -(\,{}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}})^T \tag{2.51}$$

and this proves that the angular velocity matrix is anti-symmetric.

Just as in equation 2.43 above we can take the three elements of the anti-symmetric matrix to define a corresponding vector. The vector corresponding to ${}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}}$ is the *angular velocity vector* ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}}$

$${}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}} = \begin{pmatrix} 0 & {}^{\mathrm{O}}\omega_3^{\mathrm{A}} & -{}^{\mathrm{O}}\omega_2^{\mathrm{A}} \\ -{}^{\mathrm{O}}\omega_3^{\mathrm{A}} & 0 & {}^{\mathrm{O}}\omega_1^{\mathrm{A}} \\ {}^{\mathrm{O}}\omega_2^{\mathrm{A}} & -{}^{\mathrm{O}}\omega_1^{\mathrm{A}} & 0 \end{pmatrix} \tag{2.52}$$

and this matrix contains the components of this vector in the basis $\mathbf{E}^{\mathrm{A}}$. Using this vector we can now write the result of equation 2.45 in the form

$$\frac{{}^{\mathrm{O}}d}{dt}\mathbf{E}^{\mathrm{A}} = \begin{pmatrix} \dot{\mathbf{e}}_1^{\mathrm{A}} \\ \dot{\mathbf{e}}_2^{\mathrm{A}} \\ \dot{\mathbf{e}}_3^{\mathrm{A}} \end{pmatrix} = {}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}}\,\mathbf{E}^{\mathrm{A}} = \begin{pmatrix} {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{e}_1^{\mathrm{A}} \\ {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{e}_2^{\mathrm{A}} \\ {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{e}_3^{\mathrm{A}} \end{pmatrix} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{E}^{\mathrm{A}}. \tag{2.53}$$

This equation should be compared to equation 2.44. It then shows that the small change in the basis $\mathbf{E}^{\mathrm{A}}$ during the time dt is given by $d\mathbf{E}^{\mathrm{A}} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}}dt \times \mathbf{E}^{\mathrm{A}}$ so that the basis is rotated by the infinitesimal rotation vector

$$\delta\boldsymbol{\phi} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}}dt. \tag{2.54}$$

One should note, however that the angular velocity vector is *not* the derivative of the rotation vector for finite rotation angles ϕ.

Example 2.2 Calculate the angular velocity matrix and vector of the rotation matrix

$${}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}} = \mathbf{R}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.55}$$

calculated in the example 2.1.

Solution: Using the definition 2.46 we get

$${}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}} = \left(\frac{{}^{\mathrm{O}}d}{dt}\,{}^{\mathrm{A}}\mathbf{R}^{\mathrm{O}}\right)\,{}^{\mathrm{O}}\mathbf{R}^{\mathrm{A}} = \left[\frac{{}^{\mathrm{O}}d}{dt}\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.56}$$

$$= \begin{pmatrix} -\dot\psi\sin\psi & \dot\psi\cos\psi & 0 \\ -\dot\psi\cos\psi & -\dot\psi\sin\psi & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \dot\psi & 0 \\ -\dot\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.57}$$

From this we get the angular velocity vector

$$^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = (0, 0, \dot{\psi}) = \dot{\psi}\,\mathbf{e}_3^{\mathrm{A}} \tag{2.58}$$

which thus is parallel to the axis of rotation. In this example the axis of rotation is fixed, but this is a general result; the angular velocity vector is parallel to the instantaneous axis of rotation as indicated by equation 2.54. □

2.5.1 The Time Derivative of Vectors

The time derivative of a vector $\mathbf{r}$ will, as mentioned, depend on what reference frame is used to measure its motion. By definition we have that the time derivative of the vector $\mathbf{r}$ with respect to a reference frame A in which the triad of basis vectors E^{A} is fixed can be written, using the notation of equation 2.19,

$$\frac{^{\mathrm{A}}d}{dt}\mathbf{r} = \frac{^{\mathrm{A}}d}{dt}(\mathsf{r}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}) = (\frac{d}{dt}\mathsf{r}^{\mathrm{A}})\,\mathsf{E}^{\mathrm{A}} = (\dot{x}_1^{\mathrm{A}}\ \dot{x}_2^{\mathrm{A}}\ \dot{x}_3^{\mathrm{A}}) \begin{pmatrix} \mathbf{e}_1^{\mathrm{A}} \\ \mathbf{e}_2^{\mathrm{A}} \\ \mathbf{e}_3^{\mathrm{A}} \end{pmatrix} \tag{2.59}$$

since in this basis only the components are time-dependent. If we now assume that the reference frame A rotates with respect to the observer frame O, then when taking the time derivative of the vector, as measured in O, one must take account of the fact that also the basis E^{A} is timedependent. We get

$$\frac{^{\mathrm{O}}d}{dt}\mathbf{r} = \frac{^{\mathrm{O}}d}{dt}(\mathsf{r}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}) = (\frac{d}{dt}\mathsf{r}^{\mathrm{A}})\,\mathsf{E}^{\mathrm{A}} + \mathsf{r}^{\mathrm{A}}\,\frac{^{\mathrm{O}}d}{dt}\mathsf{E}^{\mathrm{A}} = \frac{^{\mathrm{A}}d}{dt}\mathbf{r} + \mathsf{r}^{\mathrm{A}}\,{}^{\mathrm{O}}\boldsymbol{\Omega}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}. \tag{2.60}$$

connection between the time derivatives of a vector in relatively rotating reference frames

Use of this and equation 2.53 leads to the important result

$$\frac{^{\mathrm{O}}d}{dt}\mathbf{r} = \frac{^{\mathrm{A}}d}{dt}\mathbf{r} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{r}. \tag{2.61}$$

Note that this is a vector equation and that it therefore is independent of the bases that are used. The components of the four vectors appearing in this equation may be with respect to any suitable basis. When the basis is changed the components change according to equation 2.20.

2.5.2 Additivity of the Angular Velocity

It is a reasonably well known and obvious fact that relative velocities are additive. This means that if reference frame B has velocity $^{\mathrm{O}}\mathbf{v}^{\mathrm{B}}$ with respect to a fixed frame O, and reference frame A has velocity $^{\mathrm{B}}\mathbf{v}^{\mathrm{A}}$ as seen from B, then the velocity of A with respect to O is given by

$$^{\mathrm{O}}\mathbf{v}^{\mathrm{A}} = {}^{\mathrm{O}}\mathbf{v}^{\mathrm{B}} + {}^{\mathrm{B}}\mathbf{v}^{\mathrm{A}}. \tag{2.62}$$

It is much less trivial that the same thing holds for relative angular velocities. If a basis triad E^{B} has angular velocity vector $^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}}$ with respect to the reference frame O in which E^{O} is fixed, and if E^{A} has angular velocity $^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}$ with respect to the frame of E^{B}, then the angular velocity vector of A as seen from O is given by

$$^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}. \tag{2.63}$$

Proof: We use equation 2.61 three times. First we get $\frac{^{\mathrm{O}}d\mathbf{r}}{dt} = \frac{^{\mathrm{B}}d\mathbf{r}}{dt} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} \times \mathbf{r}$ and into this we insert $\frac{^{\mathrm{B}}d\mathbf{r}}{dt} = \frac{^{\mathrm{A}}d\mathbf{r}}{dt} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{r}$ to get $\frac{^{\mathrm{O}}d\mathbf{r}}{dt} = \frac{^{\mathrm{A}}d\mathbf{r}}{dt} + ({}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}) \times \mathbf{r}$. Comparison with equation 2.61 then proves the desired equality.

The result can, of course, be generalized to an arbitrary number of intermediate frames. As a concrete example one might consider time derivatives of vectors with respect to the Euler angle rotated frame A given in equation 2.34 and figure 2.6. The construction of this frame by means of a sequence of frames having simple rotations relative to each other makes it easy to find its angular velocity. Generalizing the formula 2.63 to two intermediate frames, C and B, gives

$$ {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{C}} + {}^{\mathrm{C}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}. \tag{2.64} $$

Since the relative rotation of these frames are of the simple type studied in example 2.2 reference to figure 2.6 immediately gives

$$ {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = \dot{\psi}\,\mathbf{e}_3^{\mathrm{C}} + \dot{\theta}\,\mathbf{e}_1^{\mathrm{B}} + \dot{\varphi}\,\mathbf{e}_3^{\mathrm{A}}. \tag{2.65} $$

Using

$$ \mathbf{e}_3^{\mathrm{C}} = \sin\theta\,\sin\varphi\,\mathbf{e}_1^{\mathrm{A}} + \sin\theta\,\cos\varphi\,\mathbf{e}_2^{\mathrm{A}} + \cos\theta\,\mathbf{e}_3^{\mathrm{A}}, \tag{2.66} $$

$$ \mathbf{e}_1^{\mathrm{B}} = \cos\varphi\,\mathbf{e}_1^{\mathrm{A}} - \sin\varphi\,\mathbf{e}_2^{\mathrm{A}}, \tag{2.67} $$

this gives

$$ \begin{aligned} {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = \dot{\psi}\,(\sin\theta\,\sin\varphi\,\mathbf{e}_1^{\mathrm{A}} + \sin\theta\,\cos\varphi\,\mathbf{e}_2^{\mathrm{A}} + \cos\theta\,\mathbf{e}_3^{\mathrm{A}}) + \\ \dot{\theta}\,(\cos\varphi\,\mathbf{e}_1^{\mathrm{A}} - \sin\varphi\,\mathbf{e}_2^{\mathrm{A}}) + \dot{\varphi}\,\mathbf{e}_3^{\mathrm{A}} \end{aligned} \tag{2.68} $$

for the components of the angular velocity in the body fixed basis E^{A}. We can write this in the form

$$ \begin{aligned} {}^{\mathrm{O}}\omega_1^{\mathrm{A}} &= \dot{\psi}\,\sin\theta\,\sin\varphi + \dot{\theta}\,\cos\varphi, \\ {}^{\mathrm{O}}\omega_2^{\mathrm{A}} &= \dot{\psi}\,\sin\theta\,\cos\varphi - \dot{\theta}\,\sin\varphi, \\ {}^{\mathrm{O}}\omega_3^{\mathrm{A}} &= \dot{\psi}\,\cos\theta + \dot{\varphi}. \end{aligned} \tag{2.69} $$

These equations are called *Euler's kinematic equations.*

2.6 Position and Velocity of Points of a Rigid Body

Let $\overline{\mathcal{O}\mathcal{P}_i} = \mathbf{r}_i$ $(i = 1, \ldots, N)$ be the position vector of one particle (or point) $\mathcal{P}_i$ of a rigid body A with respect to an origin $\mathcal{O}$ fixed in the observer frame O. When the body moves the different particles will move along trajectories $\mathbf{r}_i(t)$ but these will not be independent when the particles make up a rigid body. We can, in fact, parameterize the positions of all particles of the body with six coordinates. This can be done as follows. Choose one point $\mathcal{A}$ fixed in the body and a set of basis vectors E^{A} with fixed directions in the body and write

$$ \mathbf{r}_i = \overline{\mathcal{O}\mathcal{A}} + \overline{\mathcal{A}\mathcal{P}_i} = \mathbf{r}_{\mathcal{A}} + \mathbf{a}_i = \mathsf{r}_{\mathcal{A}}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} + \mathsf{a}_i^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}. \tag{2.70} $$

In this expression the lengths of the vectors $\mathbf{a}_i$ are constant since they are vectors between points of a rigid body. The components, $\mathsf{a}_i^{\mathrm{A}}$, of the vectors $\mathbf{a}_i$ with respect to the basis E^{A}, must also be constant since both follow the body in its rotation. Thus we can write the positions of the particles of the body in the form

$$ \begin{aligned} \mathbf{r}_i(t) = \mathbf{r}_{\mathcal{A}}(t) + \mathbf{a}_i(t) = \mathsf{r}_{\mathcal{A}}^{\mathrm{O}}(t)\,\mathsf{E}^{\mathrm{O}} + \mathsf{a}_i^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}(t) \\ = \mathsf{r}_{\mathcal{A}}^{\mathrm{O}}(t)\,\mathsf{E}^{\mathrm{O}} + \mathsf{a}_i^{\mathrm{A}}\;{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}(\boldsymbol{\phi}(t))\,\mathsf{E}^{\mathrm{O}}. \end{aligned} \tag{2.71} $$

The time-dependence of all N particles of the body is thus known once one knows the time-dependence of the three components $\mathsf{r}_{\mathcal{A}}^{\mathrm{O}}(t)$ of the vector $\mathbf{r}_{\mathcal{A}}$ and the time-dependence of the three parameters, $\boldsymbol{\phi}(t)$, of the rotation matrix ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$ from the observer fixed basis E^{O} to the body fixed basis E^{A}. These parameters may be chosen as the three components of the rotation vector or the three Euler angles or by other methods that we have not treated here.

2.6.1 The Connection Formula for Velocities

The velocity of the point $\mathcal{P}_i$ of the body with respect to the observer fixed frame O can now be found by differentiating the expression 2.71 with respect to time:

$$\mathbf{v}_i = \frac{{}^{\mathrm{O}}d}{dt}\mathbf{r}_i = \mathbf{v}_{\mathcal{A}} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \mathbf{a}_i. \tag{2.72}$$

Here we have used equation 2.61 and the fact that $\frac{{}^{\mathrm{A}}d}{dt}\mathbf{a}_i = \mathbf{0}$. If we now let $\mathcal{B}$=$\mathcal{P}_i$ be some, arbitrary point of the body, different from $\mathcal{A}$, and use the fact that then $\mathbf{a}_i = \overline{\mathcal{A}\mathcal{P}}_i = \overline{\mathcal{A}\mathcal{B}}$, we can rewrite this in the form

$$\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{A}} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} \times \overline{\mathcal{A}\mathcal{B}}. \tag{2.73}$$

As long as one deals with a single rigid body moving with respect to a given fixed reference frame, as will mostly be the case, there is no need for the superscripts O and A on the angular velocity vector; we then simply write $\boldsymbol{\omega}$ for the angular velocity vector of the body. The above formula now takes the simple form

connection formula for velocities in rigid body

$$\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{A}} + \overline{\mathcal{B}\mathcal{A}} \times \boldsymbol{\omega} \tag{2.74}$$

where we have used $\overline{\mathcal{A}\mathcal{B}} = -\overline{\mathcal{B}\mathcal{A}}$. This formula is analogous to the connection formulae for moments ($\mathbf{M}_{\mathcal{B}} = \mathbf{M}_{\mathcal{A}} + \overline{\mathcal{B}\mathcal{A}} \times \mathbf{F}$) and for angular momenta ($\mathbf{L}_{\mathcal{B}} = \mathbf{L}_{\mathcal{A}} + \overline{\mathcal{B}\mathcal{A}} \times \mathbf{p}$). Once the angular velocity vector and the velocity of one point are known this formula gives the velocities of all other points of the body.

2.6.2 The Instantaneous Axis of Rotation

By means of the connection formula for moments one finds that a force system defines an axis such that, at this axis, the moment and the force sum $\mathbf{F}$ are parallel to this axis. This is called the equipollent reduction of the system of forces to a wrench. Since the velocities of a rigid body are connected by a similar formula the analogous result holds. One can find an axis through the body, or rigidly connected to the body, such that for all points $\mathcal{C}$ of the body on this axis the velocity is parallel to the angular velocity vector, $\mathbf{v}_{\mathcal{C}} = v_{\mathcal{C}}\,\mathbf{e}_\omega \;\|\; \boldsymbol{\omega} = \omega\,\mathbf{e}_\omega$. We will now prove this.

Since the vector $\overline{\mathcal{B}\mathcal{A}} \times \boldsymbol{\omega}$ of formula 2.74 is perpendicular to $\boldsymbol{\omega}$ a suitable point $\mathcal{B}$=$\mathcal{C}$ in the plane through $\mathcal{A}$ perpendicular to $\boldsymbol{\omega}$, should be able to make this vector cancel the component of $\mathbf{v}_{\mathcal{A}}$ perpendicular to $\boldsymbol{\omega}$. Put

$$\mathbf{v}_{\mathcal{A}} = \mathbf{v}_{\mathcal{A}\|} + \mathbf{v}_{\mathcal{A}\perp} \tag{2.75}$$

where

$$\mathbf{v}_{\mathcal{A}\|} = (\mathbf{v}_{\mathcal{A}} \cdot \mathbf{e}_\omega)\,\mathbf{e}_\omega \tag{2.76}$$

is the vector component of $\mathbf{v}_{\mathcal{A}}$ parallel to the angular velocity vector $\boldsymbol{\omega}$ and where $\mathbf{v}_{\mathcal{A}\perp}$ is the vector component perpendicular to the angular velocity vector (see figure 2.8). If we now choose the point $\mathcal{C}$ such that

instantaneous axis of rotation goes through the point $\mathcal{C}$

$$\overline{\mathcal{A}\mathcal{C}} = (\boldsymbol{\omega} \times \mathbf{v}_{\mathcal{A}})/\omega^2 \tag{2.77}$$

we find algebraically, using 2.74 and the vector triple product, that

$$\begin{aligned} \mathbf{v}_{\mathcal{C}} \;&=\; \mathbf{v}_{\mathcal{A}} + \overline{\mathcal{C}\mathcal{A}} \times \boldsymbol{\omega} = \mathbf{v}_{\mathcal{A}} - [(\boldsymbol{\omega} \times \mathbf{v}_{\mathcal{A}})/\omega^2] \times \boldsymbol{\omega} && (2.78)\\ &= \mathbf{v}_{\mathcal{A}} + [(\mathbf{v}_{\mathcal{A}} \cdot \boldsymbol{\omega})\,\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\,\mathbf{v}_{\mathcal{A}}]/\omega^2 && (2.79)\\ &= \mathbf{v}_{\mathcal{A}} + (\mathbf{v}_{\mathcal{A}} \cdot \mathbf{e}_\omega)\,\mathbf{e}_\omega - \mathbf{v}_{\mathcal{A}} \;=\; \mathbf{v}_{\mathcal{A}\|}. && (2.80) \end{aligned}$$

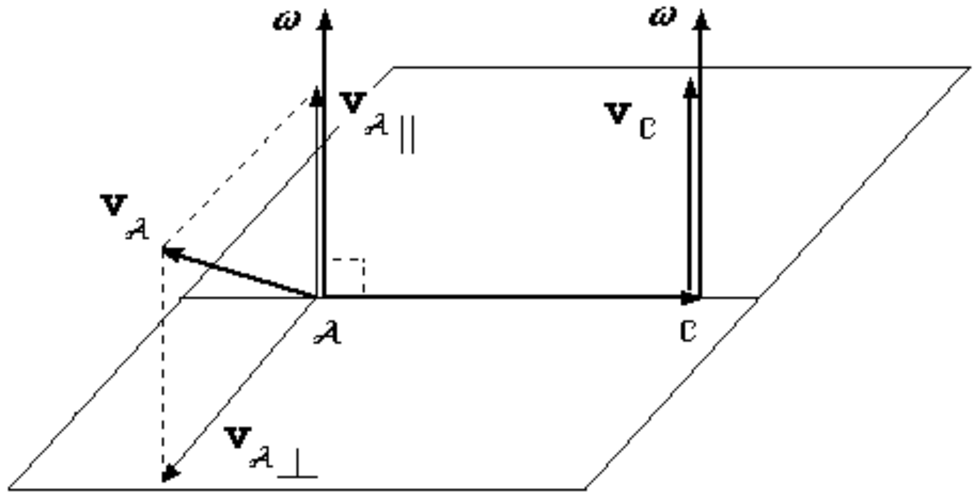


Figure 2.8: In this figure one sees the construction of the point $\mathcal{C}$ in the plane through $\mathcal{A}$ perpendicular to the angular velocity vector $\boldsymbol{\omega}$. The velocity of the point $\mathcal{C}$ is parallel to $\boldsymbol{\omega}$ and so are the velocities of all points of the body, or rigidly connected to the body, on the axis defined by $\mathcal{C}$ and $\boldsymbol{\omega}$. This axis is therefore the instantaneous axis of rotation.

At the point $\mathcal{C}$, therefore, one has $\mathbf{v}_\mathcal{C} = \mathbf{v}_{\mathcal{A}\|} = v_\mathcal{C}\, \mathbf{e}_\omega$. This concludes the proof of the existence of the instantaneous rotation axis.

We have now shown that, at a given instant of time, the velocities of a rigid body can be seen as the combination of a translational motion along the instantaneous axis of rotation, through $\mathcal{C}$ and parallel to $\boldsymbol{\omega}$, and a rotational motion around this axis. The general velocity state of the rigid body is thus said to be that of a *screw*. In the general case the direction and position of this axis changes with time and in such cases it may be very difficult to visualize the screw.

Example 2.3 The position vectors and velocities of the three points $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ of a rigid body have, at some instant of time, been found to be

$$\mathbf{r}_1 = (0,0,1)\ell, \quad \mathbf{v}_1 = (1,1,-3)v, \tag{2.81}$$
$$\mathbf{r}_2 = (0,2,0)\ell, \quad \mathbf{v}_2 = (3,2,-1)v, \tag{2.82}$$
$$\mathbf{r}_3 = (1,0,1)\ell, \quad \mathbf{v}_3 = (1,0,-3)v. \tag{2.83}$$

a) Calculate the angular velocity vector $\boldsymbol{\omega}$.
b) Show that

$$\boldsymbol{\omega} \cdot \mathbf{v}_i = \text{const.} \tag{2.84}$$

for all points $\mathcal{P}_i$ of the body.
c) Calculate the vector $\overline{\mathcal{P}_1\mathcal{C}}$ from $\mathcal{P}_1$ to the point $\mathcal{C}$ on the instantaneous rotation axis (nearest to $\mathcal{P}_1$).
d) Find the translational velocity $v_\mathcal{C}$ along the instantaneous rotation axis.
Solution:
a) The connection formula for the velocities of different points of a rigid body, 2.74, gives us

$$\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{r}_2 - \mathbf{r}_1), \tag{2.85}$$
$$\mathbf{v}_3 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{r}_3 - \mathbf{r}_1). \tag{2.86}$$

Writing out the components of these equations explicitly we get the six equations

$$\begin{array}{rcrcllr} 3v &=& 1v &+& [\omega_y \cdot (-1) &-& \omega_z \cdot 2]\ell \\ 2v &=& 1v &+& [\omega_z \cdot 0 &-& \omega_x \cdot (-1)]\ell \\ -1v &=& -3v &+& [\omega_x \cdot 2 &-& \omega_y \cdot 0]\ell \end{array} \tag{2.87}$$

$$\begin{array}{rcrcrcrl} 1v & = & 1v & + & [\omega_y \cdot 0 & - & \omega_z \cdot 0]\ell \\ 0v & = & 1v & + & [\omega_z \cdot 1 & - & \omega_x \cdot 0]\ell \\ -3v & = & -3v & + & [\omega_x \cdot 0 & - & \omega_y \cdot 1]\ell \end{array} \tag{2.88}$$

The third of these gives $\omega_x = 1\frac{v}{\ell}$, the sixth gives $\omega_y = 0\frac{v}{\ell}$, and the fifth $\omega_z = -1\frac{v}{\ell}$. Thus we get

$$\boldsymbol{\omega} = (1, 0, -1)\frac{v}{\ell} \tag{2.89}$$

for the components of the angular velocity vector.

b) Since

$$\mathbf{v}_i = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_1) \tag{2.90}$$

we get that

$$\mathbf{v}_i \cdot \boldsymbol{\omega} = \mathbf{v}_1 \cdot \boldsymbol{\omega} + [\boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_1)] \cdot \boldsymbol{\omega} = \mathbf{v}_1 \cdot \boldsymbol{\omega}. \tag{2.91}$$

This scalar product thus has the same constant value for all points of the body and this is what was to be shown.

c) Use of formula 2.77 gives us

$$\overline{\mathcal{P}_1\mathcal{C}} = (\boldsymbol{\omega} \times \mathbf{v}_1)/\omega^2 = [(1, 0, -1)\frac{v}{\ell} \times (1, 1, -3)v]/(2v^2/\ell^2) = \frac{1}{2}(1, 0, 1)\ell. \tag{2.92}$$

Since $\overline{\mathcal{P}_1\mathcal{C}} = \mathbf{r}_\mathcal{C} - \mathbf{r}_1$ the coordinates of the point $\mathcal{C}$ (the 'centre of velocity') are given by $\overline{\mathcal{P}_1\mathcal{C}} + \mathbf{r}_1 = (1, 0, 1)\ell + (0, 0, 1)\ell = (1, 0, 2)\ell$.

d) By definition $v_\mathcal{C} = \mathbf{v}_\mathcal{C} \cdot \mathbf{e}_\omega$. This can be written $v_\mathcal{C} = \mathbf{v}_\mathcal{C} \cdot \boldsymbol{\omega}/\omega$ and because of the result in part b the scalar product is the same for all points of the body so we can replace $\mathbf{v}_\mathcal{C}$ with $\mathbf{v}_1$ in this formula. Thus we get

$$v_\mathcal{C} = \mathbf{v}_\mathcal{C} \cdot \mathbf{e}_\omega = \mathbf{v}_1 \cdot \mathbf{e}_\omega = (1, 1, -3)v \cdot (1, 0, -1)/\sqrt{2} = \frac{4}{\sqrt{2}}v = 2\sqrt{2}v. \tag{2.93}$$

This is the answer to the last question of this example. □

2.6.3 Plane Motion of the Rigid Body

In the special case of rigid body motion when the angular velocity vector (and with it the instantaneous rotation axis) has a fixed direction $\mathbf{e}_\omega$ independent of time, the number of degrees of freedom is only four since two of the rotational degrees of freedom then won't be needed. The motion of the body along this fixed direction is then a pure translation. When also this translational velocity is zero ($v_\mathcal{C} = 0$) one has a special case of particular importance, namely that of *plane motion*. The velocities of the body are now all perpendicular to the fixed direction of $\boldsymbol{\omega}$, and the number of degrees of freedom reduces to three: two translational degrees of freedom in the plane perpendicular to $\boldsymbol{\omega}$, and one rotational degree of freedom (the angle of rotation around the fixed axis direction).

For the study of plane motion we will use the convention that the plane with which the velocities are parallel is the xy-plane (or 1,2-plane) and the direction of the angular velocity is the z-direction (or 3-direction). Using this we get from formula 2.77 the explicit formula for the coordinates of the point $\mathcal{C}$

$$\begin{aligned} (x_\mathcal{C} - x_\mathcal{A})\,\mathbf{e}_x + (y_\mathcal{C} - y_\mathcal{A})\,\mathbf{e}_y &= [\omega\,\mathbf{e}_z \times (\dot{x}_\mathcal{A}\,\mathbf{e}_x + \dot{y}_\mathcal{A}\,\mathbf{e}_y)]/\omega^2 \\ &= (\dot{x}_\mathcal{A}\,\mathbf{e}_y - \dot{y}_\mathcal{A}\,\mathbf{e}_x)/\omega. \end{aligned} \tag{2.94}$$

The two components become

$$x_\mathcal{C} = x_\mathcal{A} - \frac{\dot{y}_\mathcal{A}}{\omega} \quad \text{and} \quad y_\mathcal{C} = y_\mathcal{A} + \frac{\dot{x}_\mathcal{A}}{\omega}. \tag{2.95}$$

The point $\mathcal{C}$ *of the body*, or rigidly connected to the body, with coordinates constructed in this way will, for this plane case, have zero velocity. This point is therefore often called the *instantaneous centre of zero velocity.* Note that the geometric point $\mathcal{C}$ as such

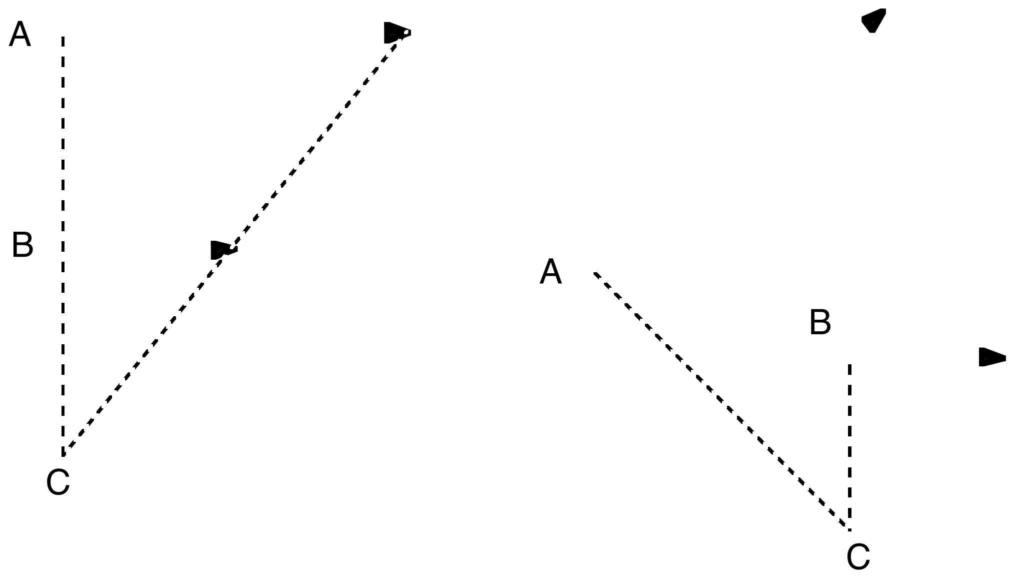


Figure 2.9: Graphic construction of the instantaneous centre of zero velocity $\mathcal{C}$ with the help of two velocities of the body at points $\mathcal{A}$ and $\mathcal{B}$. The diagram on the left is used when the velocity vectors are parallel. The one on the right if they are not.

is not at a fixed point of space, it may move and have some velocity. It is the point of the body, or rigidly connected to the body, which is at $\mathcal{C}$ at the given time which has zero velocity. The point $\mathcal{C}$ of the body really has zero velocity but, in general, it will have non-zero acceleration so it need not remain at rest.

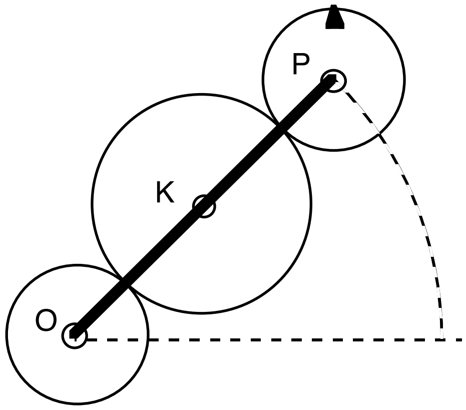
When a wheel rotates around a fixed axis the instantaneous centre $\mathcal{C}$ is at rest at the fixed point where the axis intersects the wheel. When a wheel rolls without slipping on the ground the instantaneous centre is at the point of contact with the ground. The material of the wheel must be at rest at that point since the wheel is not slipping and the ground is at rest.

If we write the connection formula 2.74 for $\mathcal{B}$= $\mathcal{C}$, with $\mathbf{v}_{\mathcal{C}} = \mathbf{0}$, as one of the points we get (in the case of plane motion)

$$\mathbf{v}_{\mathcal{A}} = \overline{\mathcal{AC}} \times \boldsymbol{\omega}. \tag{2.96}$$

This formula tells us that the velocity of a point is always perpendicular to the line from the instantaneous centre $\mathcal{C}$ to the point, and that the magnitude of the velocity grows linearly with the distance from $\mathcal{C}$ ($v_{\mathcal{A}} = |\overline{\mathcal{AC}}|\omega$). These two facts lead to convenient graphic methods for finding $\mathcal{C}$ from two known velocities of the body. This is illustrated in figure 2.9.

Consider the plane motion of a rigid body and attach one coordinate system to the body and let another coordinate system be fixed in space. The instantaneous centre $\mathcal{C}$ will then trace out (plane) curves with time in these two systems. The curve in the body fixed system is called the *polehode curve* and the curve in the space fixed system is called the *herpolehode curve*. At a given instant of time these two curves touch (meet) at the point where $\mathcal{C}$ is at that instant. The touching point of the polehode curve is then at rest since it coincides with $\mathcal{C}$ while the herpolehode curve is fixed in space so all its point are at rest. This means that the plane motion of the rigid body can be viewed as the *rolling without slipping of the polehode curve on the herpolehode curve.* As an example one might consider a circular wheel which rolls without slipping along a straight line. Then the polehode curve is then the circle which defines the circumference of the wheel and the herpolehode curve is the straight line.



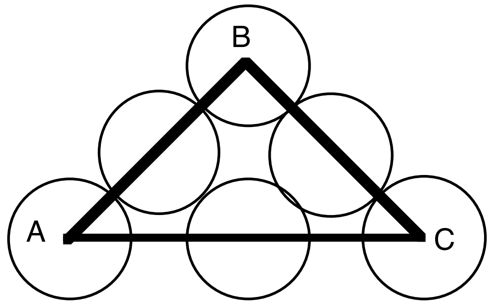


Figure 2.10: The figure on the left refers to problem 2.4. The arrow on the cog-wheel with centre at $\mathcal{P}$ must always point upwards.

Figure 2.11: The figure on the right refers to problem 2.5. Six identical cylindrical rollers are connected by the rods $\mathcal{AB}$, $\mathcal{BC}$, and $\mathcal{CA}$, so that they can rotate about their central axes and roll on each other.

2.7 Problems

Problem 2.1 Use equation 2.65 (on page 33) to find the components of the angular velocity vector ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}}$ along the fixed basis vectors E^{O}.

Problem 2.2 By making measurements on two stereographic photographs of a rigid body taken at times t and $t + \Delta t$ the position and velocity vectors of three points $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, fixed on the body, have been found to have components

$$\begin{aligned} \mathbf{r}_1 &= (1,0,0)\ell, & \mathbf{v}_1 &= (0,1,0)v, \\ \mathbf{r}_2 &= (0,2,3)\ell, & \mathbf{v}_2 &= (0,-5,4)v, \\ \mathbf{r}_3 &= (2,0,1)\ell, & \mathbf{v}_3 &= (2,3,-2)v, \end{aligned}$$

relative to a coordinate system fixed in the body. The velocities are relative to a reference frame in which the camera is fixed but their components have been determined by projection to the body fixed basis. Find the components of the velocity of the origin of the body fixed system and the components of the angular velocity vector of the body in this body fixed system.

Problem 2.3 A homogeneous circular disc of mass m and radius r is rolling and sliding in a vertical position along a straight line on a horizontal table. The uppermost point of the disc has speed v_1 and the geometric contact point with the table has speed v_2 (note that this point is *not* a material point of the disc).
a) Find the angular velocity of the disc.
b) Calculate the angular momentum of the disc with respect to the geometric contact point.

Problem 2.4 Three cog-wheels are connected as shown in figure 2.10. The wheel with centre at the fixed point $\mathcal{O}$ has radius r_o and is fixed so that it can't rotate. A second wheel of radius r_k can roll on the fixed wheel and a third cog-wheel of radius r_p can roll on the second wheel. The centres of the three wheels are connected by an arm which keep them on a straight line at fixed distances (such that they always touch). When the arm is rotated an arrow painted on the outermost (third) wheel is required to point upwards at all times. What relation between the radii r_o, r_k, and r_p must hold if this is to be the case?

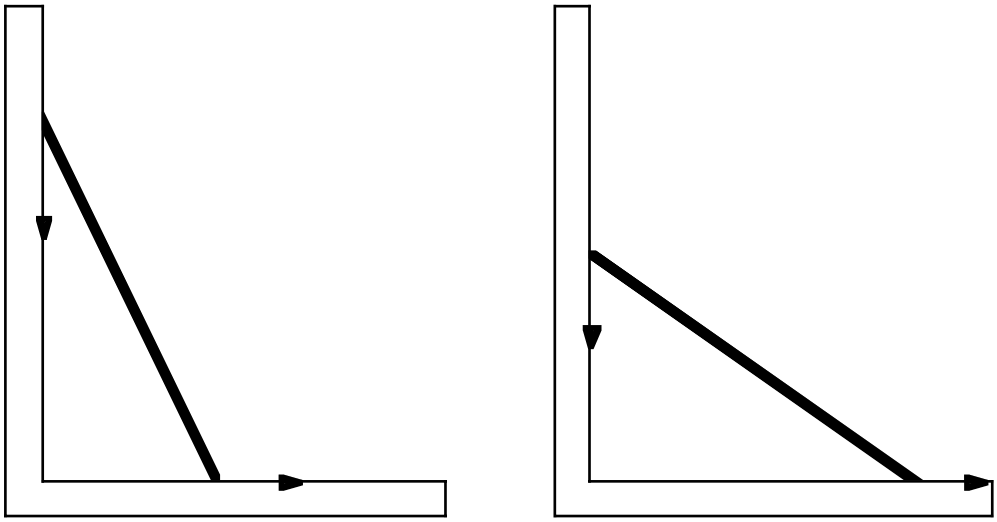


Figure 2.12: This figure refers to problem 2.7. Two different positions of the ladder sliding down are shown. The direction of the velocities of the end points of the ladder are indicated.

Problem 2.5 Six identical cylindrical rollers of radius r are connected by three rods that keep them in contact with parallel axes so that they can roll on each other. The six rollers form a pyramid with three at the base, see figure 2.11. Rods connect their central axes and two, $\mathcal{AB}$ and $\mathcal{BC}$, have length $4r$ while the one along the base, $\mathcal{AC}$, has length $6r$. Use graphical methods to determine the instantaneous centres of zero velocity for all six cylinders when the pyramid rolls to the right with speed v (with no slipping anywhere).

Problem 2.6 The position vectors and velocities of the three points $\mathcal{P}_0, \mathcal{P}_1$, and $\mathcal{P}_2$ of a rigid body have, at some instant of time, been found to be

$$\begin{aligned} \mathbf{r}_0 &= (0,0,0)\ell, \quad & \mathbf{v}_0 &= (0,0,0)v, \\ \mathbf{r}_1 &= (-2,2,\zeta)\ell, \quad & \mathbf{v}_1 &= (0,0,0)v, \\ \mathbf{r}_2 &= (1,3,-2)\ell, \quad & \mathbf{v}_2 &= (-4,2,1)v. \end{aligned}$$

a) Determine, ζ, the unknown z-component of $\mathbf{r}_1$.
b) Find the velocity of the point $\mathcal{P}_3$ with position vector $\mathbf{r}_3 = (-5,0,5)\ell$ at this time.

Problem 2.7 A ladder of length ℓ has been erected against a vertical wall. The coefficient of friction against the horizontal floor is not large enough to keep the ladder in equilibrium so it slides down as shown in figure 2.12. Determine the position of the instantaneous centre of zero velocity $\mathcal{C}$ and find the equation for the curve described by $\mathcal{C}$ when the ladder slides down,
a) with respect to a fixed coordinate system (the herpolehode curve), and
b) with respect to a coordinate system fixed in the ladder (the polehode curve).
c) Investigate how the two curves of a and b move relative to each other as the ladder slides down.

2.8 Hints and Answers

Answer 2.1 In figure 2.5 one finds that the vector $\mathbf{e}_3^{\mathrm{A}}$, which is the same as $\mathbf{e}_3^{\mathrm{B}}$, can be expressed as

$$\mathbf{e}_3^{\mathrm{A}} = \sin\theta \sin\psi \, \mathbf{e}_1^{\mathrm{O}} - \sin\theta \cos\psi \, \mathbf{e}_2^{\mathrm{O}} + \cos\theta \, \mathbf{e}_3^{\mathrm{O}}.$$

Since $\mathbf{e}_1^{\mathrm{B}} = \cos\psi \, \mathbf{e}_1^{\mathrm{O}} + \sin\psi \, \mathbf{e}_2^{\mathrm{O}}$, collecting the components gives the result

$$\begin{aligned} {}^{\mathrm{O}}\omega_x^{\mathrm{A}} &= \dot{\varphi} \sin\theta \, \sin\psi + \dot{\theta} \, \cos\psi, \\ {}^{\mathrm{O}}\omega_y^{\mathrm{A}} &= -\dot{\varphi} \sin\theta \, \cos\psi + \dot{\theta} \, \sin\psi, \\ {}^{\mathrm{O}}\omega_z^{\mathrm{A}} &= \dot{\varphi} \cos\theta + \dot{\psi}. \end{aligned} \tag{2.97}$$

for the fixed frame components of the angular velocity in terms of Euler angles.

Answer 2.2 Use the connection formula 2.74 twice for the pairs $\mathcal{P}_1\mathcal{P}_2$ and $\mathcal{P}_1\mathcal{P}_3$. This gives six equations for the three unknowns $\omega_x, \omega_y, \omega_z$. Using these one finds

$$\boldsymbol{\omega} = (1, 2, 3)\frac{v}{\ell}$$

for the angular velocity vector.

The connection formula also gives

$$\mathbf{v}_{\mathcal{O}} = \mathbf{v}_1 + \overline{\mathcal{O}\mathcal{P}}_1 \times \boldsymbol{\omega} = \mathbf{v}_1 + \mathbf{r}_1 \times \boldsymbol{\omega}$$

and use of this gives the velocity of the body fixed origin

$$\mathbf{v}_{\mathcal{O}} = (0, -2, 2)v$$

when use is made of the components of $\boldsymbol{\omega}$ determined above.

Answer 2.3 The crucial point to notice is that the speed of the geometric contact point is the speed of the centre of the disc. This is best seen by going to a reference frame that moves with the same velocity as the geometric contact point. In such a reference frame the disc will only rotate and thus its centre will be at rest. Consequently the centre must move with the speed of this reference frame i.e. the speed of the geometric contact point. The answers are:
a) $(v_1 - v_2)/r$.
b) $\frac{1}{2}mr(v_1 + v_2)$.

Answer 2.4 Use the fact that the outermost wheel must have purely translational velocity so that all its points have the same velocity. One finds that it is necessary that $r_o = r_p$ while r_k is arbitrary.

Answer 2.5 See figure 2.13. The bottom rollers must have instantaneous centres of zero velocity at the floor since they are assumed to roll without slipping. At the contact points the rollers must have equal velocities for the same reason. The direction of the velocity at the contact point with a roller of the second layer can be found by drawing a line (dashed in the figure) from the instantaneous centre of zero velocity at the floor to the contact point. The (common) velocity at the contact point must be perpendicular to this line. The centres of the rollers must all have the common translational velocity of the whole pyramid. By drawing lines perpendicular to these centre velocities (dashed vertical lines in the figure) we get lines that have to go to the instantaneous centres of zero velocity. In this way one finds that this point is at the top of the cylinders of the second layer. By a line starting at this point and going through the contact point with the top cylinder one gets the velocity of the contact point as a perpendicular to it. Finally one then gets that the instantaneous centre of zero velocity of the top cylinder is at its lowest point.

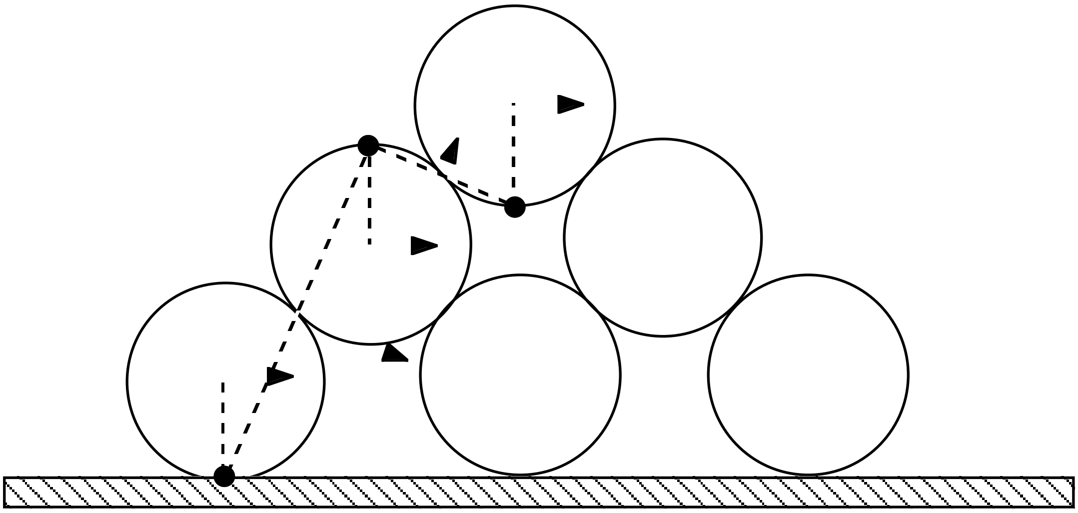


Figure 2.13: This figure shows the instantaneous centres of zero velocity of three representative rollers as black dots and refers to answer 2.5. The dashed lines are lines that are perpendicular to the velocity vectors of certain points with known velocities.

Answer 2.6 Use the connection formula for velocities in a rigid body. The answers should be:
a) $\zeta = -12$,
b) $\mathbf{v}_3 = -\frac{5}{4}(1,7,1)v$.

Answer 2.7
a) The construction of the point $\mathcal{C}$, the instantaneous centre of zero velocity, is indicated graphically in figure 2.14. It is at the intersection of the two dashed lines, since these are perpendicular to the velocities of the end points of the ladder.

If we denote the angle between the ladder and the floor α (this angle is $\pi/2$ when the ladder is vertical and 0 when it is lying on the floor) we easily get that the position of the point $\mathcal{C}$ is given by

$$\overline{\mathcal{OC}}(\alpha) = \ell(\cos\alpha\, \mathbf{e}_x^{\mathrm{O}} + \sin\alpha\, \mathbf{e}_y^{\mathrm{O}}) \tag{2.98}$$

where $\mathcal{O}$ is at the intersection of the floor and the wall and the basis vector $\mathbf{e}_x^{\mathrm{O}}$ points along the floor while $\mathbf{e}_y^{\mathrm{O}}$ points vertically upwards, see figure 2.14. ℓ is the length of the ladder. When $\alpha \in [0, \pi/2]$ this is the equation for a quarter circle with centre at $\mathcal{O}$ and radius ℓ.

b) To get an equation for the position of $\mathcal{C}$ with respect to a system fixed in the ladder we take the origin at the middle of the ladder and denote it $\mathcal{A}$. We let the basis vector $\mathbf{e}_x^{\mathrm{A}}$ point down along the ladder and $\mathbf{e}_y^{\mathrm{A}}$ be perpendicular to it. Note that the point $\mathcal{A}$ is always halfway between $\mathcal{O}$ and $\mathcal{C}$. Therefore

$$\overline{\mathcal{OA}}(\alpha) = \overline{\mathcal{AC}}(\alpha) = \frac{\ell}{2}(\cos\alpha\, \mathbf{e}_x^{\mathrm{O}} + \sin\alpha\, \mathbf{e}_y^{\mathrm{O}}). \tag{2.99}$$

We now wish to express the vector $\overline{\mathcal{AC}}$ in terms of the A-basis. In order to do this we use the expressions

$$\mathbf{e}_x^{\mathrm{A}} = \cos\alpha\, \mathbf{e}_x^{\mathrm{O}} - \sin\alpha\, \mathbf{e}_y^{\mathrm{O}}, \tag{2.100}$$
$$\mathbf{e}_y^{\mathrm{A}} = \sin\alpha\, \mathbf{e}_x^{\mathrm{O}} + \cos\alpha\, \mathbf{e}_y^{\mathrm{O}}, \tag{2.101}$$

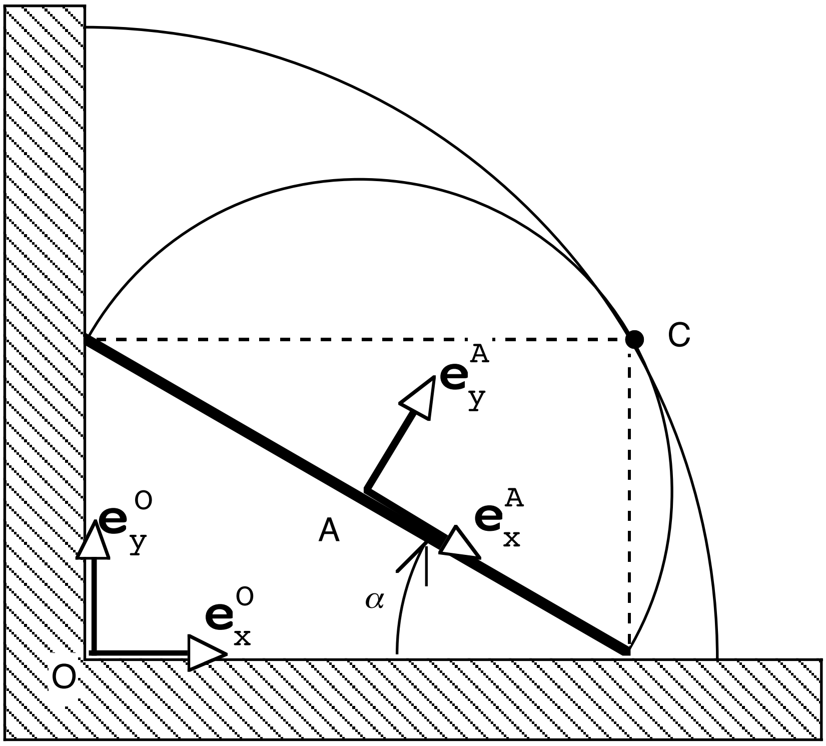


Figure 2.14: This figure shows the construction of the instantaneous centre of zero velocity $\mathcal{C}$ for the ladder of answer 2.7. It also shows the curves that $\mathcal{C}$ traces out relative to a fixed system and relative to a system fixed in the ladder when the ladder slides down. The former curve is the herpolehode curve and is a quarter circle while the latter curve, which is a half circle, is the polehode curve.

which can be read off from figure 2.14. An expression for the position vector of the point $\mathcal{C}$ in the A-system can now be found using

$$\overline{\mathcal{AC}}(\alpha) = (\overline{\mathcal{AC}} \cdot \mathbf{e}_x^{\mathrm{A}})\, \mathbf{e}_x^{\mathrm{A}} + (\overline{\mathcal{AC}} \cdot \mathbf{e}_y^{\mathrm{A}})\, \mathbf{e}_y^{\mathrm{A}} = \frac{\ell}{2}(\cos^2\alpha - \sin^2\alpha)\, \mathbf{e}_x^{\mathrm{A}} + \frac{\ell}{2}(2\sin\alpha\, \cos\alpha)\, \mathbf{e}_y^{\mathrm{A}}. \tag{2.102}$$

Standard formulae for the trigonometric functions can then be used to simplify this to the final form

$$\overline{\mathcal{AC}}(\alpha) = \frac{\ell}{2}[\cos(2\alpha)\, \mathbf{e}_x^{\mathrm{A}} + \sin(2\alpha)\, \mathbf{e}_y^{\mathrm{A}}]. \tag{2.103}$$

When the angle α varies from zero to $\pi/2$ this vector clearly describes a half circle with radius $\ell/2$. This half circle is the polehode curve.

c) From figure 2.14 it should be clear that when the ladder moves and the angle α varies the two curves move as if the polehode curve (the half circle attached to the ladder) rolls, without slipping, on the herpolehode curve (the fixed quarter circle). This is thus in agreement with the general theory of the plane motion of the rigid body.

Chapter 3

Energy of Particle Systems and Bodies

This chapter generalizes the definitions of the concepts of power, kinetic energy, work etc to systems of particles. We discuss the concept of conservative force and its connection with elasticity. Special attention is given to the special formulae obtained for rigid bodies. We discuss which forces that do work and which don't and the role of the law of conservation of energy

3.1 Power and Kinetic Energy

The power delivered by the force $\mathbf{F}_k$ acting on particle k at $\mathbf{r}_k$ is defined as the scalar quantity $P_k = \mathbf{F}_k \cdot \dot{\mathbf{r}}_k$. If there is a system of particles, $k = 1, \ldots, N$, the *power* P is defined as the sum of the P_k: power

$$P = \sum_{k=1}^{N} P_k \ = \ \sum_{k=1}^{N} \mathbf{F}_k \cdot \dot{\mathbf{r}}_k. \tag{3.1}$$

The *kinetic energy* T for a particle system is, likewise, defined as the sum of the kinetic energies, $T_k = \frac{1}{2} m_k \mathbf{v}_k^2$, of the individual particles kinetic energy

$$T = \sum_{k=1}^{N} T_k \ = \ \sum_{k=1}^{N} \frac{1}{2} m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k. \tag{3.2}$$

Since

$$\dot{T}_k = \frac{1}{2} m_k (\ddot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k + \dot{\mathbf{r}}_k \cdot \ddot{\mathbf{r}}_k) = m_k \ddot{\mathbf{r}}_k \cdot \dot{\mathbf{r}} = \mathbf{F}_k \cdot \dot{\mathbf{r}}_k, \tag{3.3}$$

where we have used $m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k$, we find immediately, that, with these definitions, the power and the kinetic energy obey the relationship

$$P = \dot{T} \tag{3.4}$$

in the same way as for a single particle.

We now split the velocities $\mathbf{v}_k$ of the system into centre of mass velocity $\mathbf{v}_{\mathcal{G}}$ and velocity $\mathbf{v}'_k$ relative to the centre of mass, as we did in subsection 1.1.1, so that $\mathbf{v}_k = \mathbf{v}_{\mathcal{G}} + \mathbf{v}'_k$, and use this in the kinetic energy expression. We get

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k \mathbf{v}_k \cdot \mathbf{v}_k = \sum_{k=1}^{N} \frac{1}{2} m_k (\mathbf{v}_{\mathcal{G}} + \mathbf{v}'_k) \cdot (\mathbf{v}_{\mathcal{G}} + \mathbf{v}'_k) \tag{3.5}$$

$$= \frac{1}{2} \mathbf{v}_{\mathcal{G}} \cdot \mathbf{v}_{\mathcal{G}} (\sum_{k=1}^{N} m_k) + \mathbf{v}_{\mathcal{G}} \cdot (\sum_{k=1}^{N} m_k \mathbf{v}'_k) + \sum_{k=1}^{N} \frac{1}{2} m_k \mathbf{v}'_k \cdot \mathbf{v}'_k. \tag{3.6}$$

Because of the centre of mass constraint the sum in the middle term of the last line here, is zero (see equation 1.10), and we thus get

the two parts of the kinetic energy or, König's theorem

$$T = \frac{1}{2}m\mathbf{v}_{\mathcal{G}} \cdot \mathbf{v}_{\mathcal{G}} + \sum_{k=1}^{N} \frac{1}{2} m_k \mathbf{v}'_k \cdot \mathbf{v}'_k = \frac{1}{2}mv_{\mathcal{G}}^2 + T'. \tag{3.7}$$

The kinetic energy can thus be thought of as having two parts, one due to the translation of the centre of mass and the rest which comes from the internal motions of the system relative to its own centre of mass system. In Germanic literature this formula is often referred to as 'König's' theorem.

3.1.1 The Kinetic Energy of Rigid Bodies

We now assume that the particle system is a rigid body and calculate the kinetic energy. According to the connection formula 2.74 for the velocities of the points of a rigid body we have

$$\mathbf{v}_k = \mathbf{v}_{\mathcal{G}} + \mathbf{v}'_k = \mathbf{v}_{\mathcal{G}} + \boldsymbol{\omega} \times \overline{\mathcal{G}\mathcal{P}}_k = \mathbf{v}_{\mathcal{G}} + \boldsymbol{\omega} \times \mathbf{r}'_k. \tag{3.8}$$

The velocity in the centre of mass system is thus given by

$$\mathbf{v}'_k = \boldsymbol{\omega} \times \mathbf{r}'_k. \tag{3.9}$$

When this is inserted into the expression T' for the internal part of the kinetic energy we get

$$T' = \sum_{k=1}^{N} \frac{1}{2} m_k \mathbf{v}'_k \cdot \mathbf{v}'_k = \sum_{k=1}^{N} \frac{1}{2} m_k (\boldsymbol{\omega} \times \mathbf{r}'_k) \cdot (\boldsymbol{\omega} \times \mathbf{r}'_k). \tag{3.10}$$

To evaluate this further we must find a formula for the scalar product and this can be done as follows

$$(\boldsymbol{\omega} \times \mathbf{r}'_k) \cdot (\boldsymbol{\omega} \times \mathbf{r}'_k) = |\boldsymbol{\omega} \times \mathbf{r}'_k|^2 = |\boldsymbol{\omega}|^2 |\mathbf{r}'_k|^2 \sin^2[\boldsymbol{\omega}, \mathbf{r}'_k] = \tag{3.11}$$

$$|\boldsymbol{\omega}|^2 |\mathbf{r}'_k|^2 (1 - \cos^2[\boldsymbol{\omega}, \mathbf{r}'_k]) = |\boldsymbol{\omega}|^2 |\mathbf{r}'_k|^2 - |\boldsymbol{\omega}|^2 |\mathbf{r}'_k|^2 \cos^2[\boldsymbol{\omega}, \mathbf{r}'_k] = \tag{3.12}$$

$$= (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}'_k \cdot \mathbf{r}'_k) - (\boldsymbol{\omega} \cdot \mathbf{r}'_k)^2. \tag{3.13}$$

The internal kinetic energy T' thus becomes

$$T' = \frac{1}{2}(\sum_{k=1}^{N} m_k \mathbf{r}'_k \cdot \mathbf{r}'_k)(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) - \frac{1}{2} \sum_{k=1}^{N} m_k (\boldsymbol{\omega} \cdot \mathbf{r}'_k)^2. \tag{3.14}$$

If we now introduce the components of the vectors in some basis, $\boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z$, and $\mathbf{r}'_k = x'_k \mathbf{e}_x + y'_k \mathbf{e}_y + z'_k \mathbf{e}_z$, we find that

internal kinetic energy of rigid body

$$T' = \frac{1}{2} J_x^{\mathcal{G}} \omega_x^2 + \frac{1}{2} J_y^{\mathcal{G}} \omega_y^2 + \frac{1}{2} J_z^{\mathcal{G}} \omega_z^2 - D_{xy}^{\mathcal{G}} \omega_x \omega_y - D_{xz}^{\mathcal{G}} \omega_x \omega_z - D_{yz}^{\mathcal{G}} \omega_y \omega_z \tag{3.15}$$

moment of inertia

where

$$J_x^{\mathcal{G}} \equiv \sum_{k=1}^{N} m_k (y'^2_k + z'^2_k), \tag{3.16}$$

product of inertia

$$D_{xy}^{\mathcal{G}} \equiv \sum_{k=1}^{N} m_k x'_k y'_k \tag{3.17}$$

and the other moments and products of inertia are obtained by cyclic replacements $x \to y \to z \to x$. We see, first of all, that if the body does not rotate then $T' = 0$ since $\boldsymbol{\omega} = \mathbf{0}$. This is the reason why a rigid body with only translational motion can be treated like a particle. One should also note that the moments and products of inertia (with respect to axes through the centre of mass) will, in general, depend on time unless the basis vectors (or equivalently, the direction of the coordinates axes) are taken as fixed in the body. When this is done these quantities become constant.

In case one point $\mathcal{C}$ of the rigid body has zero velocity, permanently due to some constraint, or instantaneously as the instantaneous centre of zero velocity that exists for plane motion, we can write the velocities of the particles as

$$\mathbf{v}_k = \boldsymbol{\omega} \times \overline{\mathcal{C}\mathcal{P}}_k \tag{3.18}$$

according to the connection formula. If we put the origin of the coordinate system at $\mathcal{C}$ this takes the form

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k. \tag{3.19}$$

The (complete) kinetic energy T now takes the form

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \mathbf{v}_k \cdot \mathbf{v}_k = \sum_{k=1}^{N} \frac{1}{2} m_k (\boldsymbol{\omega} \times \mathbf{r}_k) \cdot (\boldsymbol{\omega} \times \mathbf{r}_k). \tag{3.20}$$

This is formula is identical in form to 3.10 so the same kind of calculation gives

kinetic energy of rigid body with pure rotation

$$T = \frac{1}{2} J_x^{\mathcal{C}} \omega_x^2 + \frac{1}{2} J_y^{\mathcal{C}} \omega_y^2 + \frac{1}{2} J_z^{\mathcal{C}} \omega_z^2 - D_{xy}^{\mathcal{C}} \omega_x \omega_y - D_{xz}^{\mathcal{C}} \omega_x \omega_z - D_{yz}^{\mathcal{C}} \omega_y \omega_z. \tag{3.21}$$

This is thus the kinetic energy of a rigid body whose motion is a pure rotation around the point $\mathcal{C}$. The moments and products of inertia must now be calculated with respect to axes through the point $\mathcal{C}$. As above, they are, in general, constants only if the components of the vectors $\boldsymbol{\omega}$ and $\mathbf{r}_k$ are taken with respect to a basis that rotates with the body.

3.1.2 Matrix Form of the Rigid Body Kinetic Energy

Both expressions 3.15 and 3.21 are of the same algebraic form. They are 'quadratic forms' in the components of the angular velocity vector. We will now show how they can be written on a more compact form using matrix notation.

Introducing the notation of equation 2.18 we consider the angular velocity vector as the product of a row matrix of components and a column matrix of basis vectors

$$\boldsymbol{\omega} = \mathsf{w}\,\mathsf{E} = (\omega_x\ \omega_y\ \omega_z) \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}. \tag{3.22}$$

If we now introduce the three by three symmetric matrix, the 'inertia tensor',

$$\mathsf{J}^{\mathcal{G}} = \begin{pmatrix} J_x^{\mathcal{G}} & -D_{xy}^{\mathcal{G}} & -D_{xz}^{\mathcal{G}} \\ -D_{yx}^{\mathcal{G}} & J_y^{\mathcal{G}} & -D_{yz}^{\mathcal{G}} \\ -D_{zx}^{\mathcal{G}} & -D_{zy}^{\mathcal{G}} & J_z^{\mathcal{G}} \end{pmatrix}. \tag{3.23}$$

we see that the kinetic energy 3.15 can be written as the matrix product

$$T' = \frac{1}{2}(\omega_x\ \omega_y\ \omega_z) \begin{pmatrix} J_x^{\mathcal{G}} & -D_{xy}^{\mathcal{G}} & -D_{xz}^{\mathcal{G}} \\ -D_{yx}^{\mathcal{G}} & J_y^{\mathcal{G}} & -D_{yz}^{\mathcal{G}} \\ -D_{zx}^{\mathcal{G}} & -D_{zy}^{\mathcal{G}} & J_z^{\mathcal{G}} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \frac{1}{2}\,\mathsf{w}\,\mathsf{J}^{\mathcal{G}}\,\mathsf{w}^T. \tag{3.24}$$

The superscript $\mathcal{G}$ on the inertia tensor indicates that it depends on the the choice of origin of the coordinate system used to calculate it. This point is, for both the case of equation 3.15 ($\mathcal{G}$) and of equation 3.21 ($\mathcal{C}$) assumed to be fixed in the body. One should note that the quantities of the 'inertia tensor', as defined here, also depends on the basis E which defines the directions of the coordinate axes. Only if these also are fixed in the body are the matrix elements of the matrix in equation 3.23 guaranteed to be constants (independent of time) depending only on the shape of the body. We'll come back to these matters later.

3.1.3 Rigid Body Kinetic Energy for Fixed Axis Direction

We now assume that the angular velocity vector $\boldsymbol{\omega}$ has a direction which is fixed in space so that

$$\boldsymbol{\omega} = \omega_z\,\mathbf{e}_z = \dot{\varphi}\,\mathbf{e}_z, \tag{3.25}$$

where $\mathbf{e}_z$ has been chosen in this fixed direction. The two formulae for the kinetic energy, 3.15 and 3.21 above, reduce to simple forms. The internal energy becomes

$$T' = \frac{1}{2} J_z^{\mathcal{G}} \dot{\varphi}^2. \tag{3.26}$$

Pure rotation around $\mathcal{C}$ gives the kinetic energy

$$T = \frac{1}{2} J_z^{\mathcal{C}} \dot{\varphi}^2. \tag{3.27}$$

It can be instructive to derive this last result directly from the definition using cylindrical coordinates. Since, with the origin at $\mathcal{C}$,

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k = \dot{\varphi}\,\mathbf{e}_z \times (\rho_k\,\mathbf{e}_\rho + z_k\,\mathbf{e}_z) = \dot{\varphi}\rho_k\,\mathbf{e}_\varphi, \tag{3.28}$$

we get

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k \mathbf{v}_k \cdot \mathbf{v}_k = \sum_{k=1}^{N} \frac{1}{2} m_k (\dot{\varphi}\rho_k\,\mathbf{e}_\varphi) \cdot (\dot{\varphi}\rho_k\,\mathbf{e}_\varphi) \tag{3.29}$$

$$= \sum_{k=1}^{N} \frac{1}{2} m_k \dot{\varphi}^2 \rho_k^2 = \frac{1}{2} (\sum_{k=1}^{N} m_k \rho_k^2) \dot{\varphi}^2 \;=\; \frac{1}{2} J_z^{\mathcal{C}} \dot{\varphi}^2. \tag{3.30}$$

There is a connection between formulae 3.26 and 3.27 which is derived in the example below.

Example 3.1 Use the parallel axis theorem for moments of inertia to show that in the case of plane motion

$$\frac{1}{2} J_z^{\mathcal{C}} \dot{\varphi}^2 = \frac{1}{2} m v_{\mathcal{G}}^2 + \frac{1}{2} J_z^{\mathcal{G}} \dot{\varphi}^2. \tag{3.31}$$

and thus verify equation 3.7 ($T = \frac{1}{2} m v_{\mathcal{G}}^2 \; + T'$). The point $\mathcal{C}$ is assumed to be in the same xy-plane as $\mathcal{G}$.
Solution: The parallel axis theorem gives us

$$J_z^{\mathcal{C}} = J_z^{\mathcal{G}} + m|\overline{\mathcal{CG}}|^2 \tag{3.32}$$

so we get

$$\frac{1}{2} J_z^{\mathcal{C}} \dot{\varphi}^2 = \frac{1}{2} (J_z^{\mathcal{G}} + m|\overline{\mathcal{CG}}|^2) \dot{\varphi}^2 = \frac{1}{2} J_z^{\mathcal{G}} \dot{\varphi}^2 + \frac{1}{2} m (|\overline{\mathcal{CG}}| \dot{\varphi})^2. \tag{3.33}$$

But the connection formula for velocities gives us

$$\mathbf{v}_{\mathcal{G}} = \boldsymbol{\omega} \times \overline{\mathcal{CG}} \tag{3.34}$$

and in our plane situation this means that $|\mathbf{v}_{\mathcal{G}}| = |\boldsymbol{\omega}||\overline{\mathcal{CG}}| = |\dot{\varphi}||\overline{\mathcal{CG}}|$ so that

$$\frac{1}{2} J_z^{\mathcal{C}} \dot{\varphi}^2 = \frac{1}{2} J_z^{\mathcal{G}} \dot{\varphi}^2 + \frac{1}{2} m v_{\mathcal{G}}^2 \tag{3.35}$$

and this is what was to be shown. Note how the parallel axis theorem and the connection formula for the velocities act together to give this result. □

3.2 The Work of External and Internal Forces

When the particles of an N-particle system, $\{\{m_k, \mathbf{r}_k\};\ k = 1, \ldots, N\}$, move through space the forces acting on them do *work*. We define the work done on the system between times $t = t_1$ and $t = t_2$ as the time integral of the power delivered to the system during this time and denote it by

$$W_{1,2} \equiv \int_{t_1}^{t_2} P(t)\, dt. \tag{3.36}$$

Since $P = \dot{T}$ we get, just as for a single particle, that

work as increase of kinetic energy

$$W_{1,2} = T(t_2) - T(t_1), \tag{3.37}$$

so the work done on the system is equal to the increase in kinetic energy.

Using the definition, equation 3.1, we can rewrite 3.36 in the form

$$W_{1,2} = \int_{t_1}^{t_2} (\sum_{k=1}^{N} \mathbf{F}_k \cdot \dot{\mathbf{r}}_k)\, dt = \sum_{k=1}^{N} \int_{C_k(1,2)} \mathbf{F}_k \cdot d\mathbf{r}_k, \tag{3.38}$$

where we have introduced the line integral $\int_{C_k(1,2)} \mathbf{F}_k \cdot d\mathbf{r}_k$ along the path $C_k(1,2) : t \in [t_1, t_2] \to \mathbf{r}_k(t)$ followed by particle k between the points $\mathbf{r}_k(t_1)$ and $\mathbf{r}_k(t_2)$ during the time from t_1 to t_2. The work done on the particle system can thus also be expressed as such a sum of the work on the individual particles expressed in terms of line integrals. To simplify the notation we shall now consider this sum to be a single line integral along the curve $(x_1(t), y_1(t), z_1(t), x_2(t), \ldots, x_N(t), y_N(t), z_N(t))$ through the $3N$-dimensional *configuration space* of the system. We denote this $3N$-dimensional curve, traversed between t_1 and t_2, by $C^N(1,2)$ and thus write the work on the particle system as the single configuration space line integral

work as line integral along configuration space curve

$$W_{1,2} = \int_{C^N(1,2)} \sum_{k=1}^{N} \mathbf{F}_k \cdot d\mathbf{r}_k. \tag{3.39}$$

This is in principle a different, time-independent, way of calculating the work on the system. In some cases the forces acting are such that one really can get an explicit expression for the work done in this way and this new way of calculating it can then be combined to with the result in terms of kinetic energy increase to yield the powerful law of conservation of mechanical energy. This is possible when all forces that do work on the system are *conservative*. Unfortunately all the forces acting on systems, or bodies, in the real world rarely are conservative. There are, however, real systems that come sufficiently close to this idealization for the theory to be of interest, and since *part* of the work quite often is done by conservative forces, this type of analysis is of general importance.

We now calculate the work done on a system of particles by the external and internal forces acting on the particles. In the momentum principle and angular momentum principle only the external forces contribute but for work and energy this is *not* the case. To investigate this in more detail we introduce the notation used in subsection 1.1.2: the force on one of the particles is the sum of two contributions

$$\mathbf{F}_k = \mathbf{F}_k^{\mathrm{e}} + \mathbf{F}_k^{\mathrm{i}}, \tag{3.40}$$

where the superscript 'e' stands for external and 'i' stands for internal, and the

$$\text{force on } k \text{ from } j = \mathbf{f}_{kj} \tag{3.41}$$

so that the internal force itself is the sum

$$\mathbf{F}^{\mathrm{i}}_k = \sum_{j=1}^{N} \mathbf{f}_{kj}. \tag{3.42}$$

We assume that Newton's third law $\mathbf{f}_{jk} = -\mathbf{f}_{kj}$ holds. The total work of formula 3.39 can now be analyzed into external and internal contributions as follows

$$W_{1,2} \;=\; \int_{C^N(1,2)} \sum_{k=1}^{N} \mathbf{F}^{\mathrm{e}}_k \cdot d\mathbf{r}_k \;+\; \int_{C^N(1,2)} \sum_{k=1}^{N}\sum_{j=1}^{N} \mathbf{f}_{kj} \cdot d\mathbf{r}_k \;\equiv\; W^{\mathrm{e}}_{1,2} + W^{\mathrm{i}}_{1,2}. \tag{3.43}$$

The internal part can be transformed as the following steps indicate

$$W^{\mathrm{i}}_{1,2} = \int_{C^N(1,2)} \sum_{k=1}^{N}\sum_{j=1}^{N} \mathbf{f}_{kj} \cdot d\mathbf{r}_k = \int_{C^N(1,2)} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot d\mathbf{r}_k \tag{3.44}$$

$$= \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} (\mathbf{f}_{kj} \cdot d\mathbf{r}_k + \mathbf{f}_{jk} \cdot d\mathbf{r}_j) = \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot (d\mathbf{r}_k - d\mathbf{r}_j) \tag{3.45}$$

$$= \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot d(\mathbf{r}_k - \mathbf{r}_j) \;=\; \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot d\mathbf{r}_{kj} = W^{\mathrm{i}}_{1,2} \tag{3.46}$$

Here we have used only the 'anti-symmetry' part of Newton's third law. We now express the working external and internal forces as sums of a conservative part, which is the negative gradient of a potential energy, and a non-conservative part:

$$\mathbf{F}^{\mathrm{e}}_k = -\nabla_k \Phi_k(\mathbf{r}_k) + \mathbf{F}^{\mathrm{en}}_k, \tag{3.47}$$

$$\mathbf{f}_{kj} = -\nabla_{kj} \Phi_{kj}(\mathbf{r}_{kj}) + \mathbf{f}^{\mathrm{n}}_{kj}. \tag{3.48}$$

We can now split the external work into contributions from decrease in potential energy and non-conservative work as follows:

$$W^{\mathrm{e}}_{1,2} = \int_{C^N(1,2)} \sum_{k=1}^{N} -[\nabla_k \Phi_k(\mathbf{r}_k)] \cdot d\mathbf{r}_k + \int_{C^N(1,2)} \sum_{k=1}^{N} \mathbf{F}^{\mathrm{en}}_k \cdot d\mathbf{r}_k \tag{3.49}$$

$$= \sum_{k=1}^{N} -[\Phi_k(\mathbf{r}_k(t_2)) - \Phi_k(\mathbf{r}_k(t_1))] + W^{\mathrm{en}}_{1,2} \tag{3.50}$$

work expressed as decrease of potential energy plus non-conservative work

A similar treatment of the internal work gives

$$W^{\mathrm{i}}_{1,2} = \frac{1}{2} \sum_{k,j=1}^{N} -[\Phi_{kj}(\mathbf{r}_{kj}(t_2)) - \Phi_{kj}(\mathbf{r}_{kj}(t_1))] + W^{\mathrm{in}}_{1,2} \tag{3.51}$$

To be able to write the conservative work in a more compact way we define the external and internal *total potential energies* respectively as

$$\Phi^{\mathrm{e}}(i) \equiv \sum_{k=1}^{N} \Phi_k(\mathbf{r}_k(t_i)), \tag{3.52}$$

$$\Phi^{\mathrm{i}}(i) \equiv \frac{1}{2} \sum_{k,j=1}^{N} \Phi_{kj}(\mathbf{r}_{kj}(t_i)). \tag{3.53}$$

If we now collect all results we have the following expression for the work done on a particle system

the four types of work

$$W_{1,2} = W^{\mathrm{e}}_{1,2} + W^{\mathrm{i}}_{1,2} \;=\; \{-[\Phi^{\mathrm{e}}(2) - \Phi^{\mathrm{e}}(1)] + W^{\mathrm{en}}_{1,2}\} \;+\; \{-[\Phi^{\mathrm{i}}(2) - \Phi^{\mathrm{i}}(1)] + W^{\mathrm{in}}_{1,2}\}. \tag{3.54}$$

The total work can thus be analyzed into these four contributions from external and internal, conservative and non-conservative forces. The parts of the work which depend only on the change in total potential energies have the fundamental property that they only depend on the positions of the particles, i.e. on the position of the system in configuration space, and not on the path followed between the positions. If the system returns to the original position no net work has been done by the conservative forces. The internal total potential energy $\Phi^{\rm i}$ has the interesting property that it only depends on the relative positions and thus not on the position and orientation of the system as a whole. For rigid motions, translations and rotations, $\Phi^{\rm i}$ does not change; only changes in the shape of the system changes $\Phi^{\rm i}$. This part of the energy of a body is the *elastic* energy. It is large when the body has been deformed but can be recovered as mechanical work when the body returns to its un-loaded shape.

Example 3.2 Calculate the external total potential energy $\Phi^{\rm e}$ due to the weights of the particles of the system.
Solution: If we denote the acceleration due to gravity by $\mathbf{g}$ we have that

$$\Phi_k(\mathbf{r}_k) = -m_k\mathbf{g}\cdot\mathbf{r}_k \tag{3.55}$$

since then $-\nabla_k\Phi_k(\mathbf{r}_k) = m_k\mathbf{g}$ is the weight of particle k. We now get

$$\Phi^{\rm e} = \sum_{k=1}^{N} -m_k\mathbf{g}\cdot\mathbf{r}_k = -m\mathbf{g}\cdot\frac{\sum_{k=1}^{N} m_k\mathbf{r}_k}{\sum_{k=1}^{N} m_k} = -m\mathbf{g}\cdot\mathbf{r}_{\mathcal{G}} = mgh. \tag{3.56}$$

The total potential energy is thus completely determined by the total mass m, the acceleration due to gravity g, and the vertical projection of the centre of mass position vector. □

Example 3.3 Find the internal total potential energy $\Phi^{\rm i}$ of an elastic rubber-band assuming that the force required to stretch the band the length $\Delta\ell$ is given by $F = k\,\Delta\ell$, where k is a constant.
Solution: The work done by the internal forces during the stretching is easily found by integration and the result is that

$$\Phi^{\rm i} = \begin{cases} \frac{1}{2}k(\Delta\ell)^2 & \text{for } \Delta\ell > 0 \\ 0 & \text{for } \Delta\ell \le 0 \end{cases}. \tag{3.57}$$

Only stretching leads to internal work since the flexible rubber-band does not take up compressive forces. □

We have found two different expressions for the total work done on a particle system, namely in terms of kinetic energy in equation 3.37, and that of equation 3.54. If these two expressions are set equal we can rearrange the equation and find that

$$[T(2) + \Phi^{\rm e}(2) + \Phi^{\rm i}(2)] - [T(1) + \Phi^{\rm e}(1) + \Phi^{\rm i}(1)] = W_{1,2}^{\rm en} + W_{1,2}^{\rm in}. \tag{3.58}$$

One defines the total mechanical energy E_m to be the kinetic plus the total potential energies:

mechanical energy

$$E_m \equiv T + \Phi^{\rm e} + \Phi^{\rm i}. \tag{3.59}$$

The previous formula now states that the change in the mechanical energy is equal to the work of, external and internal, non-conservative forces. One can often arrange the external non-conservative forces to be small or negligible. Deformations of bodies, however, always entail some amount of non-conservative internal work. The best chances for mechanical energy conservation (E_m =const.) therefore come about when deformations are negligible. In the following subsection we will find that the internal work is zero for rigid bodies, but one should remember that complete rigidity is an idealization which is only approximated by nature.

Example 3.4 Consider a pole-vaulter with an elastic glass-fiber pole as a particle system and analyze the work on the system in terms of the four kinds of work in equation 3.54.

Solution: When the athlete starts his run work is done by his muscles and this must be considered as coming from internal non-conservative forces so we have the type W^{in}. At the end of the run this work has produced a lot of kinetic energy. The pole-vaulter now transforms this kinetic energy plus some more muscle work from the arms into elastic deformation energy of the pole i.e. type Φ^{i}. The pole then straightens out again and raises the athlete vertically upwards. The internal potential energy of the pole then is transformed into external potential energy Φ^{e} of the gravitational field. At the highest point of the jump all energy has become external potential energy. When he falls down it again becomes kinetic energy. This kinetic energy is then destroyed by negative work by (mainly) external non-conservative forces W^{en} from the shock absorber that bolsters his impact with the ground. □

3.2.1 The Work on a Rigid Body

In this sub-section we specialize the discussion above to rigid bodies. It turns out that the assumption of rigidity leads to considerable simplification, in particular the internal forces do no work in a rigid body.

One of the reasons for the simplification is that the displacements $d\mathbf{r}_k$ of the particles of the rigid body are connected due to the connection formula for the velocities, equation 2.74. We can write this formula $\mathbf{v}_k = \mathbf{v}_{\mathcal{A}} + (\mathbf{r}_{\mathcal{A}} - \mathbf{r}_k) \times \boldsymbol{\omega}$, where $\mathcal{A}$ is an arbitrary point fixed in the body. If we now multiply this formula by dt we get

$$d\mathbf{r}_k = d\mathbf{r}_{\mathcal{A}} + (\mathbf{r}_{\mathcal{A}} - \mathbf{r}_k) \times \delta\boldsymbol{\phi}, \tag{3.60}$$

where $\delta\boldsymbol{\phi}$ is the infinitesimal rotation vector. We now insert this in the expression for the external work as given by equation 3.43, and get

$$W^{\mathrm{e}}_{1,2} = \int_{C^N(1,2)} \sum_{k=1}^{N} \mathbf{F}^{\mathrm{e}}_k \cdot d\mathbf{r}_k = \int_{C^N(1,2)} \sum_{k=1}^{N} \mathbf{F}^{\mathrm{e}}_k \cdot (d\mathbf{r}_{\mathcal{A}} + (\mathbf{r}_{\mathcal{A}} - \mathbf{r}_k) \times \delta\boldsymbol{\phi}) = \tag{3.61}$$

$$\int_{C^N(1,2)} \sum_{k=1}^{N} \{\mathbf{F}^{\mathrm{e}}_k \cdot d\mathbf{r}_{\mathcal{A}} + \mathbf{F}^{\mathrm{e}}_k \cdot [(\mathbf{r}_{\mathcal{A}} - \mathbf{r}_k) \times \delta\boldsymbol{\phi}]\} = \int_{C^N(1,2)} (\mathbf{F}^{\mathrm{e}} \cdot d\mathbf{r}_{\mathcal{A}} + \mathbf{M}^{\mathrm{e}}_{\mathcal{A}} \cdot \delta\boldsymbol{\phi}). \tag{3.62}$$

Here we have rearranged the scalar triple product and used the definitions $\mathbf{F}^{\mathrm{e}} = \sum_{k=1}^{N} \mathbf{F}^{\mathrm{e}}_k$ and

$$\mathbf{M}^{\mathrm{e}}_{\mathcal{A}} = \sum_{k=1}^{N} (\mathbf{r}_k - \mathbf{r}_{\mathcal{A}}) \times \mathbf{F}^{\mathrm{e}}_k. \tag{3.63}$$

For the work of the internal forces we only have to replace the superscript 'e' above with i'. This tells us that $W^{\mathrm{i}}_{1,2} = \int_{C^N(1,2)} (\mathbf{F}^{\mathrm{i}} \cdot d\mathbf{r}_{\mathcal{A}} + \mathbf{M}^{\mathrm{i}}_{\mathcal{A}} \cdot \delta\boldsymbol{\phi})$ so, if we assume that the total internal force and moment are zero (see equations 1.23 and 1.32), we get that the work of the internal forces

$$W^{\mathrm{i}}_{1,2} = 0 \qquad \text{for a rigid body.} \tag{3.64}$$

work done on rigid body by external force and moment

Summarizing we have now found that the work on a rigid body is given by

$$W_{1,2} = \int_{C^N(1,2)} (\mathbf{F}^{\mathrm{e}} \cdot d\mathbf{r}_{\mathcal{A}} + \mathbf{M}^{\mathrm{e}}_{\mathcal{A}} \cdot \delta\boldsymbol{\phi}), \tag{3.65}$$

i.e. as a line-integral depending on the external force and the external moment.

It can be instructive to calculate the internal work directly in terms of inter-particle forces to see in more detail why (or when) it is zero. To do this we need an expression for $d\mathbf{r}_{kj} = d(\mathbf{r}_k - \mathbf{r}_j)$, and to get that we put $\mathbf{r}_{\mathcal{A}} = \mathbf{r}_j$ in the connection formula 3.60. Some rearrangement then gives

$$d\mathbf{r}_{kj} = \delta\boldsymbol{\phi} \times \mathbf{r}_{kj}. \tag{3.66}$$

We now use this in formula 3.46 for the internal work and get

$$W^{\mathrm{i}}_{1,2} = \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot d\mathbf{r}_{kj} = \int_{C^N(1,2)} \frac{1}{2} \sum_{k,j=1}^{N} \mathbf{f}_{kj} \cdot (\delta\boldsymbol{\phi} \times \mathbf{r}_{kj}). \tag{3.67}$$

Just as in subsection 1.1.2 we now see that this is manifestly zero only if assume that the internal forces are parallel to the inter-particle vectors: $\mathbf{f}_{kj} \parallel \mathbf{r}_{kj}$ since then $\mathbf{f}_{kj} \perp d\mathbf{r}_{kj}$. Now that we have established that the internal forces do not contribute to rigid body work we will not bother with the superscripts 'e' on forces and moments any more.

The notation $C^N(1,2)$ which we introduced for the path through configuration space of the N-particle system is not well suited for the rigid body case, since in this case the number of particles is immaterial; only the change in the six degrees of freedom $\mathbf{r}_{\mathcal{A}}, \boldsymbol{\phi}$ with time matters. The first part of the work in equation 3.65 is thus an ordinary line integral along the curve on which the point $\mathcal{A}$ moves through space, but the second part is a curve in the space of orientation (rotation) parameters (e.g. Euler angles). If we take the centre of mass as base point we can thus express the work in the form

$$W_{1,2} = \int_{C^{\mathcal{G}}(1,2)} \mathbf{F} \cdot d\mathbf{r}_{\mathcal{G}} + \int_{C^R(1,2)} \mathbf{M}_{\mathcal{G}} \cdot \delta\boldsymbol{\phi}. \tag{3.68}$$

If the external force is the gradient of a total potential energy, as is the case for gravity, the first of these two parts can be expressed with the help of decrease in potential energy in the way discussed above. Also, if there is only gravity, then by the definition of centre of mass as the point of application of the resultant, we have $\mathbf{M}_{\mathcal{G}} = \mathbf{0}$, so that the second, rotational part of the work gives zero. One then gets

$$W^{\mathrm{grav}}_{1,2} = \int_{C^{\mathcal{G}}(1,2)} -\nabla_{\mathcal{G}}\Phi(\mathbf{r}_{\mathcal{G}}) \cdot d\mathbf{r}_{\mathcal{G}} = -[\Phi(\mathbf{r}_{\mathcal{G}}(t_2)) - \Phi(\mathbf{r}_{\mathcal{G}}(t_1))] \tag{3.69}$$

for the work of gravity on a rigid body, the same as for any particle system. Here $\Phi = \Phi^{\mathrm{e}}$ is given by equation 3.56.

Should the moment, on the other hand, be non-zero there is, in general, no hope of representing the rotational part of the work with a 'rotational' potential energy since the integration element $\delta\boldsymbol{\phi}$ is not the differential of a vector. The exception is when there is a fixed direction, $\mathbf{e}_z$, of the rotation axis. In this case one has $\delta\boldsymbol{\phi} = d\phi\,\mathbf{e}_z$ so the rotational part of the work is simply

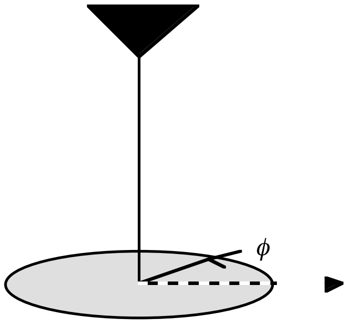
$$W^{\mathrm{rot}}_{1,2} = \int_{C^R(1,2)} M_{\mathcal{A}z}\, d\phi. \tag{3.70}$$

Should one be able to find a model for the moment in which it is a function of the rotation angle ϕ alone: $M_{\mathcal{A}z} = M_{\mathcal{A}z}(\phi)$ then there is also a rotational potential energy given by

$$\Phi^{\mathrm{rot}}(\phi) = -\int^{\phi} M_{\mathcal{A}z}(\phi')\, d\phi'. \tag{3.71}$$

The following example illustrates this.

Example 3.5 A homogeneous circular disc of mass m and radius R is attached horizontally to a thin vertical staff at its mid-point. See figure 3.1. The vertical staff has the property that when it is twisted an angle ϕ it responds with a moment in the opposite direction which is proportional to the amount of twisting: $M(\phi) = -\tau_0\phi$. Determine the period for twisting



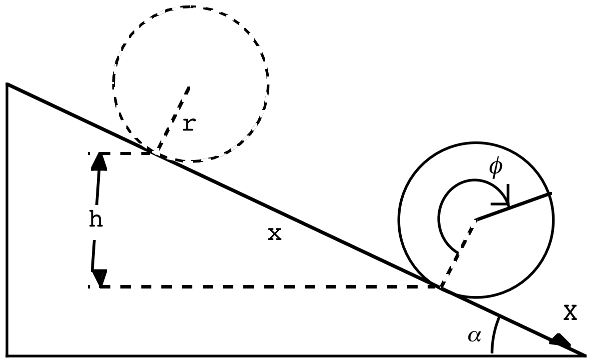


Figure 3.1: The torsion pendulum of example 3.5 is shown in the figure on the left. In the figure the disc has been twisted an angle ϕ. The line fixed on the disc is assumed to be parallel to be fixed direction of the arrow at equilibrium, when the vertical staff is not twisted.

Figure 3.2: The figure on the right shows a cylinder rolling down an incline which makes an angle α with the horizontal. In example 3.6 the speed of the cylinder, after it has sunk a vertical distance h, is calculated.

oscillations of the disc around the axis determined by the staff.
Solution: We take the vertical axis defined by the staff to be the Z-axis. The moment of inertia for the disc is $J_z = \frac{1}{2}mR^2$. This problem can be solved directly using the equation of motion 1.76 which gives $J_z\ddot{\phi} = -\tau_0\phi$ in this case. We will, however, solve it by using the conservation of mechanical energy. The rotational potential energy of equation 3.71 gives in this case

$$\Phi(\phi) = -\int_0^\phi M_{\mathcal{O}z}(\phi')\,d\phi' = -\int_0^\phi(-\tau_0\phi')\,d\phi' = \frac{1}{2}\tau_0\phi^2. \tag{3.72}$$

This potential energy is of course an elastic (internal) potential energy of the staff. The only other form of energy in the system is the kinetic energy of the rotating disc and it is given by equation 3.26 (note that $\mathcal{C} = \mathcal{G}$ in this case). The total mechanical energy of the system, which is called a 'torsion pendulum', is thus

$$E_m = T + \Phi = \frac{1}{2}J_z\dot{\phi}^2 + \frac{1}{2}\tau_0\phi^2 = \text{const.}. \tag{3.73}$$

If we take the time derivative of this we get

$$J_z\dot{\phi}\ddot{\phi} + \tau_0\phi\dot{\phi} = 0 \tag{3.74}$$

and after division with $\dot{\phi}$ this gives the equation of motion

$$\ddot{\phi} + \frac{\tau_0}{J_z}\phi = 0. \tag{3.75}$$

This equation is easily solved using the theory of linear oscillations. The angular frequency is given by $\omega_0 = \sqrt{\frac{\tau_0}{J_z}}$ and the period is thus

$$T_p = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\tau_0}{J_z}} = 2\pi\sqrt{\frac{2\tau_0}{mR^2}}. \tag{3.76}$$

In this example it was not particularly advantageous to use the conservation of mechanical energy, but in other more complicated cases it can be very useful as we'll see below. □

3.2.2 Forces that do not Work

In the previous subsection we found that the internal forces in a rigid body do not perform any work on the body. There are, however, also a number of cases when external forces do not perform work on a body which are useful to know about.

When rigid bodies are in contact they act with contact forces upon each other. Such contact forces can be classified into three types: a) *Normal forces*, b) *Static friction*, and c) *Sliding* (or kinetic) *friction*. Of these three only the last type, kinetic friction, does any net work on the combined system of bodies in contact. We now discuss why this is so.

First consider the normal force on a moving body from the surface of a fixed body. By definition of normal force it acts perpendicularly to the possible direction of motion of the moving body at the point of contact. The work $dW = \mathbf{N} \cdot d\mathbf{r}$ must then be zero since the force vector $\mathbf{N}$ and the displacement vector $d\mathbf{r}$ are perpendicular. If on the other hand both bodies move then the component of the motion along $\mathbf{N}$ will give non-zero net work on the body on which $\mathbf{N}$ acts. Because of Newton's third law, however, a reaction force of equal magnitude but opposite direction must act on the other body. Since this body has the same velocity component perpendicular to the surface of contact as the other body, the net work of force and reaction force is zero.

normal force does no work

The case of static friction forces is similar. When such a force acts from a body at rest on a moving body, the point of the moving body in contact must, by definition of static friction, be at rest. This means that the motion of the moving body must be a *rotation* about the point of contact. The work done by the force must thus be zero since the displacement vector $d\mathbf{r}$ at the point of contact is zero $d\mathbf{r} = \mathbf{0}$. If both bodies are moving the same reasoning as above, for normal forces, shows that the static friction force and its reaction force on the other body, together produce zero net work.

static friction does no work

The two non-working forces discussed above occur, for example, when there is rolling without slipping. They also occur when there are *smooth constraints*, i.e. well lubricated hinges, bearings, ball and socket joints, tracks to slide along etc. Should these devices not be well lubricated kinetic friction will occur as non-conservative internal force (with respect to the system as a whole) which does negative work. The mechanical energy then decreases (dissipates) and becomes heat, vibration and noise.

Example 3.6 A homogeneous circular cylinder of mass m and radius r rolls without slipping on an incline which makes an angle α with the horizontal. See figure 3.2 for the geometry. Assume that the cylinder starts from rest. Find the centre of mass speed of the cylinder after it has sunk the vertical distance h.
Solution: Since there is rolling without slipping the mechanical energy is conserved and only the gravitational force does work on the rolling cylinder. In figure 3.2 the angle ϕ is the angle that a line fixed on the cylinder has rotated from the initial position. Rolling without slipping means that the length x traversed along the incline must be equal to the length $r\phi$ along the circumference of the cylinder:

$$x = r\phi. \tag{3.77}$$

This is usually called the 'rolling constraint'. We now write down the energy of the cylinder using equations 3.7 and 3.26 and the expression 3.56 for the potential energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_{\mathcal{G}}\dot{\phi}^2 - mgx\sin\alpha. \tag{3.78}$$

Here $J_{\mathcal{G}} = \frac{1}{2}mr^2$ and $v_{\mathcal{G}} = \dot{x}$. If we use the rolling constraint we get $\dot{\phi} = \dot{x}/r$. At the initial position we have $\dot{x} = 0$ and $x = 0$ so the energy $E = 0$. Since $h = x\sin\alpha$ we now get

$$0 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{1}{2}mr^2(\dot{x}/r)^2 - mgh. \tag{3.79}$$

From this one easily calculates

$$\dot{x} = \sqrt{\frac{4gh}{3}} \tag{3.80}$$

and this is thus the desired speed. □

Example 3.7 A smooth stiff wire of mass $3m$ has been given the shape of a semi-circle with radius r. The semi-circle hangs vertically down from a smooth horizontal axis through small

loops at its end points. A small pearl of mass m can slide with negligible friction along the semi-circular wire. The pearl is released from rest at one of the end points of the wire in contact with the horizontal axis.
a) Show that the trajectory of the pearl is an ellipse.
b) Find the speed of the pearl in its lowest position.
Solution: a) Let the horizontal axis be the x-axis and place the origin at the initial position of the mid-point between the ends of the semi-circular wire:

$$(x_w(0), y_w(0)\,) = (0, 0).$$

The position of the pearl is then: $(x(0), y(0)\,) = (r, 0)$. We choose the y-axis vertically downwards. The absence of external horizontal forces means that the x-coordinate of the center of mass will be at rest and we get

$$x_{\mathcal{G}} = \frac{m\,x(t) + 3m\,x_w(t)}{m + 3m} = \frac{m\,x(0) + 3m\,x_w(0)}{4m} = \frac{r}{4}. \tag{3.81}$$

This directly gives the relationship

$$x_w(t) = \frac{1}{3}[r - x(t)] \tag{3.82}$$

between the x-coordinates. The fact that the pearl at (x, y) is on part of the circle with radius r and mid-point $(x_w, 0)$ gives the following relationship

$$(x - x_w)^2 + y^2 = r^2. \tag{3.83}$$

We now eliminate x_w from this using the previous equation and get

$$[x - \frac{1}{3}(r - x)]^2 + y^2 = r^2. \tag{3.84}$$

Some algebra leads to the expression

$$\left(\frac{x - r/4}{3r/4}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \tag{3.85}$$

which, clearly, is the equation for an ellipse with center at $(r/4, 0)$ and the pearl must move on this ellipse.

b) Since there is no friction in this problem the only force that does work is gravity and the energy is thus conserved. The wire cannot move vertically and must thus have constant potential energy. We get the following expression for the total energy

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}3m\dot{x}_w^2 - mgy = E. \tag{3.86}$$

Since initially everything is at rest and $y = 0$ we have $E = 0$. Differentiating the relationship $x_w(t) = \frac{1}{3}[r - x(t)]$ above with respect to time gives $\dot{x}_w = -\dot{x}/3$. Inserting this into the energy expression gives

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}3m\left(\frac{\dot{x}^2}{9}\right) - mgy = 0. \tag{3.87}$$

At its lowest position the pearl has vertical velocity so $\dot{y} = 0$ and $y = r$. This now gives for the speed $v = \dot{x}$ at this position

$$\frac{1}{2}m\left[v^2 + \left(\frac{v^2}{3}\right)\right] - mgr = 0 \tag{3.88}$$

so we find that

$$v = \sqrt{\frac{3}{2}gr}. \tag{3.89}$$

is the speed of the pearl at the lowest position. □

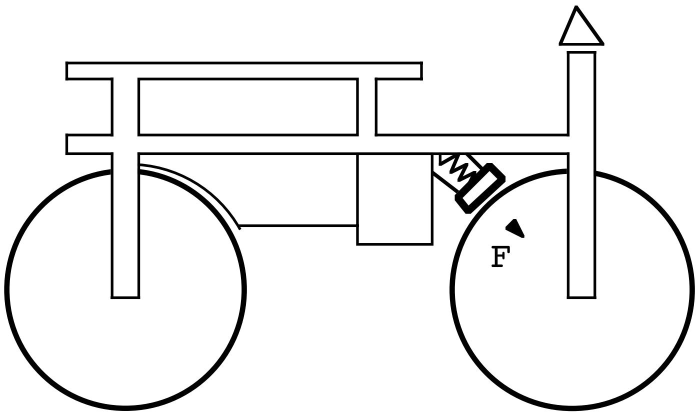


Figure 3.3: The figure refers to problem 3.3 and shows the steam roller and how the braking force is applied to its front wheel.

3.3 Problems

Problem 3.1 Find the kinetic energy of the rolling and sliding circular disc of problem 2.3 (on page 38).

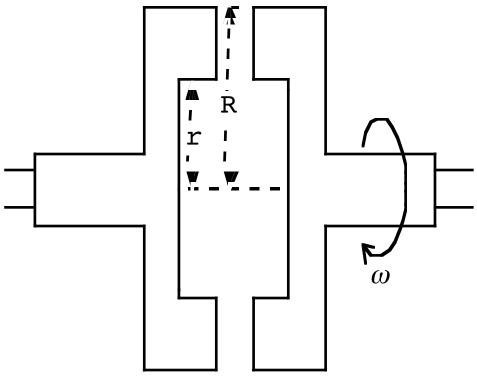
Problem 3.2 Consider the pyramid of rollers of problem 2.5 on page 39. Assume that each cylindrical roller is homogeneous and of mass m, radius r and has speed v. Calculate the kinetic energy of the pyramid
a) using König's theorem, formula 3.7, and
b) using the existence of the instantaneous centre of zero velocity in the plane motion of the rigid body.

Problem 3.3 In order to reduce the transport weight and braking distance of the steam roller shown in figure 3.3 it is suggested that the wheels are made as hollow cylindrical shells. These are then filled with water when the steam roller is to be used.

Assume that such a steam roller with empty wheels, has a braking distance of 2 m at a speed of 5 km/h. The brake consists of a chock which is pressed against the front wheel with a force of $F = 125 \cdot g\,\mathrm{N}$, the coefficient of (sliding) friction being 0.8 between wheel and chock. The wheels have a diameter of 1 m and a width of 2 m. Calculate the braking distance when the wheels are filled with
a) liquid water, and
b) ice.

Problem 3.4 One can specify the performance of a car engine by giving either the maximum power $P_{\max}$ that it can deliver, or its maximum moment (or torque) $M_{\max}$. Since $M_{\max}$ determines the acceleration of the car it is common to specify at what number of revolutions (per unit time) $n_{\max}$ that this maximum torque is achieved. Assume that the torque, M, is a known function, $M(n)$, of the number of revolutions n.
a) Derive a formula for the power as a function of n.
b) Show that the maximum power and the maximum torque cannot correspond to the same n.

Problem 3.5 One part of a friction clutch rotates freely around its axis with angular velocity ω as shown in figure 3.4. The moment of inertia of this part is J. It is suddenly pressed with force P against an identical but initially non-rotating part. The coefficient of friction between the two parts is f and the contact area is the plane annulus with



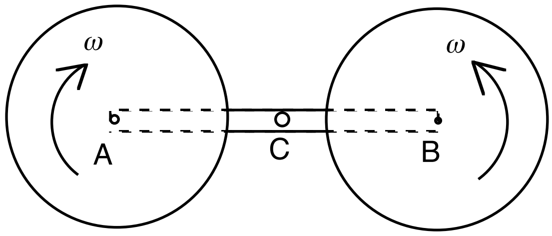


Figure 3.4: A vertical cross section through the symmetry (rotation) axis of the friction clutch of problem 3.5 is shown in the figure on the left.

Figure 3.5: The figure on the right refers to problem 3.6. Two identical discs rotate in opposite directions around parallel axes through the ends of a light bar $\mathcal{AB}$. The bar can rotate around an axis through its midpoint $\mathcal{C}$ around a fixed axis parallel to those of the discs.

inner radius r and outer radius R. All other friction is negligible. Calculate
a) the final common angular velocity of the clutch, and
b) the total heat loss in the clutch.

Problem 3.6 Two identical homogeneous circular discs, of mass m and radius R, rotate in opposite directions around parallel axes through the ends of a light bar $\mathcal{AB}$, see figure 3.5. The bar can rotate around a fixed axis, parallel to those of the discs, through its midpoint $\mathcal{C}$. The rotation axes are all perpendicular to the plane of the figure 3.5, and the friction can be neglected in the bearings at $\mathcal{B}$ and $\mathcal{C}$. Initially the bar, which has length ℓ, is at rest and the discs have angular velocity ω. Calculate the heat loss in the bearing at $\mathcal{A}$ once the rotation with respect to this bearing has stopped.

3.4 Hints and Answers

Answer 3.1
The kinetic energy is $T = \frac{1}{4}m(v_1^2 + 3v_2^2 - 2v_1v_2)$.

Answer 3.2
a) $6\left[\frac{1}{2}mv^2 + \frac{1}{2}J_{\mathcal{G}}(v/r)^2\right] = \frac{9}{2}mv^2$.
b) $6\left[\frac{1}{2}J_{\mathcal{C}}(v/r)^2\right] = \frac{9}{2}mv^2$, since the parallel axis theorem gives $J_{\mathcal{C}} = J_{\mathcal{G}} + mr^2$.

Answer 3.3 Introduce the notation:

$$
\begin{aligned}
m_0 &= \text{mass of steam roller without wheels,}\\
m_e &= \text{mass of one empty wheel,}\\
m_w &= \text{mass of water in one wheel,}\\
J_e &= \text{moment of inertia of one empty wheel,}\\
J_w &= \frac{1}{2}m_w r^2, \ \text{ moment of inertia of water in one wheel,}\\
\ell_e &= 2\,\text{m, empty braking distance,}\\
v &= 5\,\text{km/h, initial speed of steam roller,}\\
r &= 0.5\,\text{m, radius of wheel,}\\
\mu &= 0.8 \text{ coefficient of sliding friction.}
\end{aligned}
$$

Equating the work done by the braking force F to the initial kinetic energy of the steam roller we get

$$\frac{1}{2}(m_0 + 2m_e)v^2 + 2\frac{1}{2}J_e\left(\frac{v}{r}\right)^2 = \mu F \ell_e.$$

When the wheels contain water the steam roller can come to halt without dissipating the rotational energy of the water. The extra kinetic energy that the brake force must make zero is thus $2\frac{1}{2}m_w v^2$. Using the previous result, this gives

$$\mu F \ell_e + m_w v^2 = \mu F \ell_w$$

where ℓ_w is the braking distance with liquid water in the wheels. This gives

$$\ell_w = \ell_e + \frac{m_w v^2}{\mu F}$$

Putting numbers in we find that $m_w = \pi \cdot 500\,\text{kg}$, $v = (5/3.6)\,\text{m/s}$, and $\mu F = 0.8 \cdot 125 \cdot 9.81\,\text{N}$ so that the extra braking distance with water in is roughly 3 m. Since $\ell_e = 2\,\text{m}$ we thus find
a) answer $\ell_w = 5\,\text{m}$.
When the water is frozen to ice one must also dissipate the rotational energy of the water to brake the steam roller. The extra kinetic energy is then $2\frac{1}{2}J_w\left(\frac{v}{r}\right)^2$. In the same way as above one then finds that

$$\ell_i = \ell_w + \frac{1}{2}\frac{m_w v^2}{\mu F}$$

is the new braking distance. This thus adds an extra 1.5 m to give
b) answer $\ell_i = 6.5\,\text{m}$.

Answer 3.4
a) Use of

$$P = \frac{d}{dt}\int^t M\,d\varphi = M\dot{\varphi}$$

and $n = \dot{\varphi}/(2\pi)$ gives

$$P(n) = 2\pi M(n)n.$$

b) Show that the derivative of the power with respect to n cannot be zero at $n = n_{\text{max}}$.

Answer 3.5
Note that the details of the process between the initial and final state of motion is irrelevant. the answers will thus not depend on P and f.
a) The final angular velocity is $\omega/2$.
b) The heat loss is $J\omega^2/4$.

Answer 3.6
Use of L_z =const. gives the angular velocity of the bar after the braking. It is found to be

$$\Omega_{\text{final}} = -\frac{R^2}{\ell^2 + R^2}\omega,$$

where the sign assumes that the disc at $\mathcal{B}$ has positive angular velocity before the braking (as well as after). The heat loss is given by

$$T_{\text{initial}} - T_{\text{final}} = \frac{1}{4}mR^2\omega^2\left(1 - \frac{R^2}{\ell^2 + R^2}\right)$$

and is thus always positive. Note that when $\ell \to 0$ the heat loss goes to zero. The reason for this is that when $\ell = 0$ the bar will have no inertia and the disc at $\mathcal{A}$ can tranfer its angular velocity to the bar with no loss of energy. Energy is then conserved.

Chapter 4

Dynamics of Rigid Bodies

In this chapter we present the general form that the laws of linear and angular momentum take for rigid bodies. The law of angular momentum leads to Euler's dynamic equations for the rigid body. The concept of the inertia tensor is discussed and it is stressed that the equations of motion should be given in such a way that the elements of the inertia tensor are constant. Pendulums and reaction forces on rotating bodies are also treated.

4.1 The Angular Momentum of a Rigid Body

We wish to calculate the angular momentum $\mathbf{L}_{\mathcal{A}}$ for a rigid body. Only two types of base point $\mathcal{A}$ will be of interest; either the centre of mass $\mathcal{G}$ or a fixed point $\mathcal{C}$ of the body. We place the origin of the coordinate system at the relevant base point so that the definition of angular momentum gives

$$\mathbf{L} = \sum_{k=1}^{N} \mathbf{r}_k \times m_k \mathbf{v}_k. \tag{4.1}$$

Here $\mathbf{r}_k$ are the position vectors of the particles of the body with respect to the relevant base point ($\mathcal{G}$ or $\mathcal{C}$) and $\mathbf{v}_k$ are the velocities of the particles. In the case of a fixed point, $\mathcal{C}$, these are absolute velocities and are given by equation 3.19. In the case of the centre of mass we use velocities relative to the centre of mass system (so that we calculate $\mathbf{L}'_{\mathcal{G}}$) and these are given by equation 3.9. Both these equations for the velocities have the form

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k. \tag{4.2}$$

The angular momentum that we want to calculate will thus in both cases have the algebraic form

$$\mathbf{L} = \sum_{k=1}^{N} \mathbf{r}_k \times m_k(\boldsymbol{\omega} \times \mathbf{r}_k) \tag{4.3}$$

and this can be rewritten, using the formula for the triple vector product,

$$\mathbf{L} = \sum_{k=1}^{N} [\boldsymbol{\omega} m_k(\mathbf{r}_k \cdot \mathbf{r}_k) - m_k(\boldsymbol{\omega} \cdot \mathbf{r}_k)\mathbf{r}_k]. \tag{4.4}$$

The three components of this equation now give the result:

angular momentum of rigid body

$$\begin{aligned} L_x &= J_x\omega_x - D_{xy}\omega_y - D_{xz}\omega_z, \\ L_y &= J_y\omega_y - D_{yx}\omega_x - D_{yz}\omega_z, \\ L_z &= J_z\omega_z - D_{zx}\omega_x - D_{zy}\omega_y. \end{aligned} \tag{4.5}$$

If we introduce the matrix L=$(L_x\ L_y\ L_z)$ of components of $\mathbf{L}$ and use the symmetric matrix of equation 3.23 we can express these three equations with the single formula

$$\mathsf{L} = \mathsf{wJ} \Longleftrightarrow (L_x\ L_y\ L_z) = (\omega_x\ \omega_y\ \omega_z) \begin{pmatrix} J_x & -D_{xy} & -D_{xz} \\ -D_{yx} & J_y & -D_{yz} \\ -D_{zx} & -D_{zy} & J_z \end{pmatrix}. \tag{4.6}$$

One should remember that the elements of these matrices depend on the basis triad used. There is, however, an invariant geometric relationship between the two vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ expressed by this equation. We now investigate this.

4.1.1 Linear Transformations and the Inertia Tensor

We now investigate what happens to the matrix J when we change the basis. To do this we must introduce a superscript on the component matrices so that we can distinguish between the components with respect to different bases. For the vector $\mathbf{L}$ we thus write

$$\mathbf{L} = \mathsf{L}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = (L_1^{\mathrm{O}}\ L_2^{\mathrm{O}}\ L_3^{\mathrm{O}}) \begin{pmatrix} \mathbf{e}_1^{\mathrm{O}} \\ \mathbf{e}_2^{\mathrm{O}} \\ \mathbf{e}_3^{\mathrm{O}} \end{pmatrix} = \mathsf{L}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}} \tag{4.7}$$

and correspondingly for $\boldsymbol{\omega}$ and its components w. The equation 4.6 now gives

$$\mathbf{L} = \mathsf{L}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} = \mathsf{w}^{\mathrm{O}}\mathsf{J}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}} \tag{4.8}$$

in the fixed (observer) basis triad O. The same formula can also be expressed in a body fixed basis triad E^{A} so that we also have

$$\mathbf{L} = \mathsf{w}^{\mathrm{A}}\mathsf{J}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}. \tag{4.9}$$

We now make use of the formulas 2.10 and 2.20 to transform from the O to the A basis with the rotation matrix ${}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}$ and its inverse (=transpose) ${}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}$. The above equation then gives

$$\mathbf{L} = \mathsf{w}^{\mathrm{O}}\,{}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}\,\mathsf{J}^{\mathrm{A}}\,{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}\,\mathsf{E}^{\mathrm{O}}. \tag{4.10}$$

transformation of the inertia tensor under rotation

If we now compare this with equation 4.8 we find that the matrix of moments and products of inertia for the two sets of basis vectors must be related according to

$$\mathsf{J}^{\mathrm{O}} = {}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}\,\mathsf{J}^{\mathrm{A}}\,{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}}. \tag{4.11}$$

Once the moments and products of inertia have been calculated for one set of axis directions one can thus find the corresponding quantities for any other set of axes simply by multiplying with the relevant rotation matrix according to this formula.

In the chapter on rigid body kinematics we found that to each rotation matrix there corresponds a rotation operator $\hat{R}$ which operates on vectors to give new rotated vectors. The operator is a *linear operator*, i.e. it obeys $\hat{R}(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda\hat{R}\mathbf{a} + \mu\hat{R}\mathbf{b}$. Such linear operators are geometric 'objects', independent of any particular basis in which their components are given, and they are sometimes called (second rank) *tensors* or 'dyads'.

Since equation 4.6 clearly defines a linear transformation from the components of $\boldsymbol{\omega}$ to $\mathbf{L}$ there is a corresponding linear operator $\hat{J}$ which transforms the vector $\boldsymbol{\omega}$ to the vector $\mathbf{L}$

$$\mathbf{L} = \hat{J}\boldsymbol{\omega} = \mathsf{w}^{\mathrm{A}}\mathsf{J}^{\mathrm{A}}\,\mathsf{E}^{\mathrm{A}}. \tag{4.12}$$

This linear operator is thus what properly should be called the *inertia tensor* of the body even though this name is often used for the matrix of its components in some basis. Unlike the rotation operators more general linear operators change both the direction and the length of the vector on which it operates. Nor is the rotating part of the operation a rigid rotation; different vectors are rotated in different ways.

When a rigid body moves its inertia tensor will, in general, be *constant* only if it is calculated with respect to a point fixed in the body, or rigidly connected to the body, and with respect to axis directions fixed in the body. The requirement that the axis directions are fixed in the body, however, does not specify any particular directions for these axes; they can still be chosen in an infinity of different ways differing from each other by constant rotations. General mathematical theory of linear transformations, however, indicates that when such transformations are represented by real symmetric matrices there might exist unique mutually orthogonal eigen-vectors of the transformation. These eigen-vectors $\mathbf{e}_i^{\mathrm{A}'}$ are defined by the equation

$$\hat{J}\mathbf{e}_i^{\mathrm{A}'} = J'_i \mathbf{e}_i^{\mathrm{A}'} \quad i = 1, 2, 3 \tag{4.13}$$

and the numbers J'_i are the eigen-values [1]. These eigen-vectors of the inertia tensor define the *principal axis* directions of the body and the component matrix of the inertia tensor in this principal basis is diagonal with the eigen-values along the diagonal:

$$\mathsf{J'} = \begin{pmatrix} J'_x & 0 & 0 \\ 0 & J'_y & 0 \\ 0 & 0 & J'_z \end{pmatrix}. \tag{4.14}$$

These diagonal matrix elements are the *principal moments of inertia* of the body. In what follows we will normally omit the primes on the principal moments of inertia and simply write J_x etc.

The principal axes of a body can sometimes be found by means of the following two rules:

(I.) Any plane of symmetry of a body is perpendicular to one of the principal axes.

(II.) A symmetry axis of a body is a principal axis. Any pair of axis perpendicular to the symmetry axis will be principal axes and correspond to equal principal moments of inertia.

A body for which two of the principal moments of inertia are equal is said to be a *symmetric top.* The two eigen-vectors in the plane corresponding to these two moments are then not unique; any pair will do. If all three moments of inertia are equal, the inertia tensor does not single out any direction of the body as special; all vectors are eigen-vectors. Such a body might be called a 'spherical top'.

Example 4.1 A rigid body has mass m and principal moments of inertia J_x, J_y, and J_z. Show, by explicit construction that there exists a rigid four particle system with the same mass and inertia tensor.

Solution: Consider the four particle system in figure 4.1, where all particles have mass $m/4$ so that the total mass is m. The position vectors are given by

$$\mathbf{r}_1 = (\ a,\ 0,\ c), \tag{4.15}$$
$$\mathbf{r}_2 = (-a,\ 0,\ c), \tag{4.16}$$
$$\mathbf{r}_3 = (\ 0,\ b, -c), \tag{4.17}$$
$$\mathbf{r}_4 = (\ 0, -b, -c). \tag{4.18}$$

The elements of the inertia tensor matrix for this system are

$$D_{xy} = \sum m_i x_i y_i = \frac{m}{4}(a\cdot 0 - a\cdot 0 + 0\cdot b - 0\cdot b) = 0 \tag{4.19}$$
$$D_{xz} = \sum m_i x_i z_i = \frac{m}{4}(a\cdot c - a\cdot c - 0\cdot c - 0\cdot c) = 0 \tag{4.20}$$

[1] One sometimes finds that the terms 'characteristic vectors' and 'characteristic values' are used instead.

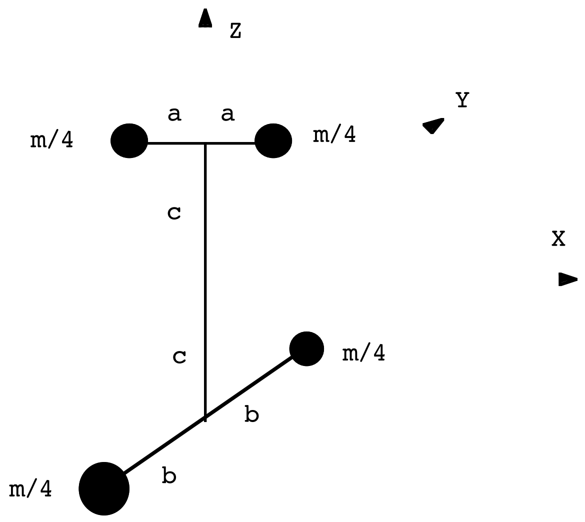


Figure 4.1: This figure shows the four particle system for which the inertia tensor is calculated in example 4.1. The total mass is m and the four identical particles are connected by light rigid rods.

$$\begin{aligned}
D_{yz} &= \sum m_i y_i z_i = \frac{m}{4}(0\cdot c + 0\cdot c - b\cdot c + b\cdot c) = 0 && (4.21)\\
J_x &= \sum m_i(y_i^2+z_i^2) = \frac{m}{4}(2b^2+4c^2) && (4.22)\\
J_y &= \sum m_i(x_i^2+z_i^2) = \frac{m}{4}(2a^2+4c^2) && (4.23)\\
J_z &= \sum m_i(x_i^2+y_i^2) = \frac{m}{4}(2a^2+2b^2). && (4.24)
\end{aligned}$$

Since the inertia tensor is diagonal the axes are the principal axes for this case. The principal moments of inertia for this four particle system are thus given by

$$J_x = \frac{m}{2}(b^2+2c^2), \qquad J_y = \frac{m}{2}(a^2+2c^2), \qquad J_z = \frac{m}{2}(a^2+b^2). \tag{4.25}$$

Since we consider the moments of inertia as given and the particle system geometry as unknown, we must solve these equations for the distances a, b, and c in terms of J_x, J_y, and J_z. Some algebra shows that

$$\begin{aligned}
J_x + J_y - J_z &= 2mc^2, && (4.26)\\
J_z + J_x - J_y &= mb^2, && (4.27)\\
J_z - J_x + J_y &= ma^2, && (4.28)
\end{aligned}$$

so the our four particle system will have the desired inertia tensor provided the distances are chosen as follows:

$$a = \sqrt{\frac{1}{m}(J_z - J_x + J_y)}, \qquad b = \sqrt{\frac{1}{m}(J_z + J_x - J_y)}, \qquad c = \sqrt{\frac{1}{2m}(J_x + J_y - J_z)}. \tag{4.29}$$

The four particles can thus be arranged to have any desired inertia tensor.

Should two principal moments of inertia be equal, $J_x = J_y$ say, one finds that this means that the two distances a and b must be equal: $a = b$. A body with two equal principal moments of inertia is said to be a *symmetric top.* All three principal moments of inertia will be equal if $a = b$ and $a = \sqrt{2}c$; in this case the inertia tensor is said to have spherical symmetry. The four particles are then at the corners of a regular tetrahedron.

The fact that a three particle system cannot produce a general inertia tensor is best seen from the fact that such a system always will be plane (the plane of the three particles). If this plane is chosen as the xy-plane the moments of inertia will necessarily obey $J_x + J_y = J_z$ and a general inertia tensor can thus not be produced. □

Example 4.2 Calculate the inertia tensor with respect to a basis which has been rotated by the angle ψ around $\mathbf{e}_3$ of the principal basis, using equation 4.11.
Solution: In the principal basis the matrix of the inertia tensor is given by equation 4.14. The new matrix in the rotated system becomes

$$\begin{aligned}\mathsf{J}^{\mathrm{O}} &= {}^{\mathrm{O}}\mathsf{R}^{\mathrm{A}}\,\mathsf{J}^{\mathrm{A}'}\,{}^{\mathrm{A}}\mathsf{R}^{\mathrm{O}} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J'_x & 0 & 0 \\ 0 & J'_y & 0 \\ 0 & 0 & J'_z \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\psi J'_x + \sin^2\psi J'_y & \cos\psi\sin\psi\,(J'_x - J'_y) & 0 \\ \cos\psi\sin\psi\,(J'_x - J'_y) & \sin^2\psi J'_x + \cos^2\psi J'_y & 0 \\ 0 & 0 & J'_z \end{pmatrix}. \end{aligned} \tag{4.30}$$

This calculation thus shows that the product of inertia in the non-principal axis system is given by

$$D^{\mathrm{O}}_{xy} = -\cos\psi\sin\psi\,(J'_x - J'_y). \tag{4.31}$$

From this it is clearly seen that in the case of a symmetric top, $J'_x = J'_y$, the directions of the X- and Y-axes, in the plane perpendicular to the Z-axis, do not matter since then $D_{xy} \equiv 0$. When two principal moments of inertia are equal rotation about the third axis will not make the inertia tensor matrix non-diagonal. ☐

In a principal axis system the rotational kinetic energy, equation 3.15, takes the simple form

$$T = \frac{1}{2}(J_x\omega_x^2 + J_y\omega_y^2 + J_z\omega_z^2). \tag{4.32}$$

The angular momentum vector components of equation 4.5 become simply

$$\mathsf{L} = (L_x\ L_y\ L_z) = (J_x\omega_x\ J_y\omega_y\ J_z\omega_z) \tag{4.33}$$

in a such a system.

4.1.2 The Structure of the Inertia Tensor

We have obtained the inertia tensor twice by now. First when we calculated the kinetic energy of the rigid body in equations 3.14 and 3.15. We also found the inertia tensor when we calculated the angular momentum of a rigid body in equations 4.4 and 4.6. From these equations we see that the inertia tensor matrix elements can be written in the form

$$J_{kl} = [\sum m(x_1^2 + x_2^2 + x_3^2)]\delta_{kl} - \sum m x_k x_l, \qquad (k, l = 1, 2, 3) \tag{4.34}$$

where we have omitted the particle index, for simplicity, and where $\delta_{kl} = 1$ when $k = l$ and $\delta_{kl} = 0$ when $k \neq l$. If we define the *spherical moment of inertia* a by

$$J_S \equiv \sum_i m_i\,\mathbf{r}_i \cdot \mathbf{r}_i \tag{4.35}$$

and then extend the definition of the products of inertia, 3.17, to diagonal terms so that e.g.

$$D_{xx} \equiv \sum_i m_i\,x_i x_i, \tag{4.36}$$

and similarly for y and z, we can rewrite the above formula for the inertia tensor matrix in the form

$$\mathsf{J} = J_S\mathbf{1} - \mathsf{D} = J_S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}. \tag{4.37}$$

We now see that the first term in this sum of matrices is invariant under the transformations 4.11. Eigen-values and eigen-vectors are thus completely determined by the second part, the symmetric matrix D of products of inertia.

The division of the inertia tensor matrix into the two terms above simplify the derivation of some of the properties of the inertia tensor. One such property is the parallel axis theorem for the inertia tensor. We have come across the parallel axis theorem for moments of inertia before. The corresponding theorem for the tensor is found in exactly the same way. It states that the inertia tensor $\mathsf{J}^{\mathcal{C}}$ calculated with respect to a point $\mathcal{C}$ of the body can be expressed in terms of the inertia tensor $\mathsf{J}^{\mathcal{G}}$ with respect to the centre of mass $\mathcal{G}$ of the body and the vector from $\mathcal{G}$ to $\mathcal{C}$: $\overline{\mathcal{GC}} = \mathbf{R} = (R_1, R_2, R_3)$ according to the formula

parallel axis theorem for inertia tensor

$$\mathsf{J}^{\mathcal{C}} = \mathsf{J}^{\mathcal{G}} + m(R^2\mathbf{1} - \mathbf{R} \otimes \mathbf{R}). \tag{4.38}$$

Here m is the total mass of the body and

$$\mathbf{R} \otimes \mathbf{R} \equiv \begin{pmatrix} R_1R_1 & R_1R_2 & R_1R_3 \\ R_2R_1 & R_2R_2 & R_2R_3 \\ R_3R_1 & R_3R_2 & R_3R_3 \end{pmatrix}. \tag{4.39}$$

The addition to the centre of mass inertia tensor when the origin is moved is thus simply the inertia tensor of a particle with the mass of the body and the position vector of the new point. Note this formula generalizes the parallel axis theorem for moments of inertia. A diagonal component of the formula, e.g. the 33 or zz-component

$$J_z^{\mathcal{C}} = J_z^{\mathcal{G}} + m(R_x^2 + R_y^2), \tag{4.40}$$

is in fact our old friend the parallel axis (or Steiner's) theorem.

4.2 Euler's Dynamic Equations

The translational motion of a rigid body is given by the equation of motion

$$m\ddot{\mathbf{r}}_{\mathcal{G}} = \mathbf{F} \tag{4.41}$$

which is a direct consequence of the principle of linear momentum. In order to determine the motion of other points of the body than the centre of mass we must find the angular velocity vector $\boldsymbol{\omega}(t)$ as a function of time. The velocities of all other points can then be found in terms of the centre of mass velocity by means of the connection formula 2.74. We will now derive the equations of motion for the rotational motion of a rigid body in the form of differential equations for the angular velocity vector components.

The starting point is now the principle of angular momentum on one of the forms

$$\dot{\mathbf{L}}'_{\mathcal{G}} = \mathbf{M}_{\mathcal{G}} \quad \text{or} \quad \dot{\mathbf{L}}_{\mathcal{C}} = \mathbf{M}_{\mathcal{C}} \tag{4.42}$$

where the first form is the general one while the second may be used when there is a point $\mathcal{C}$ of the body which is fixed in space. We write simply

$$\dot{\mathbf{L}} = \mathbf{M} \tag{4.43}$$

in what follows to simplify the notation. In order to take the time derivative of the angular momentum vector, as given by equations 4.5, we must decide which basis vectors the components refer to. In order for the inertia tensor matrix elements to be guaranteed to be constant we must use a basis fixed in the body, i.e. a rotating basis. The time derivative must thus be calculated using the formula 2.61, i.e.

$$\dot{\mathbf{L}} = \frac{{}^{\mathrm{O}}d}{dt}\mathbf{L} = \frac{{}^{\mathrm{A}}d}{dt}\mathbf{L} + \boldsymbol{\omega} \times \mathbf{L}. \tag{4.44}$$

To simplify matters further we choose, among all possible basis vectors fixed in the body, the principal basis, which we denote E^{A}, and in which the angular momentum vector has the simple form (see equation 4.33)

$$\mathbf{L} = J_1\omega_1\,\mathbf{e}_1^{\mathrm{A}} + J_2\omega_2\,\mathbf{e}_2^{\mathrm{A}} + J_3\omega_3\,\mathbf{e}_3^{\mathrm{A}}. \tag{4.45}$$

We now find that

$$\dot{\mathbf{L}} = J_1\dot{\omega}_1\,\mathbf{e}_1^{\mathrm{A}} + J_2\dot{\omega}_2\,\mathbf{e}_2^{\mathrm{A}} + J_3\dot{\omega}_3\,\mathbf{e}_3^{\mathrm{A}} + \boldsymbol{\omega}\times\mathbf{L}. \tag{4.46}$$

If we now expand the vector product in terms of components we find that the angular momentum principle gives

Euler's dynamic equations for the rigid body

$$\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 &= M_1,\\ J_2\dot{\omega}_2 + (J_1 - J_3)\omega_3\omega_1 &= M_2,\\ J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 &= M_3. \end{aligned} \tag{4.47}$$

These equations are called *Euler's dynamic equations*. Together with Euler's kinematic equations, 2.69, which give the components of the angular velocity vector, in the same body fixed frame, in terms of Euler angles, these equations determine the rotational motion of the rigid body.

4.3 Fixed Axis Rotation and Reaction Forces

In this section we first derive some results for the 'physical' pendulum. i.e. the the rotation of a rigid body around a fixed axis under the influence of gravity. We then use the equations of motion to calculate the reaction (or constraint) force on the body from the axis. This requires both the principles of linear and angular momentum, albeit only the z-component of the latter. Finally we show how the reaction force can be resolved into two contributions from two bearings if the remaining components of the angular momentum principle are used.

4.3.1 The Physical Pendulum

In particle dynamics we have come across the 'mathematical' pendulum which consists of a particle in a string acted on by gravity and the tension in the string. The physical pendulum is a rigid body which can rotate freely around a fixed axis which we conventionally take to be the Z-axis. It is acted on by the resultant of gravity, $m\mathbf{g}$, at the centre of mass $\mathcal{G}$, and we denote by h the distance between the axis and $\mathcal{G}$ (see figure 4.2). If one is only interested in the rotational motion the only equation of motion needed is the z-component of the angular momentum principle, equation 1.76 ($\dot{L}_z = M_z$). This gives

$$J_z\ddot{\varphi} = -mgh\sin\varphi. \tag{4.48}$$

If we introduce the 'reduced length' ℓ of the physical pendulum according to

$$\ell \equiv \frac{J_z}{mh} \tag{4.49}$$

we can rewrite the above equation in the form

$$\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi. \tag{4.50}$$

This is exactly the same equation as for a mathematical pendulum of mass m and length ℓ, so the problems are mathematically equivalent. For small oscillations ($\sin\varphi \approx \varphi$) one finds that the angular frequency is

$$\omega_0 = \sqrt{\frac{g}{\ell}} \tag{4.51}$$

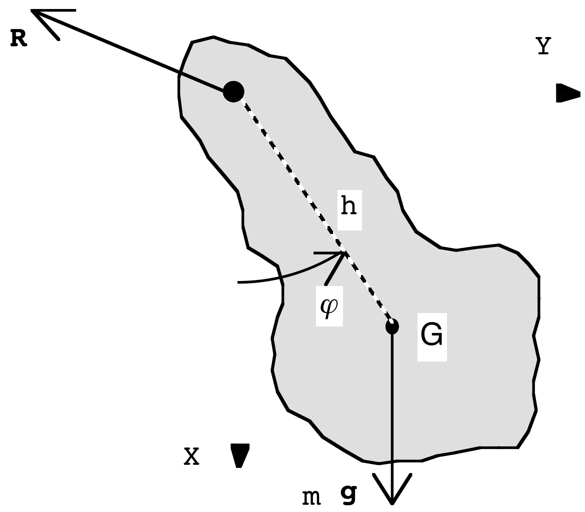


Figure 4.2: The physical pendulum is a rigid body which can rotate freely around the fixed Z-axis. It is acted on by gravity $m\mathbf{g}$ at the centre of mass $\mathcal{G}$. The reaction force $\mathbf{R}$ from the axis is due to a smooth constraint and does no work.

and the period is

$$T_p = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{J_z}{mgh}}. \tag{4.52}$$

In this formula one should note that when h is changed J_z also changes.

Example 4.3 A straight homogeneous rod of length l and mass m has been suspended so that it can swing freely about one of its ends.
a) Calculate the period for small plane oscillations around the vertical (equilibrium) position.
b) What length should a simple pendulum of the same mass have in order to have the same period?
Solution:
a) In this case the moment of inertia is $J_z = \frac{1}{3}ml^2$ and the distance from the rotation axis (at the end of the rod) and its center of mass (in the middle) is $h = l/2$. Formula 4.52 then gives

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ml^2}{mgl/2}} = 2\pi\sqrt{\frac{2l}{3g}} \tag{4.53}$$

for the period of the rod.
b) The period for a simple pendulum of length ℓ is given by $T = 2\pi\sqrt{\ell/g}$. It will thus have the same period as a rod of length l provided that $\ell = 2l/3$. Note that this is independent of the mass. □

Let us investigate how the small amplitude period changes when the position of the axis is changed. According to the parallel axis theorem

$$J_z = md_z^2 = J_z^{\mathcal{G}} + mh^2 = md_{\mathcal{G}z}^2 + mh^2 \tag{4.54}$$

where $J_z^{\mathcal{G}}$ is the moment of inertia with respect to a parallel Z-axis through the centre of mass. Here d_z and $d_{\mathcal{G}z}$ are the radii of gyration corresponding to J_z and $J_z^{\mathcal{G}}$ respectively. If we insert this into equation 4.52 we get

$$T_p(h) = 2\pi\sqrt{\frac{d_{\mathcal{G}z}^2 + h^2}{gh}}. \tag{4.55}$$

This shows that both $\lim_{h\to 0} T_p = \infty$ and $\lim_{h\to\infty} T_p = \infty$. Consequently there must be some value for h between zero and infinity for which the period is minimized. The minimum h-value is easily found by putting the derivative of T_p with respect to h equal to zero. This gives $h_{\min} = d_{\mathcal{G}z}$ and therefore

$$T_p(h_{\min} = d_{\mathcal{G}z}) = 2\pi\sqrt{2\frac{d_{\mathcal{G}z}}{g}}. \tag{4.56}$$

is the smallest possible period for small amplitude oscillations of a physical pendulum.

The energy of the physical pendulum is given by

$$\frac{1}{2}J_z\dot{\varphi}^2 - mgh\cos\varphi = E. \tag{4.57}$$

This equation can be used to find the angular velocity as a function of angle, $\dot{\varphi}(\varphi)$, as in the following example.

Example 4.4 A straight homogeneous rod of length l and mass m has been suspended so that it can swing freely about one of its ends. It is released with zero velocity in a horizontal position. Calculate the angular velocity of the rod when it is vertical.
Solution:
In this case $J_z = \frac{1}{3}ml^2$ and $h = l/2$. The initial conditions are $\varphi(0) = \pi/2$ and $\dot{\varphi}(0) = 0$. We put these into equation 4.57 and get $E = 0$. In the vertical position we then get, from the same equation, that

$$\frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\varphi}^2 - mg\left(\frac{l}{2}\right)\cos 0 = 0. \tag{4.58}$$

Using $\cos 0 = 1$ we easily solve this equation to get

$$\dot{\varphi} = \sqrt{\frac{3g}{l}} \tag{4.59}$$

for the angular velocity of the rod in the vertical position. □

Problems with small oscillations of systems involving complicated constraints, such as rolling, are sometimes best treated by deriving an equation of motion by differentiating the conservation of energy with respect to time. This is illustrated in the following example.

Example 4.5 A homogeneous circular cylinder of mass m and radius r rolls without slipping on a cylindrical track of radius $R(> r)$. See figure 4.3 for the geometry. Determine the period for small oscillations of the cylinder around the equilibrium position.
Solution: Since there is rolling without slipping the mechanical energy is conserved and only the gravitational force does work on the rolling cylinder so we can use the method of example 3.5. The kinetic energy can be found using the method of example 3.1.

In figure 4.3 two angles are introduced. The angle ψ is the angle that the vector from the centre of the track to the midpoint of the cylinder makes with the vertical. The angle ϕ is the angle between this vector and the line on the cylinder which is vertical at equilibrium. Because of the rolling without slipping constraint we have the following connection between these:

$$R\psi = r\phi. \tag{4.60}$$

This says that the length $R\psi$ along the track from the equilibrium position must be equal to the length $r\phi$ along the circumference from the point of contact at equilibrium to the current point of contact. Note, however, that the angle of rotation of the cylinder with respect to the fixed vertical direction is given by $\phi - \psi$.

The potential energy is given by $\Phi = mgh$, where h is the height of the centre of mass which is given by

$$h = (R - r)(1 - \cos\psi). \tag{4.61}$$

To get the total kinetic energy we need the angular velocity which is (note that $\psi = \frac{r}{R}\phi$)

$$\omega = \dot{\phi} - \dot{\psi} = \dot{\phi} - \frac{r}{R}\dot{\phi} = (1 - \frac{r}{R})\dot{\phi} \tag{4.62}$$

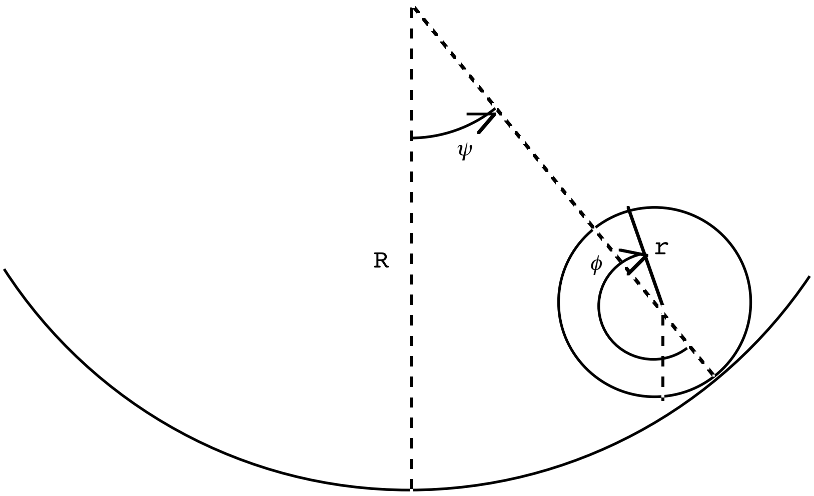


Figure 4.3: This figure shows a non-equilibrium position for a cylinder rolling on a cylindrical track. In example 4.5 the period for small oscillations of the rolling cylinder around the equilibrium at the bottom of the track, is calculated. The angle ϕ is the angle that the cylinder has turned when the vector from the center of the track to its mid-point has rotated the angle ψ.

so T is given by

$$T = \frac{1}{2}J_{\mathcal{C}}\omega^2 = \frac{1}{2}(mr^2 + \frac{1}{2}mr^2)(1 - \frac{r}{R})^2\dot{\phi}^2. \tag{4.63}$$

Using $\psi = \frac{r}{R}\phi$ again we now get for the total energy

$$E_m = \frac{1}{2}(\frac{3}{2}mr^2)(1 - \frac{r}{R})^2\dot{\phi}^2 + mg(R - r)[1 - \cos(\frac{r}{R}\phi)]. \tag{4.64}$$

We can now differentiate this with respect to time. When the resulting expression is divided by $\dot{\phi}$ we get an equation of motion. In this equation one then makes use of the assumption of small amplitude and put $\sin(\frac{r}{R}\phi) \approx \frac{r}{R}\phi$. One then gets the usual equation of motion for harmonic oscillations and the period becomes

$$T_p = \pi\sqrt{\frac{6(R - r)}{g}}. \tag{4.65}$$

Note correct behavior $T_p \to \infty$ in the limit when $R \to \infty$. $\square$

4.3.2 The Reaction Force from the Axis

In this subsection we calculate the reaction force $\mathbf{R}$ from the axis on the rigid body of figure 4.2. This can be done if one assumes that the rotational motion $\varphi(t)$ is known since then the motion of the centre of mass is given and thus the total force on the system. The momentum principle gives

$$\mathbf{F} = m\ddot{\mathbf{r}}_{\mathcal{G}} \tag{4.66}$$

and here $\mathbf{F} = \mathbf{F}^a + \mathbf{R}$ where $\mathbf{F}^a$ is the 'applied' force. In figure 4.2 the only applied force is gravity so $\mathbf{F}^a = m\mathbf{g}$ but in the general case there may be other known external forces applied to the body. If we now write down the three Cartesian components of the equation of motion above we get

$$\begin{aligned} R_x &= m\ddot{x}_{\mathcal{G}} - F^a_x, &(4.67)\\ R_y &= m\ddot{y}_{\mathcal{G}} - F^a_y, &(4.68)\\ R_z &= -F^a_z. &(4.69)\end{aligned}$$

In order to make use of the known rotational motion it is, however, better to introduce cylindrical coordinates and take the components along the position dependent basis vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\varphi$. The acceleration of the centre of mass, for which $\rho = h$, is then given by

$$\ddot{\mathbf{r}}_\mathcal{G} = -h\dot{\varphi}^2\,\mathbf{e}_\rho + h\ddot{\varphi}\,\mathbf{e}_\varphi. \tag{4.70}$$

The ρ- and φ-components of the equations of motion above now give

$$\begin{aligned} R_\rho &= -mh\dot{\varphi}^2 - F^a_\rho, \\ R_\varphi &= mh\ddot{\varphi} - F^a_\varphi. \end{aligned} \tag{4.71}$$

These equations thus determine the interesting components of the reaction force on the body from the axis. The force $-\mathbf{R}$ is then, of course, the force on the axis and its bearings from the body.

We now calculate the right hand sides as explicitly as possible for the case of figure 4.2. We have that

$$\mathbf{F}^a = F^a_\rho\,\mathbf{e}_\rho + F^a_\varphi\,\mathbf{e}_\varphi = m\mathbf{g} = mg(\cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi). \tag{4.72}$$

The angular acceleration $\ddot{\varphi}$ is determined as a function of φ by equation 4.48 and is thus

$$\ddot{\varphi} = -\frac{mgh}{J_z}\sin\varphi. \tag{4.73}$$

The angular velocity squared is determined as a function of φ once the total energy E is known since

$$E = \frac{1}{2}J_z\dot{\varphi}^2 + mgh(1-\cos\varphi) \tag{4.74}$$

gives

$$\dot{\varphi}^2 = \frac{2}{J_z}[E - mgh(1-\cos\varphi)]. \tag{4.75}$$

If we now put all these results together into equations 4.71 we get

$$R_\rho(\varphi) = \frac{2mh}{J_z}(mgh - E) - \left(1 + \frac{2mh^2}{J_z}\right) mg\cos\varphi, \tag{4.76}$$

$$R_\varphi(\varphi) = \left(1 - \frac{mh^2}{J_z}\right) mg\sin\varphi, \tag{4.77}$$

and this is thus the final result. Once the initial conditions are known one can calculate E and get numbers out of these equations. Note that the small amplitude approximation has not been needed in this derivation.

4.3.3 The Reaction Forces at Two Bearings

We now again consider a rigid body rotating around a fixed axis, the Z-axis. We again assume that the rotational motion $\varphi(t)$ of the body can be determined from $J_z\ddot{\varphi} = M_z$ and that it is known. In the previous subsection we calculated the total reaction force $\mathbf{R}$ on the body from the axis. We now assume that the axis is mounted in bearings at two points $\mathcal{A}$ and $\mathcal{O}$, as shown in figure 4.4, and we wish to calculate the contributions, $\mathbf{R}_1$ and $\mathbf{R}_2$, to the total force $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$, from each of these bearings.

To solve this problem we can use the momentum principle as above and rederive the equations 4.71 for this case. We take the coordinate system $\mathcal{O}XYZ$ to be fixed in the body, see figure 4.4, and choose it so that $\mathcal{G}$ is in the xz-plane. Then $\mathbf{e}_x$ plays the same role as $\mathbf{e}_\rho$ did when we found 4.71, so we get

$$\begin{aligned} R_{1x} + R_{2x} &= -mh\dot{\varphi}^2 - F^a_x, \\ R_{1y} + R_{2y} &= mh\ddot{\varphi} - F^a_y. \end{aligned} \tag{4.78}$$

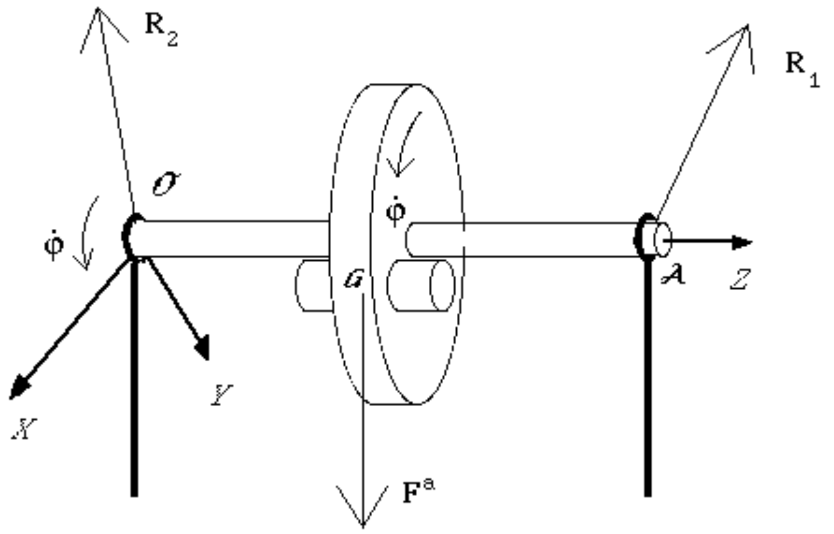


Figure 4.4: A rigid body can rotate around a fixed Z-axis. The axis is held in place by bearings at $\mathcal{A}$ and at $\mathcal{O}$. The reaction forces at these are $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively. There is also an applied force $\mathbf{F}^{\mathrm{a}}$, normally its weight, acting on the body.

There are, however, now four unknowns but still only two equations, so this does not suffice to determine the separate bearing reactions.

We'll now show that the missing equations are provided by the two unused x- and y-components of the principle of angular momentum $\dot{\mathbf{L}}_{\mathcal{O}} = \mathbf{M}_{\mathcal{O}}$. Since the angular velocity vector is $\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{e}_z$ equations 4.5 give us (as usual we leave out the base point index $\mathcal{O}$)

$$\begin{aligned} L_x = J_x\omega_x - D_{xy}\omega_y - D_{xz}\omega_z &= -D_{xz}\dot{\varphi}, \\ L_y = J_y\omega_y - D_{yx}\omega_x - D_{yz}\omega_z &= -D_{yz}\dot{\varphi}, \\ L_z = J_z\omega_z - D_{zx}\omega_x - D_{zy}\omega_y &= J_z\dot{\varphi}. \end{aligned} \tag{4.79}$$

To find the time derivative $\dot{\mathbf{L}}$ we use equation 4.44 and find for the x- and y-components

$$\dot{L}_x = \frac{d}{dt}(-D_{xz}\dot{\varphi}) + \omega_y L_z - \omega_z L_y = -D_{xz}\ddot{\varphi} + D_{yz}\dot{\varphi}^2, \tag{4.80}$$

$$\dot{L}_y = \frac{d}{dt}(-D_{yz}\dot{\varphi}) + \omega_z L_x - \omega_x L_z = -D_{yz}\ddot{\varphi} - D_{xz}\dot{\varphi}^2. \tag{4.81}$$

We now need the corresponding components of the moment. Reference to figure 4.4 immediately gives ($z_{\mathcal{A}}$ is the distance between the two bearings)

$$M_x = M_x^a - z_{\mathcal{A}} R_{1y} \tag{4.82}$$

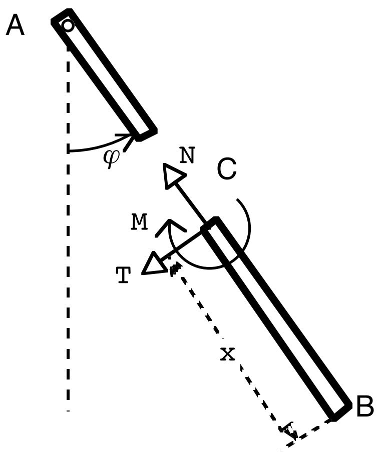
$$M_y = M_y^a + z_{\mathcal{A}} R_{1x} \tag{4.83}$$

so the angular momentum principle now provides us with the two equations for the reaction force at bearing $\mathcal{A}$:

$$R_{1y} = \frac{1}{z_{\mathcal{A}}}(D_{xz}\ddot{\varphi} - D_{yz}\dot{\varphi}^2 + M_x^a), \tag{4.84}$$

$$R_{1x} = -\frac{1}{z_{\mathcal{A}}}(D_{yz}\ddot{\varphi} + D_{xz}\dot{\varphi}^2 + M_y^a). \tag{4.85}$$

These two equations together with the equations 4.78 completely solve the problem i.e. determine the four components, perpendicular to the axis, of the the two reaction forces



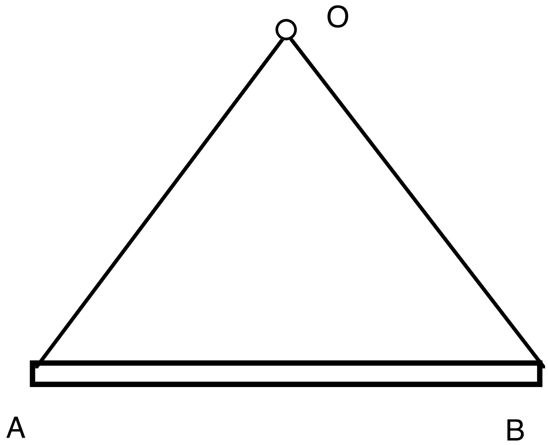


Figure 4.5: The figure on the left refers to problem 4.1. The notation used in the problem is indicated. The rod can swing around a horizontal axis through $\mathcal{A}$.

Figure 4.6: The figure on the right refers to problem 4.2. The straight narrow rod, which hangs in two strings of equal length, one from each end, can swing as a pendulum in a vertical plane.

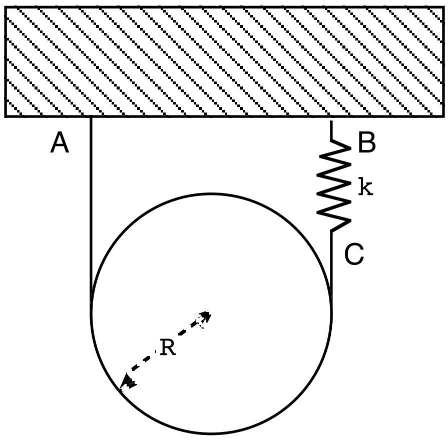
at the bearings. The force along the axis can be determined if it is assumed to arise from one of the bearings, in which case it is simply the negative of the z-component of the applied force. Should both bearings apply forces along the axis the problem is statically indeterminate.

If one wishes to minimize the reaction forces on the bearings it is obvious from equations 4.71 that one should make $h = 0$, i.e. put the centre of mass on the axis, since then only the applied force will contribute to the net reaction force $\mathbf{R}$. A rotating system for which this has been done is sometimes called 'statically balanced'. If the total force is made small in this way there is, however, still no guarantee that the individual forces on the two bearings of this section are small. To achieve this one must also make the products of inertia D_{xz} and D_{yz} zero, according to our findings above. This means that the rotation axis should be a principal axis of the body. The system is then said to be 'dynamically balanced'. Wheels that rotate rapidly should thus be both statically and dynamically balanced. An example of a system which is statically but not dynamically balanced is the one of example 1.1 on page 6.

4.4 Problems

Problem 4.1 A straight narrow homogeneous $\mathcal{AB}$ rod of mass m and length ℓ can swing, with negligible friction, around a fixed horizontal axis at $\mathcal{A}$, as shown in figure 4.5. The instantaneous angle that it makes with the vertical is φ and the maximum value for this angle is α. Calculate, as functions of φ and x, the tension N, the shear T, and the bending moment M on the part $\mathcal{CB}$ of the rod in an imaginary cut at $\mathcal{C}$ a distance x from $\mathcal{B}$.

Problem 4.2 A straight narrow rod $\mathcal{AB}$, hangs in two strings of equal length, one from each end and both fixed at $\mathcal{O}$. The rod can swing as a pendulum in a vertical plane around a horizontal axis through $\mathcal{O}$, see figure 4.6. The lengths of the rod and the strings have the ratio $6/5(= |\mathcal{AB}|/|\mathcal{AO}|)$. What maximum value of the angle of deflection is allowed if the strings are to remain taut at all times?



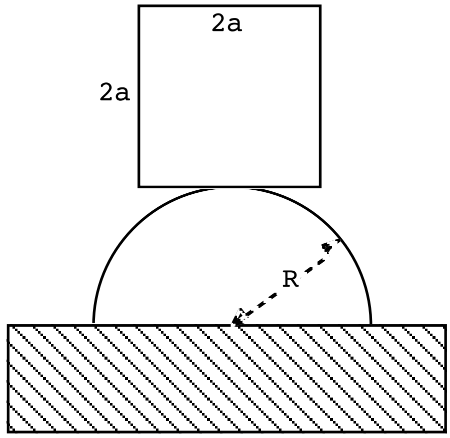


Figure 4.7: The figure on the left refers to problem 4.3. A circular disc hangs in a string fixed at the two points $\mathcal{A}$ and $\mathcal{B}$. The string goes round the circumference of the disc. The part $\mathcal{BC}$ of the string has been replaced by a spring.

Figure 4.8: The figure on the right refers to problem 4.4. A cube stands on a semi-cylinder.

Problem 4.3 A circular homogeneous circular disc of mass m and radius R hangs vertically in a string fixed at the two points $\mathcal{A}$ and $\mathcal{B}$ of a horizontal ceiling. The points are a distance $2R$ apart. The string goes round the circumference of the disc one and a half times. The part $\mathcal{BC}$ of the string has been replaced by a spring of stiffness k, as shown in figure 4.7. The string will not slip on the circumference of the disc. Calculate the period for small vertical oscillations.

Problem 4.4 A homogeneous cube of side length $2a$ stands on a rough semi-cylinder of radius R, see figure 4.8. The semi-cylinder has horizontal axis and four of the edges of the cube are parallel to this axis. At equilibrium the bottom face of the cube and the cylinder touch along a straight horizontal line in the middle of the bottom face of the cube. Calculate the period for small oscillations of the cube near this equilibrium.

Problem 4.5 A tall factory chimney made of brick is being demolished. It is severed at the base by means of an explosion an starts to fall. Assume constant thickness. Assume that it breaks again somewhere during the fall. Use the results of problem 4.1 to calculate where this is likely to happen and in which direction.

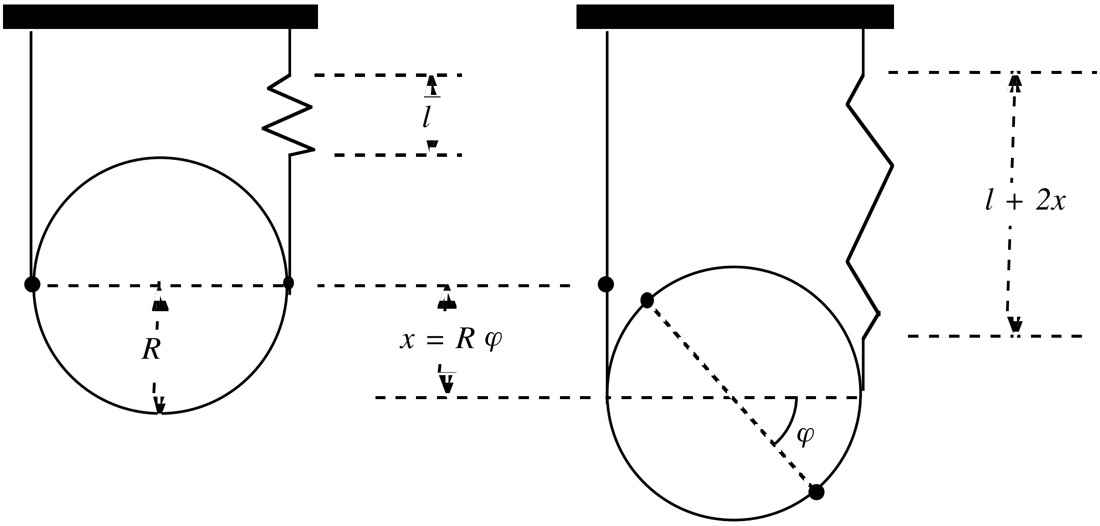


Figure 4.9: This figure refers to answer 4.3. It shows why the elongation of the spring is twice the vertical displacement, x, of the disc.

4.5 Hints and Answers

Answer 4.1 Use conservation of energy to get $\dot{\varphi}$ as a function of φ and α. This gives

$$N(\varphi, x) = mg\frac{x}{2\ell^2}[(8\ell - 3x)\cos\varphi - 3(2\ell - x)\cos\alpha],$$

and

$$T(\varphi, x) = mg\frac{x}{4\ell^2}(2\ell - 3x)\sin\varphi.$$

To find the moment of force, use $\dot{\mathbf{L}}_{\mathcal{G}} = \mathbf{M}_{\mathcal{G}}$ for the piece of length x. This relation gives

$$J_{\mathcal{G}}(x)\ddot{\varphi} = -M(x) + \frac{x}{2}T(x),$$

from which one finds

$$M(\varphi, x) = mg\left(\frac{x}{2\ell}\right)^2(\ell - x)\sin\varphi.$$

One notes that $\frac{dM}{dx} = T(x)$, a well known relation in strength of materials.

Answer 4.2 Do the experiment at home and observe on which side the string first becomes slack. The maximum allowed angle is given by

$$\alpha = \arctan\frac{19}{4}$$

and this is the answer.

Answer 4.3 The string has fixed length and must go one and a half times round the disc at all times. The disc then translates and rotates in such a way that the vertical displacement is R times the rotation angle, see figure 4.9. The elongation of the spring must be twice the vertical displacement of the disc. One finds that

$$T = \pi\sqrt{\frac{3m}{2k}}$$

is the period of the oscillations.

Answer 4.4 The kinetic energy of the cube is

$$T = \frac{1}{2} J_{\mathcal{C}}(\varphi)\dot{\varphi}^2,$$

where $\mathcal{C}$ is a contact point and $J_{\mathcal{C}}$ is the instantaneous moment of inertia of the cube for an axis through a contact point parallel to the axis of the cylinder. The potential energy is

$$\Phi = mg(R\cos\varphi + R\varphi\sin\varphi + a\cos\varphi),$$

where φ is the tilt angle of the cube. The period is found to be

$$T = 2\pi\sqrt{\frac{5a^2}{3g(R-a)}}.$$

Answer 4.5 The chimney is likely to break where the bending moment $M(x,\varphi)$ has its maximum. This happens at $x = 2\ell/3$.

Chapter 5

Three Dimensional Motion of Rigid Bodies

In this chapter we discuss the general non-planar rotational motion of rigid bodies. In particular we discuss the free motion (no external moment) and the motion of the symmetric top in the gravitational field.

The general rotational motion of an asymmetric top rigid body is in principle given by solving Euler's dynamic equations 4.47 and Euler's kinematic equations 2.69 together as a system of coupled non-linear differential equations for the three functions $\psi(t)$, $\theta(t)$, and $\varphi(t)$. In the general case, for some given external moment $\mathbf{M}$ acting on the body, the only method for finding a solution is by numerical techniques. In the free case when the external moment is zero $\mathbf{M} = \mathbf{0}$ it is possible to find a solution of Euler's dynamic equations in terms of elliptic functions but we will not go into this. In order to get reasonably simple closed form solutions we concentrate on the symmetric top with principal moments of inertia given by $J_1 = J_2 \neq J_3$. For the free asymmetric top we discuss the Poinsot construction and stability of rotation around the principal axes.

5.1 The Spherical Top

The free motion of the spherical top is the easiest of all to treat. The inertia tensor matrix for such a body is by definition diagonal and proportional to the unit matrix in all basis triads: $\mathsf{J} = J_1\mathbf{1}$. Note that a body need *not* be spherical in order to have such an inertia tensor. For example a body with the symmetry of a cube or tetrahedron (see example 4.1, page 61) will be a 'spherical top'.

For the free spherical top the total external moment $\mathbf{M} = \mathbf{0}$. The equation of motion $\dot{\mathbf{L}} = \mathbf{0}$ together with the relation $\mathbf{L} = \hat{J}\boldsymbol{\omega} = J_1\boldsymbol{\omega}$ immediately gives $\dot{\boldsymbol{\omega}} = \mathbf{0}$. The solution is

solution of the free spherical top

$$\boldsymbol{\omega} = \mathbf{L}/J_1 = \text{const.} \tag{5.1}$$

so the body spins with an angular velocity which is constant both in magnitude and direction and parallel to $\mathbf{L}$.

For bodies that really have spherical shape there are many interesting rolling problems that can be attacked by means of the basic equations of mechanics. One such problem is treated in the following example.

Example 5.1 A sphere of mass m and radius R rolls on a rough inclined plane. The angle of inclination is β, see figure 5.1. The moment of inertia of the sphere is J for any axis through its center of mass (=centroid) $\mathcal{G}$. Let the X-axis point downwards along the incline.
a) Find the trajectory of the sphere on the plane for arbitrary initial conditions.
b) Use conservation of mechanical energy to find an equation connecting x and $\dot{x}$.
Solution:
a) Since the sphere rolls without slipping the point $\mathcal{C}$ on the sphere in contact with the plane is

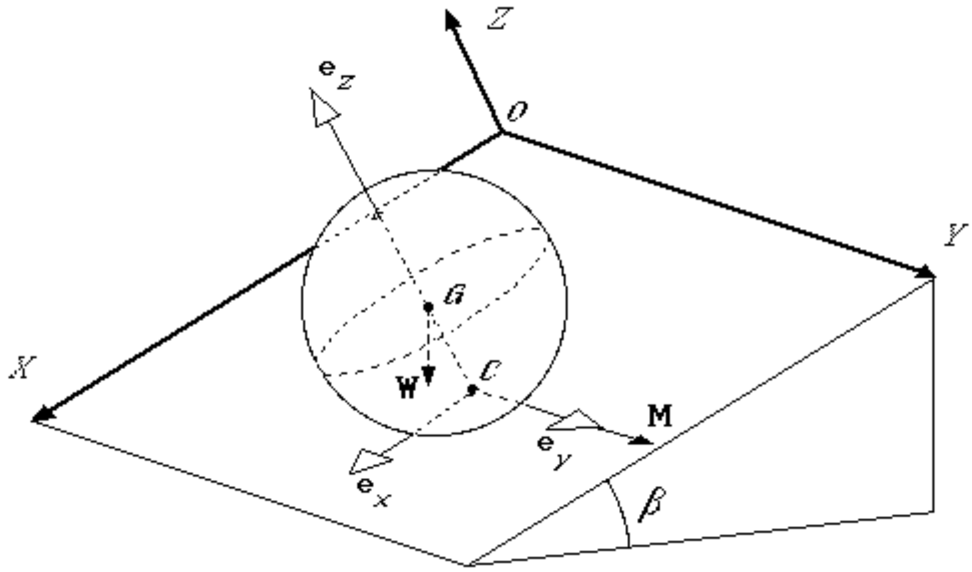


Figure 5.1: The notation used in example 5.1 is defined in this figure. The sphere rolls without slipping on the inclined plane.

instantaneously at rest. The motion of the sphere is thus pure rotation around $\mathcal{C}$. The moment of inertia of the sphere with respect to any axis in the plane through $\mathcal{C}$ is, according to the parallel axes theorem,

$$J_{\mathcal{C}} = J + mR^2. \tag{5.2}$$

We will use the equation of motion $\dot{\mathbf{L}}_{\mathcal{C}} = \mathbf{M}_{\mathcal{C}}$ and we thus need expressions for $\mathbf{L}_{\mathcal{C}}$ and $\mathbf{M}_{\mathcal{C}}$. These are found to be (see equation 1.12)

$$\mathbf{L}_{\mathcal{C}} = \overline{\mathcal{CG}} \times \mathbf{p} + \mathbf{L}'_{\mathcal{G}} = R\,\mathbf{e}_z \times m\mathbf{v}_{\mathcal{G}} + J\boldsymbol{\omega}, \tag{5.3}$$

and

$$\mathbf{M}_{\mathcal{C}} = \mathbf{M} = \overline{\mathcal{CG}} \times \mathbf{W} = Rmg\sin\beta\,\mathbf{e}_y \tag{5.4}$$

where $\mathbf{W} = -mg(\cos\beta\,\mathbf{e}_z - \sin\beta\,\mathbf{e}_x)$ is the weight of the sphere as indicated in figure 5.1. Since $\mathbf{v}_{\mathcal{C}} = \mathbf{0}$ the connection formula for velocities in a rigid body, equation 2.74, gives us

$$\mathbf{v}_{\mathcal{G}} = \mathbf{v} = \boldsymbol{\omega} \times \overline{\mathcal{CG}} = (\omega_x\,\mathbf{e}_x + \omega_y\,\mathbf{e}_y + \omega_z\,\mathbf{e}_z) \times R\,\mathbf{e}_z \tag{5.5}$$

for the center of mass velocity, $\mathbf{v}$, of the sphere. The three components of this equation can be written

$$\dot{x} = R\omega_y \tag{5.6}$$
$$\dot{y} = -R\omega_x \tag{5.7}$$
$$\dot{z} = 0 \tag{5.8}$$

so that $\mathbf{v} = R\omega_y\,\mathbf{e}_x - R\omega_x\,\mathbf{e}_y$. Use of this in the expression for $\mathbf{L}_{\mathcal{C}}$ gives

$$\mathbf{L}_{\mathcal{C}} = mR\,\mathbf{e}_z \times (R\omega_y\,\mathbf{e}_x - R\omega_x\,\mathbf{e}_y) + J\boldsymbol{\omega} = \tag{5.9}$$
$$= (J + mR^2)\omega_x\,\mathbf{e}_x + (J + mR^2)\omega_y\,\mathbf{e}_y + J\omega_z\,\mathbf{e}_z, \tag{5.10}$$

and the equation of motion $\dot{\mathbf{L}}_{\mathcal{C}} = \mathbf{M}$ thus yields

$$(J + mR^2)\dot{\omega}_x\,\mathbf{e}_x + (J + mR^2)\dot{\omega}_y\,\mathbf{e}_y + J\dot{\omega}_z\,\mathbf{e}_z = Rmg\sin\beta\,\mathbf{e}_y. \tag{5.11}$$

The components of this vector equation are

$$(J + mR^2)\dot{\omega}_x = 0 \tag{5.12}$$
$$(J + mR^2)\dot{\omega}_y = Rmg\sin\beta \tag{5.13}$$
$$J\dot{\omega}_z = 0 \tag{5.14}$$

If we put

$$\alpha \equiv \dot{\omega}_y = \frac{Rmg\sin\beta}{J+mR^2} \tag{5.15}$$

for the angular acceleration around the Y-axis, we can solve these equation for the angular velocity components as follows

$$\omega_x = \omega_x(0) = \text{const.}, \tag{5.16}$$
$$\omega_y = \omega_y(0) + \alpha t, \tag{5.17}$$
$$\omega_z = \omega_z(0) = \text{const.} \tag{5.18}$$

This can be inserted into equation 5.6 above and integrated to give

$$\dot{x} = R\omega_y = R(\omega_y(0) + \alpha t) \tag{5.19}$$
$$\Rightarrow \quad \int_0^t R(\omega_y(0) + \alpha t')\, dt' = x(t) - x(0) \tag{5.20}$$
$$\Rightarrow \quad R\omega_y(0)\, t + R\frac{1}{2}\alpha t^2 = x(t) - x(0), \tag{5.21}$$
$$\Rightarrow \quad x(t) = x(0) + R\omega_y(0)\, t + \frac{1}{2}R\left(\frac{Rmg\sin\beta}{J+mR^2}\right)t^2. \tag{5.22}$$

To find the time dependence of the y-coordinate we use equation 5.7 and proceed in a similar way. The result is

$$y(t) = y(0) - R\omega_x(0)\, t. \tag{5.23}$$

We have thus found the trajectory of the sphere on the inclined plane in terms of the initial conditions $x(0), y(0), \omega_x(0)$, and $\omega_y(0)$.
b) Use of equation 3.21 gives us the expression

$$T = \frac{1}{2}[(J+mR^2)\omega_x^2 + (J+mR^2)\omega_y^2 + J\omega_z^2] \tag{5.24}$$

for the kinetic energy of the rolling sphere. The potential energy is as usual given by mg times the height of the center of mass of the sphere. This gives us

$$\Phi = -mgx\sin\beta. \tag{5.25}$$

Note that gravity is the only force doing work on the sphere since the other forces, normal force and (static) friction, do no work. Thus the mechanical energy

$$E = T + \Phi \tag{5.26}$$

is conserved. Since ω_x and ω_z are also constants use of equation 5.6 allows us to write

$$\text{const.} = \left(E + \frac{1}{2}(J+mR^2)\omega_x^2 + \frac{1}{2}J\omega_z^2\right) = \frac{1}{2}(J+mR^2)\left(\frac{\dot{x}}{R}\right)^2 - mgx\sin\beta. \tag{5.27}$$

This concludes our treatment of the sphere rolling down a rough incline. □

5.2 The Symmetric Top

In this section we treat the motion of the symmetric top in some detail. One of the facts that make it easier to treat the symmetric top is that the inertia tensor matrix is constant, not only in a body fixed basis $\mathbf{E}^{\mathrm{A}}$, but also in any basis where one basis vector is along the axis with different moment of inertia J_3, as for example the basis $\mathbf{E}^{\mathrm{B}}$ of figure 5.2. This axis is the symmetry axis of the body's inertia tensor. Relative to the reference frame B of this basis the angular velocity vector of the body is

$$^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}} = \dot{\varphi}\,\mathbf{e}_3^{\mathrm{B}}. \tag{5.28}$$

The angular velocity of the frame B relative to the fixed frame O is given by

$$^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} = \dot{\psi}\,\mathbf{e}_3^{\mathrm{O}} + \dot{\theta}\,\mathbf{e}_1^{\mathrm{B}} = \dot{\theta}\,\mathbf{e}_1^{\mathrm{B}} + \dot{\psi}(\sin\theta\,\mathbf{e}_2^{\mathrm{B}} + \cos\theta\,\mathbf{e}_3^{\mathrm{B}}). \tag{5.29}$$

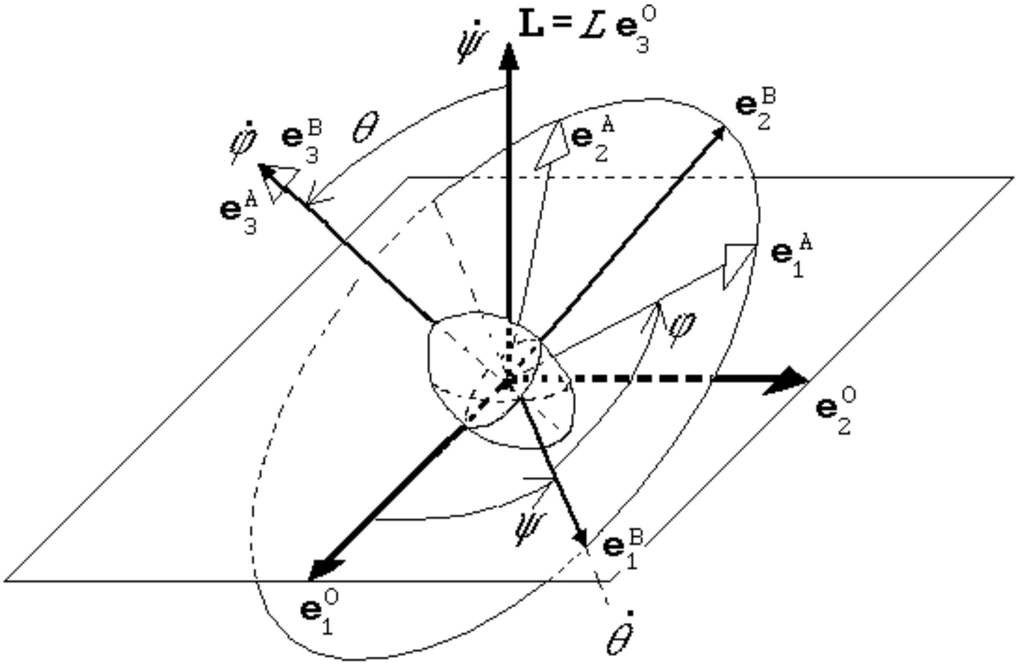


Figure 5.2: To study the motion of the symmetric top there is no need to use the body fixed basis E^{A}. Because of the equality of two of the moments of inertia all elements of the inertia tensor matrix are constant already in the basis E^{B} where the 1,2-plane corresponds to the plane of the equal moments of inertia. For the free symmetric top the origin $\mathcal{O}$ is taken at the center of mass ($\mathcal{O} = \mathcal{G}$) and the third axis of the observer fixed system, $\mathbf{e}_3^{\mathrm{O}}$, is taken along the constant angular momentum vector $\mathbf{L}$.

According to the theorem on the additivity of angular velocities the angular velocity of the body relative to the fixed reference frame is then

$$\boldsymbol{\omega} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{A}} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}} = \dot\theta\,\mathbf{e}_1^{\mathrm{B}} + \dot\psi\sin\theta\,\mathbf{e}_2^{\mathrm{B}} + (\dot\psi\cos\theta + \dot\varphi)\,\mathbf{e}_3^{\mathrm{B}} \tag{5.30}$$

(compare equation 2.65 where the same vector is given in the A basis).

5.2.1 The Free Symmetric Top

For the free symmetric top, see figure 5.2, we write down the components of the vector equation $\mathbf{L} = \hat{J}\boldsymbol{\omega}$ in the basis E^{B}, i.e. the elements of the matrix equation $\mathsf{L}^{\mathrm{B}} = \mathsf{w}^{\mathrm{B}}\mathsf{J}^{\mathrm{B}}$. The angular velocity is given above in equation 5.30 so we have that

$$\mathsf{w}^{\mathrm{B}}\mathsf{J}^{\mathrm{B}} = (\dot\theta \;\; \dot\psi\sin\theta \;\; \dot\psi\cos\theta + \dot\varphi)\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_3 \end{pmatrix}. \tag{5.31}$$

We select the fixed 3-axis along the (constant) angular momentum vector so that $\mathbf{L} = L\,\mathbf{e}_3^{\mathrm{O}}$. The components of $\mathbf{L}$ in the B basis are then given by

$$\mathsf{L}^{\mathrm{B}} = (0 \;\; L\sin\theta \;\; L\cos\theta). \tag{5.32}$$

so we get the result

$$\begin{aligned} 0 &= J_1\dot\theta, \\ L\sin\theta &= J_1\dot\psi\sin\theta, \\ L\cos\theta &= J_3(\dot\psi\cos\theta + \dot\varphi). \end{aligned} \tag{5.33}$$

The first of these equations says that $\theta = \theta_0$ =const. I.e. the axis of the top makes a constant angle with the angular momentum vector $\mathbf{L}$. The second of the three equations gives us

$$\dot{\psi} = \frac{L}{J_1} \tag{5.34}$$

unless $\theta_0 = 0$. This is the angular velocity with which the axis of the top rotates around $\mathbf{L}$. The third equation gives

$$\dot{\varphi} = \frac{L}{J_1} \cos\theta_0 \frac{J_1 - J_3}{J_3} \tag{5.35}$$

for the angular velocity of the body in the B frame. Note that the sign of this angular velocity depends on the shape of the body; it is positive when $J_1 > J_3$ as is the case for the 'prolate' body in figure 5.2 (assuming it is homogeneous) but it would be negative for an 'oblate' (flattened) body.

In the case $\theta_0 = 0$, when the axis of the top is parallel to the angular momentum, the third of our equations give $L/J_3 = \dot{\psi} + \dot{\varphi}$ and this is simply the magnitude of the angular velocity of the top around the common fixed direction of $\mathbf{L}$ and its axis. In this case one cannot resolve the two angular velocities since the B frame becomes undefined.

This last result, that if the symmetric top spins around its axis, then this axis is parallel to the angular momentum vector and is fixed in space, is the principle behind the gyroscope. A symmetric top body which has been mounted in a pair of gimbal rings so that the external moment on it is negligible will be a free symmetric top. If it spins around its symmetry axis this axis will have a fixed direction in space so the system can be used as a compass.

Example 5.2 Consider Euler's dynamic equations 4.47 for the free symmetric top:

$$\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_1)\omega_2\omega_3 &= 0, \\ J_1\dot{\omega}_2 - (J_3 - J_1)\omega_1\omega_3 &= 0, \\ J_3\dot{\omega}_3 &= 0. \end{aligned} \tag{5.36}$$

Here ω_i, $(i = 1, 2, 3)$ are the body fixed components of the angular velocity vector. Find the time dependence of these.

Solution:

The third of the equations 5.36 immediately gives

$$\omega_3(t) = \omega_3(0) = \text{const.} \tag{5.37}$$

The first two can then be rewritten in the form

$$\dot{\omega}_1 + \beta\omega_2 = 0, \tag{5.38}$$

$$\dot{\omega}_2 - \beta\omega_1 = 0 \tag{5.39}$$

where

$$\beta \equiv \frac{J_3 - J_1}{J_1}\omega_3(0). \tag{5.40}$$

If we now multiply the equation for ω_2 by the imaginary unit i and add the result to the equation for ω_1 we find that the quantity

$$\zeta \equiv \omega_1 + i\omega_2 \tag{5.41}$$

obeys

$$\dot{\zeta} - i\beta\zeta = 0. \tag{5.42}$$

The solution to this simple differential equation is

$$\zeta(t) = A\exp(i\beta t) \tag{5.43}$$

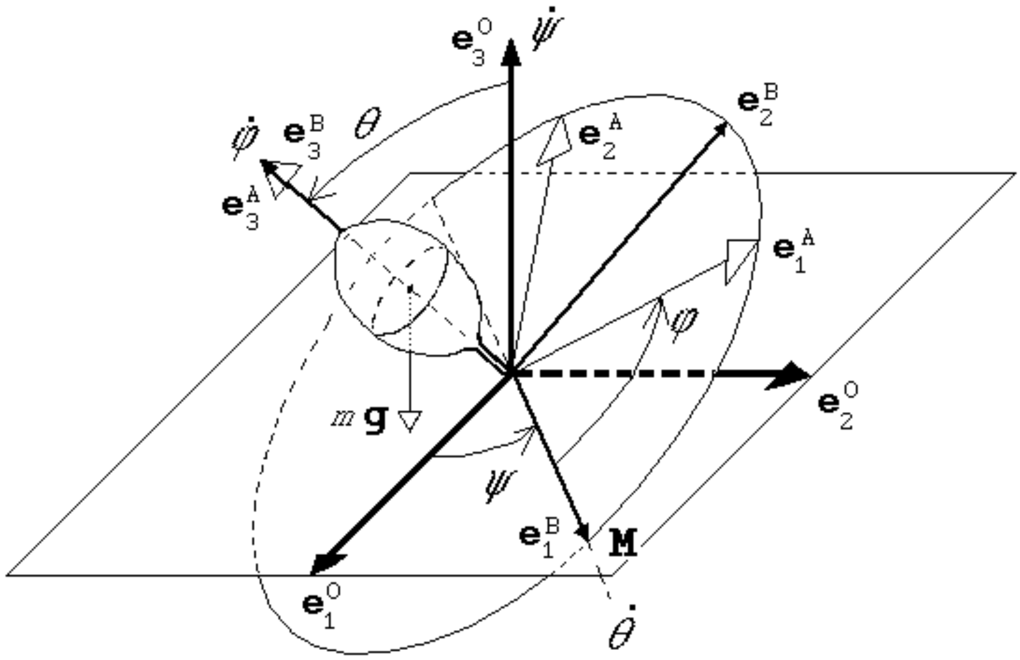


Figure 5.3: Just as the free symmetric top the heavy symmetric top is best studied in the basis E^{B}. The origin $\mathcal{O}$ is taken at the (fixed) point of contact, $\mathcal{C}$, with the ground ($\mathcal{O} = \mathcal{C}$) and the third axis of the observer fixed system, $\mathbf{e}_3^{\mathrm{O}}$, is taken along the vertical direction. The moment is given by $\mathbf{M} = \overline{\mathcal{CG}} \times m\mathbf{g} = -\ell\,\mathbf{e}_3^{\mathrm{B}} \times mg\,\mathbf{e}_3^{\mathrm{O}} = mg\ell \sin\theta\,\mathbf{e}_1^{\mathrm{B}}$.

where A is a complex constant that depends on the initial conditions. The result is thus that

$$\omega_1(t) = |A|\cos(\beta t + \alpha), \quad (5.44)$$
$$\omega_2(t) = |A|\sin(\beta t + \alpha), \quad (5.45)$$

where α is the argument of $A = |A|\exp(i\alpha)$. This means that the angular velocity vector, in the body fixed system, rotates with angular velocity β, around the third principal axis (the one with different moment of inertia). $\square$

5.2.2 The Heavy Symmetric Top

The heavy symmetric top is essentially a model of the well known toy top with which everyone has played at one time or another. It is set spinning by rotating an axis on it between the thumb and the other fingers. This axis goes through the top and has a relatively pointed end which is in contact with the floor or table surface on which the top is left spinning after starting it. Usually a fairly oblate (flat) body is mounted on this axis since this makes the resulting motion more stable as we'll see below.

Our model is that of a rigid body which can rotate with negligible friction around a fixed point $\mathcal{C}$, which does not coincide with the center of mass, $\mathcal{G}$, of the body. The well defined principal axis of the body goes through both $\mathcal{C}$ and $\mathcal{G}$. Any pair of perpendicular directions in the plane perpendicular to this axis, the axis of the top, will define principal axes of the body with equal principal moments of inertia. In a suitably chosen coordinate system with axis directions given by the basis E^{B} and origin at $\mathcal{C}$, see figure 5.3, the inertia tensor matrix of the body will be

$$\mathsf{J}^{\mathcal{C}} = \mathsf{J}^{\mathcal{G}} + \begin{pmatrix} m\ell^2 & 0 & 0 \\ 0 & m\ell^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} J_1 + m\ell^2 & 0 & 0 \\ 0 & J_1 + m\ell^2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} = \begin{pmatrix} J_1' & 0 & 0 \\ 0 & J_1' & 0 \\ 0 & 0 & J_3 \end{pmatrix}. \quad (5.46)$$

Here m is the mass of the body and ℓ is the distance between $\mathcal{G}$ and $\mathcal{C}$. We have used formula 4.38 in which we have put that the components of $\mathbf{R}$ in the B coordinate system are $(R_1\ R_2\ R_3) = (0\ 0\ \ell)$. Note that this inertia tensor in the B coordinate system is exactly the same as in the body fixed A system because of the symmetry (i.e. equality of the principal moments of inertia), just as we found in our treatment of the free symmetric top. The reason, in both cases, being that the body fixed frame is obtained by a rotation of the B frame an angle φ around the axis of the top.

The angular velocity vectors of the frame B and the body are given by the same expressions as in equation 5.30 and those above it. We now calculate the equations of motion $\dot{\mathbf{L}} = \mathbf{M}$ and we wish the components in the B frame. We get

$$\dot{\mathbf{L}} = \frac{{}^{\mathrm{B}}d\mathbf{L}}{dt} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} \times \mathbf{L} \tag{5.47}$$

according to equation 2.61 and we have that

$$\mathbf{L} = \hat{J}\boldsymbol{\omega} = \hat{J}({}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}) = J_1'(\dot{\theta}\,\mathbf{e}_1^{\mathrm{B}} + \dot{\psi}\sin\theta\,\mathbf{e}_2^{\mathrm{B}}) + J_3(\dot{\psi}\cos\theta + \dot{\varphi})\mathbf{e}_3^{\mathrm{B}}. \tag{5.48}$$

Here we have introduced $J_1' \equiv J_1 + m\ell^2$. If we now use the expression in equation 5.29 for ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}}$ we can calculate the vector product that we need for the time derivative of $\mathbf{L}$ in the B frame. The moment of the weight of the body can be read off directly from figure 5.3 and is

$$\mathbf{M} = mg\ell\sin\theta\,\mathbf{e}_1^{\mathrm{B}}. \tag{5.49}$$

Finally the angular momentum law, $\dot{\mathbf{L}} = \mathbf{M}$, now gives us

$$\begin{aligned}
\frac{d}{dt}(J_1'\dot{\theta}) + [(J_3 - J_1')\dot{\psi}^2\sin\theta\cos\theta + J_3\dot{\psi}\dot{\varphi}\sin\theta] &= mg\ell\sin\theta && (5.50)\\
\frac{d}{dt}(J_1'\dot{\psi}\sin\theta) - [(J_3 - J_1')\dot{\psi}\dot{\theta}\cos\theta + J_3\dot{\theta}\dot{\varphi}] &= 0 && (5.51)\\
\frac{d}{dt}[J_3(\dot{\psi}\cos\theta + \dot{\varphi})] &= 0 && (5.52)
\end{aligned}$$

Of these complicated equations only the third gives some simply useful information. It tells us that

$$L_3 \equiv J_3(\dot{\psi}\cos\theta + \dot{\varphi}) = \text{const.} \tag{5.53}$$

i.e. that the component of the angular momentum along the axis of the top is conserved. The reason for this is that neither the vector product ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} \times \mathbf{L}$ nor the moment $\mathbf{M}$ have components along $\mathbf{e}_3^{\mathrm{B}}$, and this depends crucially on the fact that the two first moments of inertia are equal. The corresponding conserved quantity therefore does not exist for an asymmetric top.

Since the moment $\mathbf{M}$ always lies in the horizontal plane, $\mathbf{M} = M_x\,\mathbf{e}_1^{\mathrm{O}} + M_y\,\mathbf{e}_2^{\mathrm{O}}$, one can conclude that the vertical component of $\mathbf{L}$ also must be conserved; just consider the third (Z-component) of $\dot{\mathbf{L}} = \mathbf{M}$ in the fixed O-system. Use of equation 5.48 gives us the following expression for this component of the angular momentum vector

$$\mathbf{L}\cdot\mathbf{e}_3^{\mathrm{O}} = \mathbf{L}\cdot(\sin\theta\,\mathbf{e}_2^{\mathrm{B}} + \cos\theta\,\mathbf{e}_3^{\mathrm{B}}) = \dot{\psi}(J_1'\sin^2\theta + J_3\cos^2\theta) + J_3\dot{\varphi}\cos\theta. \tag{5.54}$$

We thus also have the following conserved quantity

$$L_z \equiv (J_1'\sin^2\theta + J_3\cos^2\theta)\dot{\psi} + J_3\dot{\varphi}\cos\theta = \text{const.} \tag{5.55}$$

for the heavy symmetric top. The constraint force acting on the top at $\mathcal{C}$ does no work so the only work done on the body is done by the conservative force of gravity. This means that the mechanical energy of the top is conserved. We'll use these conserved quantities later to study the 'nutation'.

5.2.3 Precession of the Heavy Symmetric Top

Everyone who has played with a top has observed that it normally moves in a special way. The top spins rapidly around its axis while the axis rotates slowly around the vertical direction while maintaining a constant angle with the vertical. This slow rotation of the axis is called 'precession'. Since this is an observed fact it is natural to check whether this type of motion can be a solution to the equations of motion 5.50 - 5.52. We thus put

$$\Omega = \dot{\psi} = \text{const.}, \tag{5.56}$$

$$\theta_0 = \theta = \text{const.} \tag{5.57}$$

for the constant angular velocity of precession and the constant angle between axis and vertical respectively. When this is inserted in equation 5.53 one immediately finds that also the angular velocity of the top around its own axis must be constant

$$\dot{\varphi} = \frac{L_3}{J_3} - \Omega \cos\theta_0 = \text{const.} \tag{5.58}$$

The third equation 5.52 is then automatically satisfied. The second equation, 5.51, is easily seen to satisfied by this type of motion; all its terms become zero. The first equation (5.50) gives after some algebraic manipulation that

$$\dot{\varphi} = \frac{1}{J_3}\left[\frac{mg\ell}{\Omega} - \Omega(J_3 - J_1')\cos\theta_0\right]. \tag{5.59}$$

We thus see that provided the value of L_3 is given by

$$L_3 = \frac{mg\ell}{\Omega} + J_1'\Omega\cos\theta_0. \tag{5.60}$$

all these results agree and constitute one possible solution of the equations of motion. One finds the following expression for the angular velocity of precession

$$\Omega = \frac{mg\ell}{J_3\dot{\varphi} - (J_3 - J_1')(\dot{\varphi} - L_3/J_3)}. \tag{5.61}$$

One might call this kind of motion 'pure' precession. It is characterized by the constancy of the angle θ. Changes in θ are called nutations and are treated below.

It is fairly easy to see that the precessional motion always is a good approximation to the exact motion when the top spins rapidly enough. If $\dot{\varphi}$ is very large one should have $\mathbf{L} \approx J_3\dot{\varphi}\,\mathbf{e}_3^{\mathrm{B}}$. Then however $\mathbf{L}$ is parallel to the moment arm of the weight and we get from $\dot{\mathbf{L}} = \mathbf{M}$ that (in the fixed reference frame)

$$\frac{{}^{\mathrm{O}}d\mathbf{L}}{dt} + \frac{m\ell}{L}\mathbf{g} \times \mathbf{L} = \mathbf{0} \tag{5.62}$$

after moving the moment to the left hand side. This, however, is the equation (see equation 2.61)

$$\frac{{}^{\mathrm{P}}d\mathbf{L}}{dt} = \frac{{}^{\mathrm{O}}d\mathbf{L}}{dt} + {}^{\mathrm{P}}\boldsymbol{\omega}^{\mathrm{O}} \times \mathbf{L} = \mathbf{0}. \tag{5.63}$$

that $\mathbf{L}$ would obey if it was at rest in a reference frame P with respect to which O rotates with the angular velocity

$${}^{\mathrm{P}}\boldsymbol{\omega}^{\mathrm{O}} = \frac{m\ell}{L}\mathbf{g} = -\frac{m\ell g}{J_3\dot{\varphi}}\mathbf{e}_3^{\mathrm{O}}. \tag{5.64}$$

The conclusion must be that **L** and with it the axis of the top, rotates around the vertical axis with an angular velocity ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{P}}$ which is the negative of ${}^{\mathrm{P}}\boldsymbol{\omega}^{\mathrm{O}}$. This angular velocity is thus

$$\boldsymbol{\Omega} = \frac{m\ell g}{J_3\dot{\varphi}}\,\mathbf{e}_3^{\mathrm{O}}. \tag{5.65}$$

If we compare with the result 5.61 above we see that these agree reasonably with each other when $\dot{\varphi} \approx L_3/J_3$ and according to equation 5.58 this happens when $|\Omega\cos\theta_0| \ll |L_3/J_3|$. It is thus exact also when $\theta_0 = \pi/2$.

5.2.4 Nutation of the Heavy Symmetric Top

The energy of the heavy symmetric top is the sum of the kinetic energy, which can be calculated from equation 4.32, and the potential energy. One finds

$$E = \frac{1}{2}J_1'(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + \frac{L_3^2}{2J_3} + mg\ell\cos\theta \tag{5.66}$$

where the constant L_3 is given in equation 5.53. However, using the two conserved components of the angular momentum in equations 5.53 and 5.55 one can easily find that

$$\dot{\psi} = \frac{L_z - L_3\cos\theta}{J_1'\sin^2\theta} \tag{5.67}$$

and thus also eliminate $\dot{\psi}$ from the expression for the energy. The result is that

$$E' = \frac{1}{2}J_1'\dot{\theta}^2 + \Phi_{\text{eff}}(\theta) \tag{5.68}$$

where E' is the constant

$$E' \equiv E - \frac{L_3^2}{2J_3} - mg\ell \tag{5.69}$$

and

$$\Phi_{\text{eff}}(\theta) \equiv \frac{(L_z - L_3\cos\theta)^2}{2J_1'\sin^2\theta} - mg\ell(1-\cos\theta). \tag{5.70}$$

Conservation of energy has thus given us a one-dimensional effective energy conservation law for the θ-motion.

We can now find the limits (turning points) of the θ-motion as the roots of the equation

$$E' - \Phi_{\text{eff}}(\theta) = 0. \tag{5.71}$$

It turns out that there are in general two roots, $\theta_1 \leq \theta_2$, of this equation between which 'nutation' takes place. When the two roots coincide one has the special case of pure precession.

Example 5.3 Find the condition for the rotation of the heavy symmetrical top in a vertical position to be stable.
Solution: For $\theta = 0$ the two vectors $\mathbf{e}_3^{\mathrm{B}}$ and $\mathbf{e}_3^{\mathrm{O}}$ coincide so we must have that $L_3 = L_z$. We also see that $E' = 0$. The rotation will be stable if the function $\Phi_{\text{eff}}(\theta)$ has a minimum at $\theta = 0$. To check if this is the case one puts $L_3 = L_z$ in and then approximate the trigonometric functions according to $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. The result is that

$$\Phi_{\text{eff}}(\theta) \approx \left(\frac{L_3^2}{8J_1'} - \frac{mg\ell}{2}\right)\theta^2. \tag{5.72}$$

This function thus has a minimum when $L_3^2 > 4J_1'mg\ell$. So when the angular momentum is large enough this motion is stable, but when friction has reduced the angular momentum to this limiting value the top starts to wobble and soon falls to the ground. □

5.2.5 The Resal System

Sometimes the motion of a rigid body of symmetric top type is partly known. In this section we will consider the problem of the symmetric top for the case that the motion of the B-system of figure 5.3 is known. The angles ψ and θ thus have some known time dependence and the angular velocity vector ${}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}}$ of formula 5.29 is assumed known. In this case the equations of motion are equations connecting $\varphi(t)$ and the three components of the moment $\mathbf{M}$. If one of these is known one can solve for the other three.

We now denote the known angular velocity of the B-system, the so called Resal[1] system, by $\boldsymbol{\Omega}$. Since the angular velocity of the A-system with respect to the B-system is $\dot{\varphi}\,\mathbf{e}_3^{\mathrm{B}}(=\dot{\varphi}\,\mathbf{e}_3^{\mathrm{A}})$, equation 5.30, where now $\boldsymbol{\Omega} = {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}}$, gives

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\varphi}\,\mathbf{e}_3^{\mathrm{B}} \tag{5.73}$$

for the total angular velocity vector of the body.

To find the equations connecting $\mathbf{M}$ and $\dot{\varphi}$ we now consider $\dot{\mathbf{L}} = \mathbf{M}$ and calculate the components of this vector equation in the B-system. In this system we have

$$\mathsf{L}^{\mathrm{B}} = (\Omega_1,\ \ \Omega_2,\ \ \Omega_3+\dot{\varphi})\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_3 \end{pmatrix} = (J_1\Omega_1,\ \ J_1\Omega_2,\ \ J_3(\Omega_3+\dot{\varphi})) \tag{5.74}$$

To get the B-system components we also need

$$\dot{\mathbf{L}} = \frac{{}^{\mathrm{B}}d\mathbf{L}}{dt} + \boldsymbol{\Omega}\times\mathbf{L} = \mathbf{M}, \tag{5.75}$$

and the B-componenets of this equation are

$$(J_1\dot{\Omega}_1,\ \ J_1\dot{\Omega}_2,\ \ J_3(\dot{\Omega}_3+\ddot{\varphi})) + \boldsymbol{\Omega}\times(J_1\Omega_1,\ \ J_1\Omega_2,\ \ J_3(\Omega_3+\dot{\varphi})) = (M_1, M_2, M_3). \tag{5.76}$$

The three component equations thus are

$$\begin{aligned} J_1\dot{\Omega}_1 + (J_3-J_1)\Omega_2\Omega_3 + J_3\Omega_2\dot{\varphi} &= M_1, \\ J_1\dot{\Omega}_2 - (J_3-J_1)\Omega_1\Omega_3 - J_3\Omega_1\dot{\varphi} &= M_2, \\ J_3\dot{\Omega}_3 + J_3\ddot{\varphi} &= M_3, \end{aligned} \tag{5.77}$$

and they are clearly linear in $\mathbf{M}(t)$ and $\varphi(t)$. If $\varphi(t)$ is known one can calculate the moment $\mathbf{M}$, or if one component of $\mathbf{M}$ is known one can calculate $\varphi(t)$ and the other two components of $\mathbf{M}$.

Example 5.4 On a light horizontal axis $\mathcal{OB}$, of length ℓ, is mounted a bicycle wheel of mass m that can rotate freely around it at $\mathcal{B}$, see figure 5.4. The moment of inertia of the wheel, which can be considered to be thin, with respect to the axis $\mathcal{OB}$ is J and the axis is connected by a ball and socket joint at $\mathcal{O}$ to a fixed vertical axis. This vertical axis rotates with constant angular velocity Ω. By means of an arm, which has a smooth ring at $\mathcal{A}$ through which $\mathcal{OB}$ goes, it keeps the axis $\mathcal{OB}$ horizontal, and carries it around in the rotation. The angular velocity of the bicycle wheel around the axis $\mathcal{OB}$ is $\dot{\varphi}$. Calculate the force $\mathbf{F}$ at $\mathcal{A}$ from the ring on the axis $\mathcal{OB}$, assuming that the distance $|\mathcal{OA}| = d\ (<\ell)$.

Solution:

The symmetric top in this case is the bicycle wheel together with the axis $\mathcal{OB}$ which also is the symmetry axis. We introduce a B-system just as in figure 5.3 so that $\mathbf{e}_3^{\mathrm{B}}$ is along $\mathcal{OB}$, the angle $\theta = \frac{\pi}{2}$ and $\Omega = \dot{\psi}$. The angular velocity vector of the B-system is thus

$$\boldsymbol{\Omega} = \Omega_1\,\mathbf{e}_1^{\mathrm{B}} + \Omega_2\,\mathbf{e}_2^{\mathrm{B}} + \Omega_3\,\mathbf{e}_3^{\mathrm{B}} = \Omega\,\mathbf{e}_2^{\mathrm{B}} \tag{5.78}$$

[1]The name refers to the french nineteenth century mathematician H. Resal.

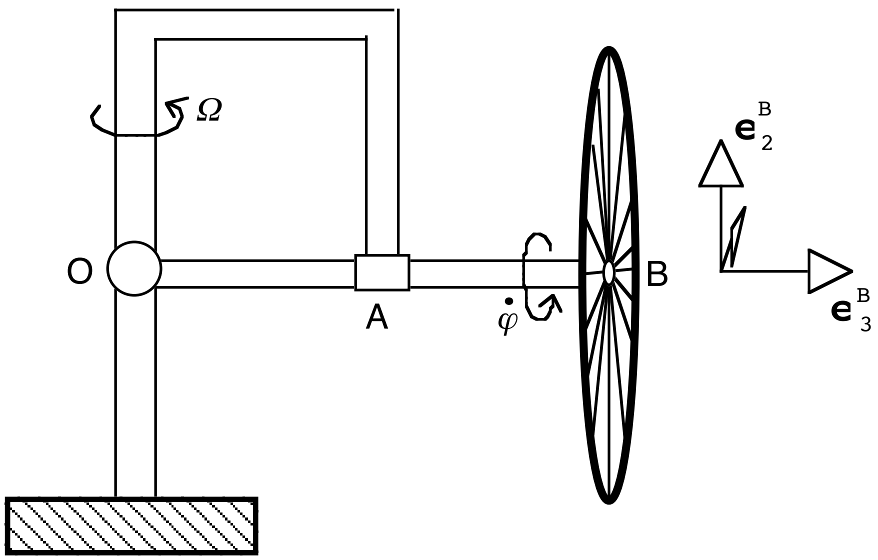


Figure 5.4: This figure shows the rotating bicycle wheel discussed in example 5.4. The vertical axis through $\mathcal{O}$ together with its arm to $\mathcal{A}$ rotates with angular velocity Ω around the vertical direction. The bicycle wheel rotates with angular velocity $\dot{\varphi}$ on the light horizontal axis $\mathcal{OB}$.

so that $\Omega_1 = \Omega_3 = 0$, $\Omega_2 = \Omega$ while all time derivatives are zero: $\dot{\Omega}_1 = \dot{\Omega}_2 = \dot{\Omega}_3 = 0$. Only the third term in the first two of equations 5.77 will thus survive. Since the wheel is thin one must have $J_1 = J_2 = J/2$ for axes through $\mathcal{B}$, in the plane of the wheel. The fixed base point is at $\mathcal{O}$, however, so we really need $J_1' = J_1 + m\ell^2$. In this case therefore the quantity $(J_3 - J_1)$ of equations 5.77 is given by $J - (J/2 + m\ell^2) = J/2 - m\ell^2$. However, we found above that we don't need this quantitiy, only $J_3 = J$ is needed. The system of equations 5.77 gives

$$\begin{aligned} J\Omega\dot{\varphi} &= M_1, \\ 0 &= M_2, \\ J\ddot{\varphi} &= M_3. \end{aligned} \tag{5.79}$$

The moment with respect to $\mathcal{O}$ is given by

$$\mathbf{M} = M_1\,\mathbf{e}_1^{\mathrm{B}} + M_2\,\mathbf{e}_2^{\mathrm{B}} + M_3\,\mathbf{e}_3^{\mathrm{B}} = \overline{\mathcal{OB}} \times (-mg\,\mathbf{e}_2^{\mathrm{B}}) + \overline{\mathcal{OA}} \times \mathbf{F}, \tag{5.80}$$

where $\mathbf{F}$ is the unknown force acting at $\mathcal{A}$. Since the ring is smooth this force cannot have any component along $\mathcal{OB}$ and can thus be expressed in the form

$$\mathbf{F} = F_1\,\mathbf{e}_1^{\mathrm{B}} + F_2\,\mathbf{e}_2^{\mathrm{B}}. \tag{5.81}$$

Since $\overline{\mathcal{OB}} = \ell\,\mathbf{e}_3^{\mathrm{B}}$ and $\overline{\mathcal{OA}} = d\,\mathbf{e}_3^{\mathrm{B}}$ we now find that

$$\mathbf{M} = (mg\ell - F_2 d)\,\mathbf{e}_1^{\mathrm{B}} + F_1 d\,\mathbf{e}_2^{\mathrm{B}}. \tag{5.82}$$

The three equations of motion thus become

$$\begin{aligned} J\Omega\dot{\varphi} &= mg\ell - F_2 d, \\ 0 &= F_1 d, \\ J\ddot{\varphi} &= 0, \end{aligned} \tag{5.83}$$

The force is thus

$$\mathbf{F} = \frac{(mg\ell - J\Omega\dot{\varphi})}{d}\,\mathbf{e}_2^{\mathrm{B}} \tag{5.84}$$

Do this experiment yourself with a bicycle wheel on an axis, by putting one hand at $\mathcal{O}$ and the other at $\mathcal{A}$ while you rotate around around a vertical axis. Note how the rotation of the wheel reduces or increases the necessary force at $\mathcal{A}$, depending on relative sign of Ω and $\dot{\varphi}$. $\square$

5.3 The Free Asymmetric Top

5.3.1 The Inertia Ellipsoid

The energy, E, as well as the angular momentum, $\mathbf{L}$, of the free asymmetric top are conserved. Using components of $\boldsymbol{\omega}$ along the body fixed principal axes system the energy, E, is given by (see equation 4.32)

$$2E = J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2. \tag{5.85}$$

Note that this equation defines a quadratic surface in 'angular velocity space'. Since the inertia tensor is positive definite this surface is an ellipsoid which is called the *inertia ellipsoid*. Let us denote the components of the angular velocity, and the inertia tensor with respect to a space fixed basis by ω_α and $J_{\alpha\beta}$ respectively, where $\alpha, \beta = x, y, z$. If we now put

$$f(\boldsymbol{\omega}, t) \equiv \sum_{\alpha\beta} J_{\alpha\beta}(t)\,\omega_\alpha\omega_\beta, \tag{5.86}$$

where we we have indicated explicitly the time dependence of the elements of the inertia tensor (due to the rotation of the body), we can write the conservation of energy

$$f(\boldsymbol{\omega}, t) = 2E = \text{const.} \tag{5.87}$$

For each value of the time this expression represents a quadratic surface in angular velocity space, namely the inertia ellipsoid rotating with the body.

5.3.2 The Poinsot Construction

The gradient, in angular velocity space, of the function $f(\boldsymbol{\omega}, t)$, will at each point of this space be perpendicular to the ellipsoidal level surfaces ($f(\boldsymbol{\omega}, t)$=const.). This gradient has the components

$$\frac{\partial f}{\partial \omega_\alpha} = \sum_\beta 2J_{\alpha\beta}\omega_\beta, \tag{5.88}$$

where the symmetry of the inertia tensor has been used. If we compare with equations 4.5 we find that these components are, in fact, two times the components of the angular momentum

$$\frac{\partial f}{\partial \omega_\alpha} = 2L_\alpha. \tag{5.89}$$

But the space fixed components of the (conserved) angular momentum $\mathbf{L}$ must be constants, so this shows that the time dependence of the inertia tensor components are such that they cancel the time dependence of the angular velocity components in the expression $L_\alpha = \sum_\beta J_{\alpha\beta}(t)\omega_\beta(t) = \text{const.}$

The energy equation can also be written

$$2E = \mathbf{L} \cdot \boldsymbol{\omega} = \sum_\alpha L_\alpha\,\omega_\alpha(t). \tag{5.90}$$

This is the equation for a (fixed) plane in angular velocity space, with $\mathbf{L}$ as normal vector. This plane is called the *invariable plane*. Since $\mathbf{L}$ is perpendicular to the inertia ellipsoid, at the point $\boldsymbol{\omega}(t)$, the invariable plane must be tangent to the ellipsoid at this point, see figure 5.5. The angular velocity vector, which can be regarded as defining the instantaneous axis of rotation of the body (passing through its center of mass), is thus seen to move in such a way that its tip always is both on the inertia ellipsoid and on the invariable plane. These surfaces are thus always in contact and the tip of the angular velocity vector traces out a curve on each surface, the two curves touching at

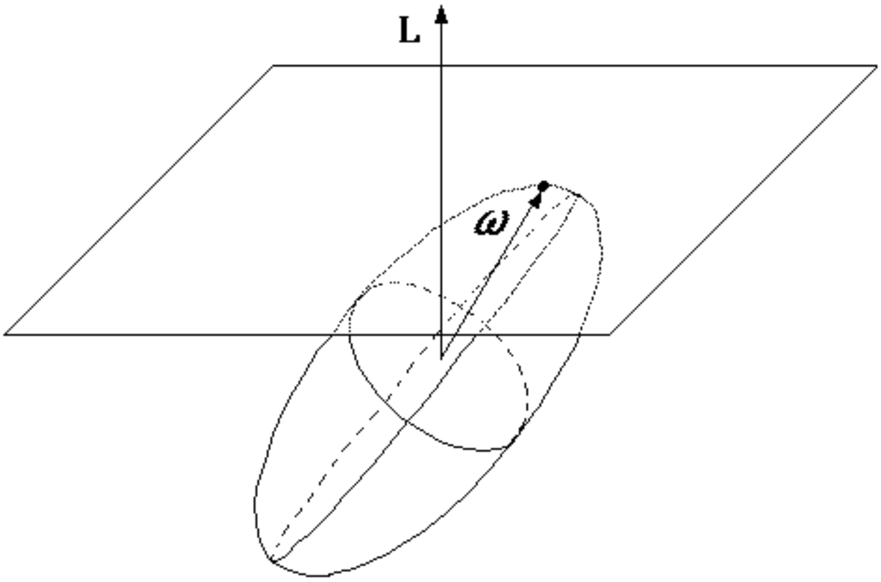


Figure 5.5: This figure illustrates the Poinsot construction. The inertia ellipsoid is in contact with the invariable plane which has **L** as fixed normal vector.

the point of contact. The curve on the inertia ellipsoid is called the *polehode* and the curve on the invariable plane is called the *herpolehode*.

In summary we have thus found that the motion of the free asymmetric top can be described as the rolling (without slipping) of the inertia ellipsoid on the invariable plane. The angular velocity vector goes from the center of the inertia ellipsoid (the origin and the center of the body) to the instantaneous point of contact. This geometric view of the dynamics was first given by Poinsot in 1834.

5.3.3 Stability of Rotation around the Principal Axes

Consider the free rotation of an asymmetric top. As we have just seen this motion can be fairly complex but with suitable initial conditions the rotation can also be quite simple as we'll now show. For the free top ($\mathbf{M} = \mathbf{0}$) Euler's dynamic equations (4.47) are

$$\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 &= 0, \\ J_2\dot{\omega}_2 + (J_1 - J_3)\omega_3\omega_1 &= 0, \\ J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 &= 0. \end{aligned} \tag{5.91}$$

Using these we can check whether there are solutions of the form

$$\boldsymbol{\omega} = \omega_i(t)\,\mathbf{e}_i^{\mathrm{A}} \tag{5.92}$$

where $i = 1, 2,$ or 3. When this 'ansatz' is inserted into the equations above two of them give $0 = 0$ and the third, the i-equation, gives

$$J_i\dot{\omega}_i = 0. \tag{5.93}$$

This equation is satisfied if ω_i =constant and we have thus found that rotation around one of the principal axes, with constant angular velocity, is one possible motion of the free top. For this simple rotation the angular velocity vector is parallel to the angular momentum, $\mathbf{L} = J_i\omega_i\,\mathbf{e}_i^{\mathrm{A}}$. Consequently this principal axis then has a fixed direction in space.

In practice there will always be small perturbations on a body and one might ask whether the principal axis parallel rotations, found above, are stable. For the motion of satellites in space, which may be required to point in some fixed direction, this is a question of practical interest. To investigate this stability we rewrite the system of equations 5.91 in the form

$$\dot{\omega}_i = C_i \omega_j \omega_k, \qquad i, j, k = 1 \to 2 \to 3 \to 1. \tag{5.94}$$

Here we assume that the axes have been labeled so that

$$J_1 < J_2 < J_3 \tag{5.95}$$

so the constants C_i are given by

$$C_1 \equiv \frac{J_3 - J_2}{J_1} < 0, \quad C_2 \equiv \frac{J_3 - J_1}{J_2} > 0, \quad C_3 \equiv \frac{J_2 - J_1}{J_3} < 0. \tag{5.96}$$

We now assume a perturbed rotation around the j-axis: $\boldsymbol{\omega}(t) = \omega_j^0\, \mathbf{e}_j^{\mathrm{A}} + \delta\boldsymbol{\omega}(t)$. Here $\delta\boldsymbol{\omega}$ is a small perturbation to the exact unperturbed solution. The components of this $\boldsymbol{\omega}(t)$ are

$$\omega_i(t) = \omega_j^0 \delta_{ji} + \delta\omega_i(t) \tag{5.97}$$

where δ_{ij} is the Kronecker delta (see equation 2.12) and summation over j from 1 to 3 is implied. When this is inserted into the equations 5.94 one gets

$$\begin{aligned} \delta\dot{\omega}_j &= C_j \delta\omega_i \delta\omega_k, & (5.98)\\ \delta\dot{\omega}_i &= C_i(\omega_j^0 + \delta\omega_j)\delta\omega_k, & (5.99)\\ \delta\dot{\omega}_k &= C_k(\omega_j^0 + \delta\omega_j)\delta\omega_i. & (5.100) \end{aligned}$$

If we differentiate the second of these equations with respect to time and insert the third equation in the resulting expression we get

$$\delta\ddot{\omega}_i = C_i \delta\dot{\omega}_j \delta\omega_k + C_i(\omega_j^0 + \delta\omega_j) C_k(\omega_j^0 + \delta\omega_j)\delta\omega_i. \tag{5.101}$$

If we neglect higher powers of the perturbations and their time derivatives this gives

$$\delta\ddot{\omega}_i \approx C_i C_k (\omega_j^0)^2 \delta\omega_i. \tag{5.102}$$

This equation shows that the perturbation will have an oscillatory behavior provided $C_i C_k < 0$, otherwise it will be exponential. For $j = 1$ this constant is $C_2 C_3$ and negative, for $j = 3$ it is $C_1 C_2$ and also negative, but for $j = 2$ the constant is $C_1 C_3$ and consequently positive. This shows that rotation around a principal axis with constant angular velocity is *stable* for the axes corresponding to the maximum and minimum moments of inertia but *unstable* when the axis corresponds to the middle moment of inertia.

5.4 Problems

Problem 5.1 The following problem illustrates the behavior of a bowling ball. A homogeneous solid sphere of radius R is, at time $t = 0$, given a center of mass velocity $\mathbf{v}(0) = v_0\, \mathbf{e}_x$ and an angular velocity $\boldsymbol{\omega}(0) = \omega_0\, \mathbf{e}_x$ and is placed on a rough horizontal floor that coincides with the xy-plane of the coordinate system. Calculate the center of mass velocity $\mathbf{v}$ of the sphere when is has started to roll without sliding.

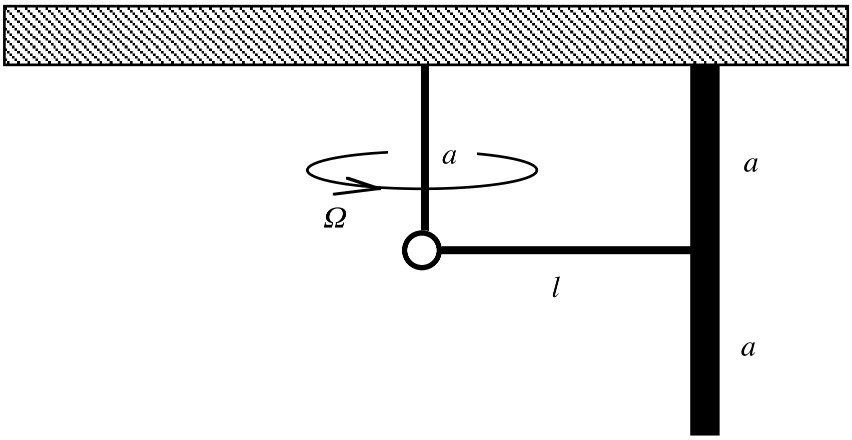


Figure 5.6: This figure refers to problem 5.3 and shows the wheel, of radius a, from the edge.

Problem 5.2 An astronaut has managed to set a thin flat homogeneous metal square spinning freely in space with no translational velocity. The component of the angular velocity around the axis through the center of the square and perpendicular to it, is $\dot{\varphi}$. This axis precesses (rotates) around a fixed direction in space with which it makes an angle $\theta = \pi/4$. What is the angular velocity of this precession?

Problem 5.3 A wheel in the shape of a homogeneous circular disc of mass m and radius a can rotate around a light axis of length l. The other end of this axis is attached to a vertical axis by a hinge, so that it can rotate freely about a horizontal axis, see figure 5.6, but when the vertical axis rotates the axis of the wheel must follow and have the same vertical component of angular velocity. The hinge on the vertical axis is at a distance a below a rough horizontal plane.

Assume that the vertical axis rotates with angular velocity Ω and that the wheel rolls without slipping on the rough plane above it. Clearly, if Ω is too small this cannot happen, and if Ω is zero the wheel and its axis will hang straight down. What is the minimum value of Ω needed for this rolling to take place?

Problem 5.4 A (millstone) wheel of radius r and mass m is mounted on a light axis $\mathcal{OG}$ as shown in figure 5.7. The axis is connected to a fixed ball and socket joint at $\mathcal{O}$ and rotates around a vertical axis with constant angular velocity Ω. The angle between the axis $\mathcal{OG}$ and the vertical has the constant value $\beta = \pi/3$. The wheel, which has moment of inertia J_3 with respect to the axis $\mathcal{OG}$ and $J = J_1 = J_2$ for all perpendicular axes through $\mathcal{O}$, rolls without slipping on the horizontal ground so that the geometric contact point traces out a circle of radius r.
a) Find the magnitude, ω, of the angular velocity vector $\boldsymbol{\omega}$ of the wheel.
b) Find the moment $\mathbf{M} = \mathbf{M}_{\mathcal{O}}$ acting on the wheel.
c) Calculate the force N acting on the wheel from the ground at the point $\mathcal{A}$ in terms of Ω, β, and mg.

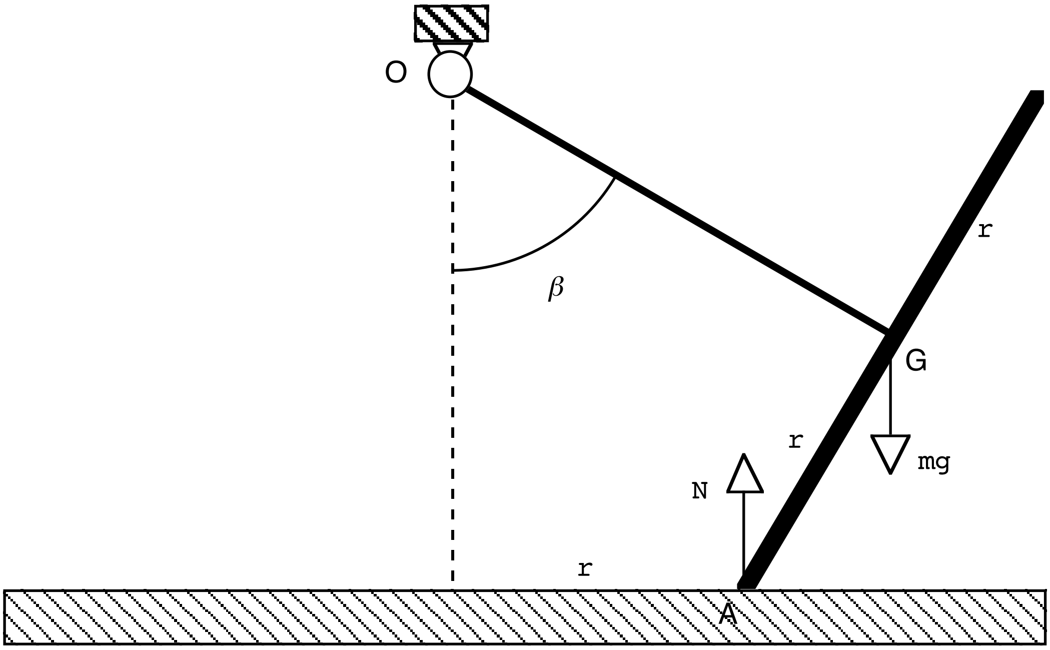


Figure 5.7: The figure refers to problem 5.4 and shows the wheel, of radius r, head on. It also indicates the geometry of the problem. Note that the point of contact of the wheel with the ground is at a distance r from the point on the ground below the ball and socket joint at $\mathcal{O}$.

5.5 Hints and Answers

Answer 5.1 If we denote by $\mathcal{C}$ the point of the sphere in contact with the floor we have the following basic equations for this problem

$$\begin{aligned} m\dot{\mathbf{v}} &= \mathbf{F}, \\ J\dot{\boldsymbol{\omega}} &= -R\,\mathbf{e}_z \times \mathbf{F}, \\ \mathbf{v}_{\mathcal{C}} &= \mathbf{v} + R\,\mathbf{e}_z \times \boldsymbol{\omega}, \end{aligned}$$

where $J = 2mR^2/5$ is the moment of inertia of the solid sphere, and $\mathbf{F}$ is the friction force from the floor. Eliminating $\mathbf{F}$ by putting the first equation into the second we get

$$J\dot{\boldsymbol{\omega}} = -R\,\mathbf{e}_z \times m\dot{\mathbf{v}}$$

This vector equation has the components

$$\begin{aligned} J\dot{\omega}_x &= Rm\ddot{y}, \\ J\dot{\omega}_y &= -Rm\ddot{x}, \\ J\dot{\omega}_z &= 0. \end{aligned}$$

Integration of the x- and y-components gives

$$\begin{aligned} \frac{2R}{5}[\omega_x(t) - \omega_x(0)] &= [\dot{y}(t) - \dot{y}(0)], \\ \frac{2R}{5}[\omega_y(t) - \omega_y(0)] &= -[\dot{x}(t) - \dot{x}(0)]. \end{aligned}$$

The connection formula for the velocities has the components

$$\begin{aligned} \dot{x} &= \dot{x}_{\mathcal{C}} + R\omega_y, \\ \dot{y} &= \dot{y}_{\mathcal{C}} - R\omega_x, \\ \dot{z} &= \dot{z}_{\mathcal{C}}. \end{aligned}$$

Using the initial conditions at $t = 0$: $\dot{x}(0) = v_0$, $\dot{y}(0) = 0$, $\omega_x(0) = \omega_0$, and, $\omega_y(0) = 0$, we get form this that

$$\dot{x}_{\mathcal{C}}(0) = v_0, \text{ and } \dot{y}_{\mathcal{C}}(0) = R\omega_0.$$

The components of the connection formula for the velocities also give

$$\omega_y(t) = \frac{1}{R}[\dot{x}(t) - \dot{x}_{\mathcal{C}}(t)], \text{ and } \omega_x(t) = -\frac{1}{R}[\dot{y}(t) - \dot{y}_{\mathcal{C}}(t)].$$

When these expressions for the components of $\boldsymbol{\omega}$ are inserted into the two integrated equations above one finds (again using the initial conditions)

$$\begin{aligned} -\frac{2}{5}[\dot{y}(t) - \dot{y}_{\mathcal{C}}(t)] - \frac{2R}{5}\omega_0 &= [\dot{y}(t) - 0], \\ \frac{2}{5}[\dot{x}(t) - \dot{x}_{\mathcal{C}}(t)] &= -[\dot{x}(t) - v_0]. \end{aligned}$$

and these can be simplified to

$$\begin{aligned} \frac{7}{5}\dot{y}(t) &= \frac{2}{5}\dot{y}_{\mathcal{C}}(t) - \frac{2}{5}R\omega_0, \\ \frac{7}{5}\dot{x}(t) &= \frac{2}{5}\dot{x}_{\mathcal{C}}(t) + v_0. \end{aligned}$$

As time increases the velocity of point $\mathcal{C}$ of the sphere in contact with the ground slows down and when the sphere rolls without slipping, at $t > T$, this velocity is zero so $\dot{x}_{\mathcal{C}}(t) = \dot{y}_{\mathcal{C}}(t) = 0$ for $t > T$. These last two equations then give the final result

$$\dot{x}(t) = \frac{5}{7}v_0, \text{ and } \dot{y}(t) = -\frac{2}{7}R\omega_0$$

for $t > T$. Thus

$$\mathbf{v} = \frac{1}{7}(5v_0\,\mathbf{e}_x - 2R\omega_0\,\mathbf{e}_y)$$

is the final center of mass velocity of the sphere.

Answer 5.2 Here we have $J_1 = J_2 = \frac{1}{12}ma^2$ and $J_3 = \frac{1}{6}ma^2$ if m is the mass of the square and a its side length. Use of equation 5.35 now gives

$$\dot{\varphi} = \frac{L}{J_1}\cos(\pi/4)\frac{J_1 - J_3}{J_3} = \frac{L}{J_1}\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)$$

so that $L/J_1 = -\dot{\varphi}2\sqrt{2}$. When this is inserted into equation 5.34 one finds that

$$\dot{\psi} = \frac{L}{J_1} = -\dot{\varphi}2\sqrt{2}$$

is the angular velocity of the precession.

Answer 5.3 Use the methods of example 5.4. The rolling condition is

$$l\Omega = a\omega$$

if $\dot{\varphi} = \omega$ is the angular velocity of the wheel about its horizontal axis.

The moment (with respect to the hinge) is

$$\mathbf{M} = l\,\mathbf{e}_3^{\mathrm{B}} \times (N + mg)(-\mathbf{e}_2^{\mathrm{B}}) = l(N + mg)\,\mathbf{e}_1^{\mathrm{B}},$$

where N is the normal force from the plane on the wheel. The angular velocity of the B-system is, just as in example 5.4, given by

$$\boldsymbol{\Omega} = \Omega\,\mathbf{e}_2^{\mathrm{B}}.$$

The first of equations 5.77 now gives us

$$\Omega\omega J = l(N + mg).$$

Here $J = \frac{1}{2}ma^2$ is the moment of inertia of the wheel with respect to its axis. The condition that the normal force is positive ($N > 0$), and the rolling condition ($l\Omega = a\omega$) now give

$$0 < lN = \Omega\left(\frac{l}{a}\Omega\right)\left(\frac{1}{2}ma^2\right) - lmg.$$

Thus one finds that

$$\Omega > \sqrt{\frac{2g}{a}}$$

is the answer.

Answer 5.4
a) $\omega = \sqrt{2(1 + \cos\beta)}\Omega = \sqrt{3}\Omega$,
b) Using the standard basis of the (B) Resal system (see figure 5.3) we find that $\mathbf{M} = [J_3(1 + \cos\beta) - J\cos\beta]\Omega^2 \sin\beta\, \mathbf{e}_1^{\mathrm{B}}$.
c) $N = mg(1 + \cos\beta) + \frac{1}{r}[J_3(1 + \cos\beta) - J\cos\beta]\Omega^2 \sin\beta$.

Chapter 6

Impact

Impact is characterized by the occurrence of large forces acting for short times and is the technical term used to describe things like collisions, bounces and similar phenomena where there are large, rapid velocity changes. As an idealized limit one can then use the approximation that the time is zero and the force infinite so that a finite impulse results. The velocities of particles and rigid bodies then change instantaneously with no change in position. Momentum and angular momentum are useful concepts in dealing with impact phenomena. Energy, on the other hand, is normally *not* conserved since the large forces involved usually produce sound, heat and irreversible deformation.

6.1 The Impact Phenomenon

We define *impact* as a mechanical process in which the velocity state of the system changes in such a short time that the corresponding change in position can be neglected. We will mainly consider impacts involving rigid bodies and we will normally assume that the bodies are in contact at a point during the impact. We will call this point the *point of impact* and denote it by $\mathcal{S}$. At this point then the bodies act on each other with very large contact forces, $\mathbf{K}(t)$, for the short duration, τ, of the impact. The force $\mathbf{K}$ thus obeys

$$\mathbf{K}(t) = \begin{cases} \mathbf{0} & \text{for } \; t < t_i \\ \text{large} & \text{for } \; t_i < t < t_f = t_i + \tau \\ \mathbf{0} & \text{for } \; t_f < t \end{cases} \tag{6.1}$$

The meaning of the words 'large' for $\mathbf{K}$ and 'short' for τ can be made more precise by demanding that the contribution of ordinary forces to the impulse delivered during impact should be negligible. The impulse of the force $\mathbf{K}$ is given by (see equation 1.55)

$$\mathbf{S} = \int_{t_i}^{t_i+\tau} \mathbf{K}(t')\, dt'. \tag{6.2}$$

If there is also an ordinary force, $\mathbf{F}^{\mathrm{a}}$, acting, the total impulse, or change of momentum, is

$$\Delta\mathbf{p} = \mathbf{p}(t_f) - \mathbf{p}(t_i) = \int_{t_i}^{t_i+\tau} \mathbf{F}(t')\, dt' = \tag{6.3}$$

$$\int_{t_i}^{t_i+\tau} [\mathbf{F}^{\mathrm{a}}(t') + \mathbf{K}(t')]\, dt' \approx \mathbf{F}^{\mathrm{a}}(t_i)\, \tau + \mathbf{S}. \tag{6.4}$$

Since we assume that the force $\mathbf{F}^{\mathrm{a}}$ remains of normal magnitude during the impact the limit $\tau \to 0$ gives

$$\mathbf{p}(t_f) - \mathbf{p}(t_i) = \mathbf{S} \tag{6.5}$$

We can thus say that the the words 'large' for $\mathbf{K}$ and 'short' for τ apply when this approximation, the 'impulse approximation', is valid.

The *mean force* during impact is defined by

$$\mathbf{K}_{\mathrm{av}} = \mathbf{S}/\tau. \tag{6.6}$$

If the force $\mathbf{K}$ is constant during the impact it will have the value $\mathbf{K}_{\mathrm{av}}$ but normally the impact force rises to a maximum and the goes to zero again.

Example 6.1 A ball of mass m falls vertically. It bounces against the floor and rises vertically again. Assume that the speed just before the bounce is 12 m/s and that the speed just after the bounce is 10 m/s. The time during which the ball is in contact with the floor is $\tau = 10^{-3}$s. Calculate the relative size of weight and mean force, i.e. the ratio mg/K_{av}.
Solution: If we choose a vertical X-axis we have the equation of motion

$$m\ddot{x} = -mg + K. \tag{6.7}$$

Time integration gives

$$m[\dot{x}(t_f) - \dot{x}(t_i)] = -mg\tau + K_{\mathrm{av}}\tau. \tag{6.8}$$

If we insert numbers we get

$$-g + K_{\mathrm{av}}/m = \{[10 - (-12)]/10^{-3}\}\mathrm{m/s}^2 = 22 \cdot 10^3\mathrm{m/s}^2 \tag{6.9}$$

Since $g = 9.81\mathrm{m/s}^2 \ll 22 \cdot 10^3\mathrm{m/s}^2$ we get that $K_{\mathrm{av}}/m \approx 22 \cdot 10^3\mathrm{m/s}^2$ and thus that the ratio is

$$mg/K_{\mathrm{av}} \approx \frac{9.81}{22 \cdot 10^3} \approx 0.4 \cdot 10^{-3} \tag{6.10}$$

The weight can thus be neglected during the bounce and we have a case where the impact approximation is excellent. □

To summarize, an impact is characterized by the following idealizations

$$\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i = \mathbf{0}, \tag{6.11}$$

$$|\Delta\mathbf{v}| = |\mathbf{v}_f - \mathbf{v}_i| \neq 0, \tag{6.12}$$

$$|\mathbf{a}| \to \infty, \text{ and } \tau \to 0. \tag{6.13}$$

I.e. no change in position, finite change in velocity, and infinite acceleration for zero time.

We now classify the impact, or collision, of two bodies. When two bodies, 1 and 2, collide they are in contact at the point of impact $\mathcal{S}$. If at least one of the bodies is smooth one can define a tangent plane which contains the point $\mathcal{S}$ and which is tangent to the surface of the body. The normal (perpendicular line) to this plane is the *normal of the impact*. The unit vector parallel to this normal and pointing into body 1 we denote $\mathbf{e}_n$. The impact between the two bodies is said to be *central* if both centres of mass, $\mathcal{G}_1$ and $\mathcal{G}_2$, lie on the normal of the impact through the point $\mathcal{S}$. (see figure 6.1). Otherwise the impact is said to be *eccentric*. When the velocities of the bodies at $\mathcal{S}$, both before and after impact, are parallel to $\mathbf{e}_n$ the impact is said to be *direct*, otherwise it is *oblique*.

6.2 Impact and the Momentum Principles

Consider two colliding bodies, 1 and 2, moving under the influence of the total external force $\mathbf{F}^{\mathrm{e}}$. The momentum principle then gives

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{F}^{\mathrm{e}}. \tag{6.14}$$

The impact forces with which the two bodies act on each other are internal forces of the system and thus do not contribute on the right hand side. If this equation is integrated from the initial time t_i to the final time $t_f = t_i + \tau$ of the impact, we obtain

$$[\mathbf{p}_1(t_f) + \mathbf{p}_2(t_f)] - [\mathbf{p}_1(t_i) + \mathbf{p}_2(t_i)] = \mathbf{0}, \tag{6.15}$$

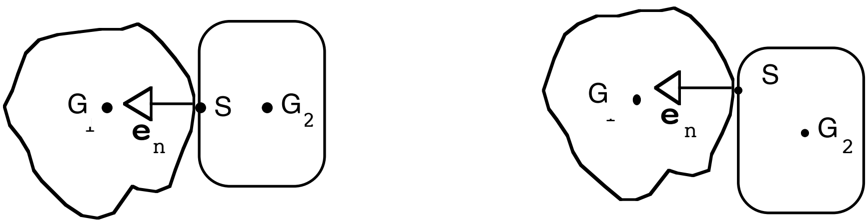


Figure 6.1: This figure illustrates some concepts presented in the text. It shows the point of impact $\mathcal{S}$ and the normal of impact $\mathbf{e}_n$. The impact to the left is central and that to the right is eccentric.

in the limit when $\tau \to 0$. With suitable notation we can rewrite this result as follows

$$\mathbf{p}_{1f} + \mathbf{p}_{2f} = \mathbf{p}_{1i} + \mathbf{p}_{2i}, \tag{6.16}$$

i.e. the total linear momentum just before the impact is the same just after the impact.

The corresponding calculation based on the angular momentum principle,

$$\frac{d}{dt}(\mathbf{L}_1 + \mathbf{L}_2) = \mathbf{M}^{\mathrm{e}}, \tag{6.17}$$

gives in the same way

$$\mathbf{L}_{1f} + \mathbf{L}_{2f} = \mathbf{L}_{1i} + \mathbf{L}_{2i}. \tag{6.18}$$

Here any continuously moving base point may be used since the extra term arising from the velocity of the base point, see equation 1.50, gives zero contribution when $\tau \to 0$.

Equation 6.5 can be written for body 1, on the form

$$\mathbf{S}_{12} = \mathbf{p}_{1f} - \mathbf{p}_{1i} \tag{6.19}$$

where $\mathbf{S}_{12}$ is the impulse of the impact force from body 2 on body 1. According to the law of action and reaction (or Newton's third) one has that $\mathbf{S}_{12} = -\mathbf{S}_{21}$. For body 1 we also have

$$\dot{\mathbf{L}}_1 = \mathbf{M}_1^{\mathrm{a}} + \mathbf{r}_{\mathcal{S}} \times \mathbf{K}_{12} \tag{6.20}$$

where the origin is base point. Time integration of this from t_i to $t_i + \tau$ gives

$$\mathbf{r}_{\mathcal{S}} \times \mathbf{S}_{12} = \mathbf{L}_{1f} - \mathbf{L}_{1i} \tag{6.21}$$

when $\tau \to 0$ if we take into consideration the fact that $\mathbf{r}_{\mathcal{S}}$, the position vector of the point of impact, does not move during the impact. There is, of course, a corresponding equation for body 2. If one chooses the point of impact $\mathcal{S}$ as origin (base point) this equation gives the simple result

$$\mathbf{L}_{\mathcal{S}1f} = \mathbf{L}_{\mathcal{S}1i}. \tag{6.22}$$

Provided that one knows the impulse of impact $\mathbf{S}_{12}$ one can use the three formulae above to get the velocity state of the body just after impact in terms of the state just before impact.

Example 6.2 A bat in the form of a straight narrow homogeneous rod of mass m and length ℓ is hit by a ball when at rest. The ball imparts an impulse S to the bat perpendicular to it at the distance x from one of its ends, see figure 6.2.
a) Calculate the kinetic energy $T(x)$ of the bat after the hit.
b) Is there a point on the bat which is at rest immediately after the impact? If so, where is it?
c) Which values of x correspond to maximal speed of the point of impact after the hit?
Solution:

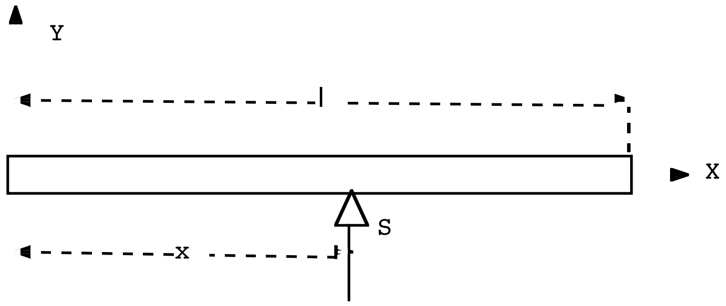


Figure 6.2: A bat in the form of a narrow homogeneous rod receives an impulse S at the distance x from one end. The properties of the subsequent motion are discussed in example 6.2.

a) To find the kinetic energy we need to know the translational and angular velocities after the impact. Use of equations 6.19 and 6.21 give us directly

$$\begin{aligned} S\,\mathbf{e}_y &= m\mathbf{v}_{\mathcal{G}}, && (6.23)\\ (x-\frac{\ell}{2})\,\mathbf{e}_x \times S\,\mathbf{e}_y &= J_{\mathcal{G}}\dot{\varphi}\,\mathbf{e}_z, && (6.24) \end{aligned}$$

where $J_{\mathcal{G}} = \frac{1}{12}m\ell^2$. Thus $v_{\mathcal{G}} = S/m$ and $\dot{\varphi} = 12S(x-\ell/2)/(m\ell^2)$. The kinetic energy can now be calculated from

$$T = \frac{1}{2}mv_{\mathcal{G}}^2 + \frac{1}{2}J_{\mathcal{G}}\dot{\varphi}^2 \qquad (6.25)$$

and we get

$$T(x) = \frac{1}{2}m\left(\frac{S}{m}\right)^2 + \frac{1}{2}\left(\frac{1}{12}m\ell^2\right)\left(\frac{12S(x-\ell/2)}{m\ell^2}\right)^2 = \frac{S^2}{2m}\left[1+12\left(\frac{x}{\ell}-\frac{1}{2}\right)^2\right]. \qquad (6.26)$$

This shows directly that the kinetic energy has a minimum for $x = \ell/2$. This happens when the ball hits in the middle of the rod so that only translational and no rotational motion results.
b) Equation 2.95 gives the x-coordinate (and the y-coordinate) of a point rigidly connected to the body which has zero velocity. If we take the point $\mathcal{A}$ to be the centre of mass of the rod we get

$$x_{\mathcal{C}} = x_{\mathcal{G}} - \frac{\dot{y}_{\mathcal{G}}}{\omega} = \frac{\ell}{2} - \frac{S/m}{\dot{\varphi}}. \qquad (6.27)$$

For this case then some algebra gives the result

$$x_{\mathcal{C}}(x) = \frac{\ell}{3}\frac{(3x-2\ell)}{(2x-\ell)} \qquad (6.28)$$

A small table of this function looks as follows

$$\begin{aligned} x_{\mathcal{C}}(0) &= 2\ell/3, && (6.29)\\ x_{\mathcal{C}}(\ell/3) &= \ell, && (6.30)\\ x_{\mathcal{C}}(\ell/2) &= \pm\infty, && (6.31) \end{aligned}$$

so one sees that for $0 \le x \le \ell/3$ there is a point on the bat that is at rest immediately after the hit. This is, of course, a point at which it is good to hold one's hand since then no part of the impulse from the ball is imparted to the hand. For $\ell/3 < x < \ell/2$ the instantaneous centre of zero velocity will lie to the right of the rod ($\ell < x_{\mathcal{C}}$).
c) The velocities of the points on the bat after the hit are given by the connection formula for velocities of a rigid body. For the point $\mathcal{P}$ on the bar with x-coordinate $x_{\mathcal{P}}$ we find the velocity

$$\mathbf{v}_{\mathcal{P}} = \mathbf{v}_{\mathcal{G}} + \overline{\mathcal{PG}} \times \boldsymbol{\omega} = \frac{S}{m}\,\mathbf{e}_y + (x_{\mathcal{G}} - x_{\mathcal{P}})\,\mathbf{e}_x \times \dot{\varphi}\,\mathbf{e}_z = \left[\frac{S}{m} - \left(\frac{\ell}{2} - x_{\mathcal{P}}\right)\dot{\varphi}\right]\mathbf{e}_y \qquad (6.32)$$

in terms of the centre of mass velocity. Some algebra and use of results above gives

$$\dot{y}_{\mathcal{P}} = \frac{S}{m}\left[1+12\left(\frac{x}{\ell}-\frac{1}{2}\right)\left(\frac{x_{\mathcal{P}}}{\ell}-\frac{1}{2}\right)\right] \qquad (6.33)$$

To get the speed of the point of impact we simply put $x_{\mathcal{P}} = x$ and get

$$\dot{y} = \frac{S}{m}\left[1 + 12\left(\frac{x}{\ell} - \frac{1}{2}\right)^2\right]. \tag{6.34}$$

For $0 \le x \le \ell$ this function clearly takes maximum values at the end points so the answer is that hits at $x = 0$ or $x = \ell$ gives maximum speed of the point of impact. □

6.3 The Coefficient of Restitution

As we have mentioned mechanical energy is normally not conserved when impact occurs since the available energy is lost as energy of sound, heat, and deformation. Since the position of the system does not change during impact the potential energy, of external or internal forces, cannot change so the energy loss can be expressed as a loss as kinetic energy. We can thus write

$$Q \equiv T_i - T_f \ge 0 \tag{6.35}$$

for the loss of mechanical energy during impact.

For direct impact, i.e. impacts in which the velocities just before and just after the impact are parallel to the impact normal, one can characterize the energy loss by a single number. We take the X-axis along the normal and define the *coefficient of restitution*, e, for direct impact of bodies 1 and 2, as the number

$$e \equiv -\frac{\Delta\dot{x}_f}{\Delta\dot{x}_i} = -\frac{\dot{x}_{2f} - \dot{x}_{1f}}{\dot{x}_{2i} - \dot{x}_{1i}}. \tag{6.36}$$

The negative sign insures that the coefficient always is positive (because of the reversal of the relative velocity at impact). One also sees that it normally will be less than one ($e < 1$) since the relative velocity $\Delta\dot{x} = \dot{x}_2 - \dot{x}_1$ after impact is less that before due to the energy loss.

The coefficient of restitution has a status similar to that of the coefficients of static and kinetic friction. It may be a 'constant' that characterizes the collision of two bodies with a given pair of materials, but it is certainly not constant in any other sense of the word. It will, for example, always decrease with increasing relative speed.

Let us calculate the energy loss Q in terms of the coefficient of restitution e. We assume that there is no rotation. Equation 6.16 gives

$$m_1\dot{x}_{1f} + m_2\dot{x}_{2f} = m_1\dot{x}_{1i} + m_2\dot{x}_{2i} \tag{6.37}$$

while the definition of e can be written

$$\dot{x}_{1f} - \dot{x}_{2f} = e\dot{x}_{2i} - e\dot{x}_{1i}. \tag{6.38}$$

This set of equations can be used to express the final velocities $\dot{x}_f$ in terms of the initial ones, $\dot{x}_i$, and e. The result can then be inserted into the expression for the kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{6.39}$$

and the change $Q = T_i - T_f$ calculated. It is more instructive, however, to use the ideas introduced in connection with the two particle problem. There we showed that the kinetic energy for two particles can be expressed as follows

$$T = \frac{1}{2}m\dot{x}_{\mathcal{G}}^2 + \frac{1}{2}\mu(\Delta\dot{x})^2 \equiv \frac{1}{2}(m_1+m_2)\left(\frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1+m_2}\right)^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(\dot{x}_2-\dot{x}_1)^2. \tag{6.40}$$

That is, it can be expressed in terms of the centre of mass velocity $\dot{x}_{\mathcal{G}}$ and relative velocity $\Delta\dot{x}$ as if there were two particles with masses $m = m_1 + m_2$ and $\mu = \frac{m_1m_2}{m_1+m_2}$

respectively, having these velocities. μ is called the reduced mass. It is now trivial to calculate the energy change since the centre of mass term will be unchanged while the relative velocity change is directly given by the definition of e. One finds

$$Q = T_i - T_f \;=\; (1 - e^2)\frac{1}{2}\mu(\Delta\dot{x}_i)^2. \tag{6.41}$$

From this one draws the conclusion that when $e = 1$ the energy loss is zero and the energy is conserved. Such collisions are said to be *elastic*. Maximal energy loss occurs at the opposite extreme when $e = 0$. The relative velocity after the impact is then zero and the collision is said to be totally *inelastic* (or plastic).

Example 6.3 Consider a direct central collision between two bodies A and B in which the coefficient of restitution is e. During the time interval $[-\delta_1, 0]$ the two bodies are in contact and have negative relative velocities (i.e. they approach each other). From time $t = 0$ to $t = \delta_2$ they are in contact but recede from one another (i.e. they have positive relative velocity), see figure 6.3. The total collision time is thus $\tau = \delta_1 + \delta_2$.

The motion is assumed to take place along an X-axis. The (x-component of the) impulse on body A, S_A, can be split into two parts corresponding to the approaching stage (1) and the receding stage (2): $S_\mathrm{A} = S_\mathrm{A1} + S_\mathrm{A2}$ and similarly for body B. Show that

$$\frac{S_\mathrm{A2}}{S_\mathrm{A1}} = \frac{S_\mathrm{B2}}{S_\mathrm{B1}} = e \tag{6.42}$$

i.e. that the ratio of the receding stage impulse to the approaching stage impulse is the same as the coefficient of restitution.

Solution: According to formula 6.19 we have

$$S_\mathrm{A1} = m_\mathrm{A}[\dot{x}_\mathrm{A}(0) - \dot{x}_\mathrm{A}(-\delta_1)] \quad \text{and} \quad S_\mathrm{A2} = m_\mathrm{A}[\dot{x}_\mathrm{A}(\delta_2) - \dot{x}_\mathrm{A}(0)], \tag{6.43}$$

$$S_\mathrm{B1} = m_\mathrm{B}[\dot{x}_\mathrm{B}(0) - \dot{x}_\mathrm{A}(-\delta_1)] \quad \text{and} \quad S_\mathrm{B2} = m_\mathrm{B}[\dot{x}_\mathrm{B}(\delta_2) - \dot{x}_\mathrm{B}(0)]. \tag{6.44}$$

According to the definition of the coefficient of restitution, equation 6.36 we have

$$e = -\frac{\dot{x}_\mathrm{A}(\delta_2) - \dot{x}_\mathrm{B}(\delta_2)}{\dot{x}_\mathrm{A}(-\delta_1) - \dot{x}_\mathrm{B}(-\delta_1)} \tag{6.45}$$

Since we assume that there are no external forces acting on the two bodies the law of action and reaction (Newton's third) gives us

$$S_\mathrm{A1} + S_\mathrm{B1} = 0, \quad \text{and} \quad S_\mathrm{A2} + S_\mathrm{B2} = 0. \tag{6.46}$$

From these two there follows that $S_\mathrm{A1} = -S_\mathrm{B1}$ and $S_\mathrm{A2} = -S_\mathrm{B2}$ and thus that

$$\frac{S_\mathrm{A2}}{S_\mathrm{A1}} = \frac{-S_\mathrm{B2}}{-S_\mathrm{B1}} = \frac{S_\mathrm{B2}}{S_\mathrm{B1}}. \tag{6.47}$$

There remains to show that this ratio is equal to e.

Since there is no relative velocity at $t = 0$ both bodies must move with the same velocity. This velocity must be that of the centre of mass so that $\dot{x}_\mathrm{A}(0) = \dot{x}_\mathrm{B}(0) = v_\mathcal{G}$. When this is inserted into equations 6.43 and 6.44, they, together with equation 6.47, give

$$\left\{\frac{S_\mathrm{A2}}{S_\mathrm{A1}} = \frac{m_\mathrm{A}[\dot{x}_\mathrm{A}(\delta_2) - v_\mathcal{G}]}{m_\mathrm{A}[v_\mathcal{G} - \dot{x}_\mathrm{A}(-\delta_1)]}\right\} = \left\{\frac{S_\mathrm{B2}}{S_\mathrm{B1}} = \frac{m_\mathrm{B}[\dot{x}_\mathrm{B}(\delta_2) - v_\mathcal{G}]}{m_\mathrm{B}[v_\mathcal{G} - \dot{x}_\mathrm{B}(-\delta_1)]}\right\}. \tag{6.48}$$

From this we get

$$-\frac{S_\mathrm{A2}}{S_\mathrm{A1}} = \frac{\dot{x}_\mathrm{A}(\delta_2) - v_\mathcal{G}}{\dot{x}_\mathrm{A}(-\delta_1) - v_\mathcal{G}} = \frac{\dot{x}_\mathrm{B}(\delta_2) - v_\mathcal{G}}{\dot{x}_\mathrm{B}(-\delta_1) - v_\mathcal{G}}. \tag{6.49}$$

But if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-c}{b-d} = \frac{c}{d}$. (Proof: $\frac{a}{b} = \frac{c}{d} \Rightarrow c = ka$ and $d = kb$, but then $\frac{a-c}{b-d} = \frac{a-ka}{b-kb} = \frac{a(1-k)}{b(1-k)} = \frac{a}{b}$, Q.E.D.) Use of this identity gives us

$$-\frac{S_\mathrm{A2}}{S_\mathrm{A1}} = \frac{\dot{x}_\mathrm{A}(\delta_2) - v_\mathcal{G} - [\dot{x}_\mathrm{B}(\delta_2) - v_\mathcal{G}]}{\dot{x}_\mathrm{A}(-\delta_1) - v_\mathcal{G} - [\dot{x}_\mathrm{B}(-\delta_1) - v_\mathcal{G}]} = \frac{\dot{x}_\mathrm{A}(\delta_2) - \dot{x}_\mathrm{B}(\delta_2)}{\dot{x}_\mathrm{A}(-\delta_1) - \dot{x}_\mathrm{B}(-\delta_1)} = -e \tag{6.50}$$

according to equation 6.45. This is what we wanted to show. □

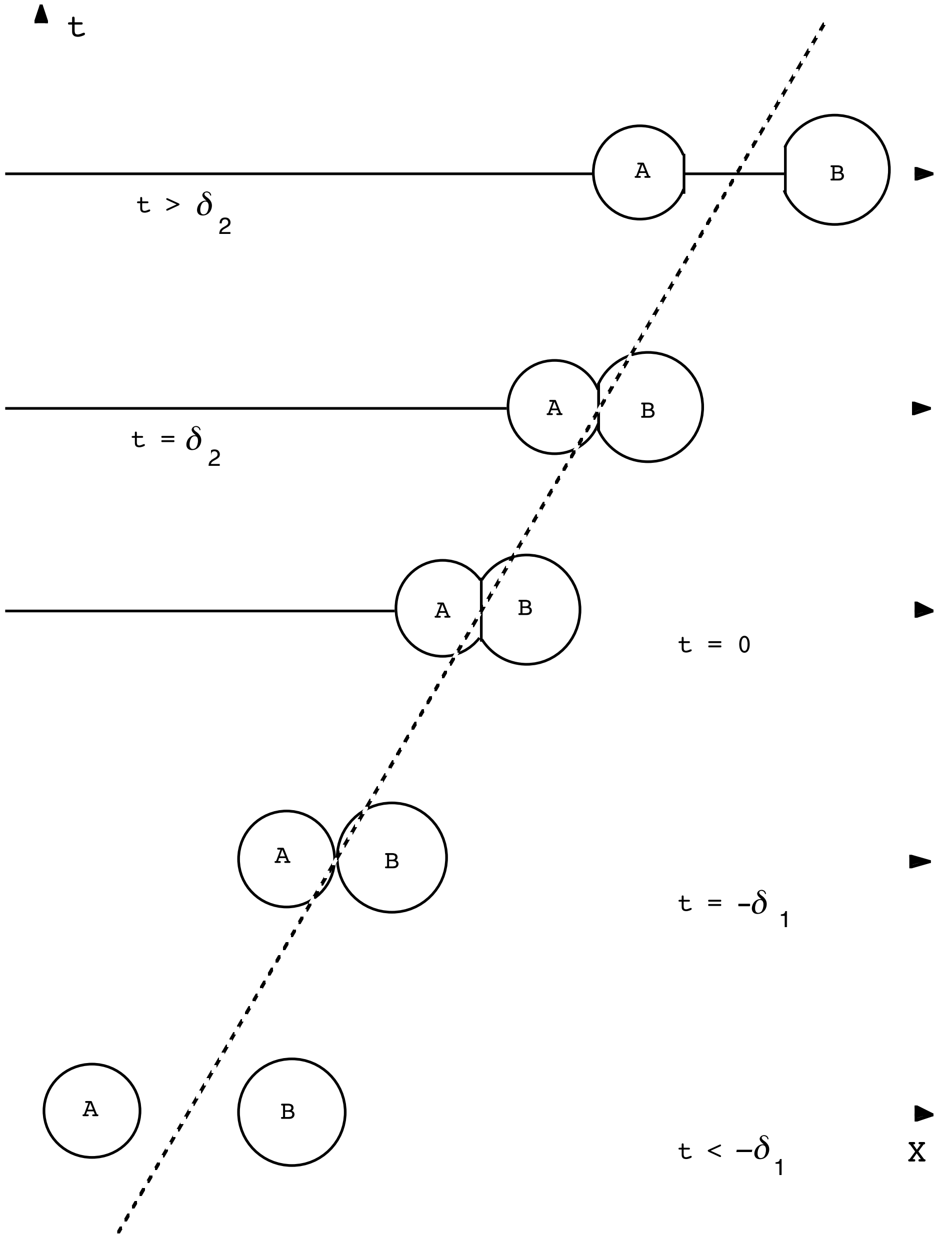
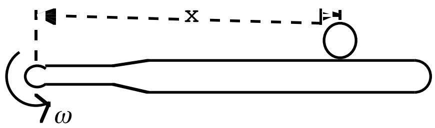


Figure 6.3: This figure shows an inelastic bounce between bodies A and B moving along the X-axis. Time increases upwards and the time $t = 0$ is taken to be the instant at which the two centres of mass of the bodies are closest together. The duration of contact between the bodies is $\tau = \delta_1 + \delta_2$ where δ_1 corresponds to a compression of the bodies while δ_2 is the time during which they decompress. The dashed line indicates the motion of the common centre of mass of the bodies. In example 6.3 it is shown that the coefficient of restitution e can be expressed as the ratio of the impulses on the bodies during these two time intervals.



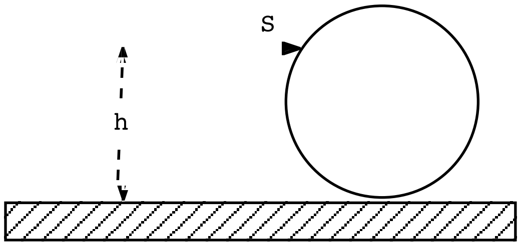


Figure 6.4: The figure on the left refers to problem 6.3. The bat rotates around the end with the handle, with angular velocity ω, when it hits the ball at a distance x from this end.

Figure 6.5: The figure on the right refers to problem 6.4. A football is given a horizontal impulse S at height h.

6.4 Problems

Problem 6.1 A straight narrow homogeneous rod of mass $m = 75\,\mathrm{kg}$ and length $\ell = 180\,\mathrm{cm}$ stands vertically on horizontal ground. It is given a horizontal impulse S at height h above the ground. For what values of h and S will the top end of the rod hit the ground first in the ensuing motion?

Problem 6.2 A tugboat of mass 800 tonnes is connected to a ship of mass 23000 tonnes (1 tonne $= 10^3$ kg) by a rope of mass 375 kg. The tugboat reaches a speed of 3 knots (i.e. 5.5 km/h) when the rope suddenly becomes taut so water spurts out of it and the ship starts to move. Since neither boat moves very far while the rope is taut and transfers force the process can be approximated as an impact. Assume that a reasonable value for the relevant coefficient of restitution is $e = 0.5$.
a) Calculate the speed (in knots) of the ship after the impact.
b) Calculate the average force in the rope while it spurts water if this goes on for 1.5 s, and express the force in terms of an equivalent mass by dividing by the acceleration due to gravity $g = 9.8\,\mathrm{kg/s^2}$.

Problem 6.3 A bat rotates around the end with the handle, with angular velocity ω, when it hits a ball at a distance x from this end, as shown in figure 6.4. The moment of inertia of the bat with respect to the axis of rotation (perpendicular to the bat through the end with the handle) is J and its length is ℓ. The ball is assumed to be at rest before the hit. Determine the maximal speed of the ball, as a function of x, assuming that the coefficient of restitution is e.

Problem 6.4 A football, which can be thought of as a homogeneous spherical shell of mass m and radius r, is given a kick that is parallel to the horizontal ground, see figure 6.5. The kick imparts a horizontal impulse S to the ball at height h above the ground.
a) For what value of h will the the ball have pure translational motion immediately after the kick?
b) How large fraction of the kinetic energy of the ball remains when it has started to roll without sliding (due to the friction from the ground)?

Problem 6.5 A homogeneous rectangular door leaf of height $h = 2\,\mathrm{m}$, width $b = 0.8\,\mathrm{m}$, and mass m is mounted on hinges at $\mathcal{A}$ and $\mathcal{B}$, see figure 6.6, so that it can rotate freely around a vertical axis. The hinges are placed symmetrically and are a distance 0.25 m from the upper and lower edge respectively (the distance d in figure 6.6 is thus 0.75 m). When the door rotates around its axis, with angular velocity ω, it hits a knob $\mathcal{C}$ in the

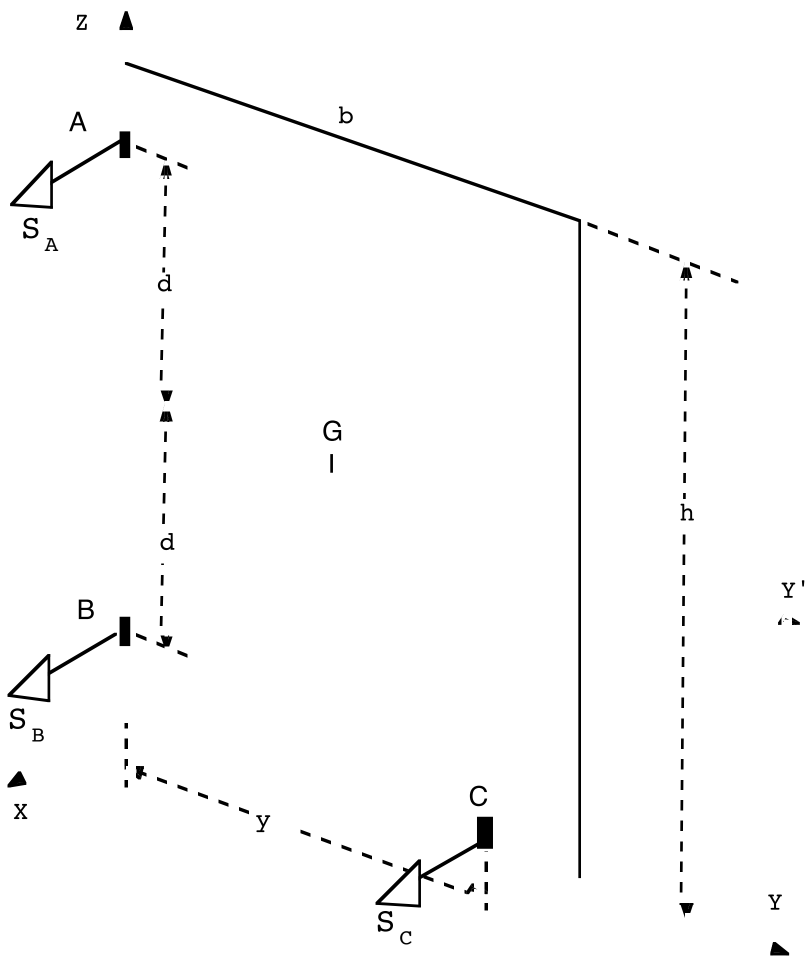


Figure 6.6: This figure refers to problem 6.5 and shows a door hinged at $\mathcal{A}$ and $\mathcal{B}$ which hits a door stop (knob in the floor) at $\mathcal{C}$.

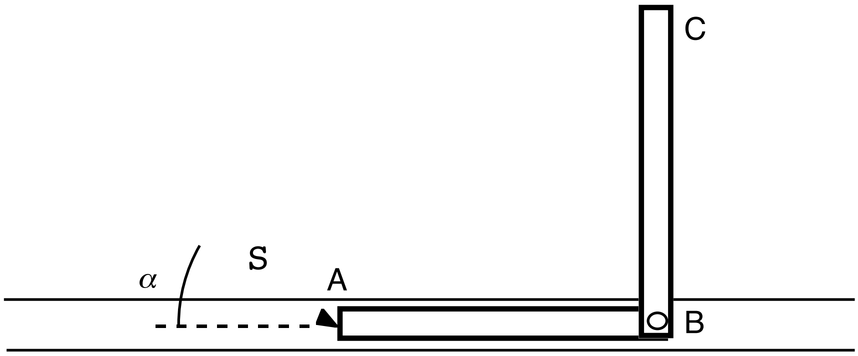


Figure 6.7: This figure refers to problem 6.6 and shows the two hinged rods and the impulse S at $\mathcal{A}$.

floor (a door stop) at a distance y from the axis along the lower edge of the door. This impact gives the door a horizontal impulse S_C normal to the plane of the door such that the door comes to rest.
a) Determine the reaction impulses S_A and S_B from the hinges on the door at the impact.
b) Can one choose y so that one of the reaction impulses vanish?
c) Can one choose y so that the magnitudes of the reaction impulses are equal?

Problem 6.6 Two straight narrow homogeneous rods $\mathcal{AB}$ and $\mathcal{BC}$ are smoothly hinged at $\mathcal{B}$. The two rods are lying on a smooth horizontal plane and the rod $\mathcal{AB}$ is restricted to slide parallel to itself along a smooth track in the plane. Suddenly an impulse S is delivered at $\mathcal{A}$ making an angle α with the rod, see figure 6.7. Calculate the resulting reaction impulse in the hinge $\mathcal{B}$.

6.5 Hints and Answers

Answer 6.1 After the impact the centre of mass of the rod will move along a parabola while the rod will rotate around the centre of mass with constant angular velocity. One finds that the height must obey

$$1.8\,\mathrm{m} \geq h \geq 0.9\,\mathrm{m} + \frac{74\,\mathrm{Nsm}}{S}$$

where one must have $S \geq 82\,\mathrm{Ns}$.

Answer 6.2
a) 0.15 knots.
b) The rope will spurt water while the tension in it increases. This period corresponds to the time interval $t \in [-\delta_1, 0]$ of example 6.3. One finds that the force corresponds to the weight of a mass of 80 tonnes.

Answer 6.3 If $\sqrt{J/m} \leq \ell$ then the optimal distance is

$$x = \sqrt{J/m}$$

and the speed attained is $v_{\max} = \frac{1}{2}\sqrt{\frac{J}{m}}\omega(1+e)$. If $\sqrt{J/m} > \ell$ (is this really possible?) then $x = \ell$ and $v_{\max} = \frac{\ell\omega(1+e)}{1+m\ell^2/J}$.

Answer 6.4
a) This happens at $h = r$ since the friction force from the ground has no time to produce any effect on the motion.
b) The ratio of remaining kinetic energy to initial is 3/5.

Answer 6.5 The basic equation 6.19 gives $(S_A + S_B + S_C)\,\mathbf{e}_x = \mathbf{0} - (-m\frac{b}{2}\omega\,\mathbf{e}_x)$. Here $-(b/2)\omega\,\mathbf{e}_x$ is the centre of mass velocity of the door before the impact. The equation 6.21 gives $\mathbf{r}_{\mathcal{A}} \times S_A\mathbf{e}_x + \mathbf{r}_{\mathcal{B}} \times S_B\mathbf{e}_x + \mathbf{r}_{\mathcal{C}} \times S_C\mathbf{e}_x = -J_z\omega\,\mathbf{e}_z$ for this problem, since there are three points at which the door impacts. It is convenient to move the origin to the intersection of the Z and Y' axes of figure 6.6; the position vectors of the impact points $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, then become

$$\begin{aligned}
\mathbf{r}_{\mathcal{A}} &= d\,\mathbf{e}_z,\\
\mathbf{r}_{\mathcal{B}} &= -d\,\mathbf{e}_z,\\
\mathbf{r}_{\mathcal{C}} &= y\,\mathbf{e}_y - \frac{h}{2}\,\mathbf{e}_z.
\end{aligned}$$

The y- and z-components of the angular momentum principle equation then yield

$$\begin{aligned}
d(S_A - S_B) - \frac{h}{2}S_C &= 0,\\
yS_C &= J_z\omega,
\end{aligned}$$

respectively, where $J_z = mb^2/3$. From these two equations, plus

$$S_A + S_B + S_C = mb\omega/2$$

obtained from 6.19 above, one can calculate the answers that follow.

a) The impulse from the door stop is found to be $S_C = \frac{2}{3}\frac{b}{y}(m\frac{b}{2}\omega)$. The answer is

$$S_A = \frac{1}{2}\left[\left(1 - \frac{2}{3}\frac{b}{y}\right) + \frac{1}{3}\frac{h}{d}\frac{b}{y}\right]\left(m\frac{b}{2}\omega\right),$$

and

$$S_B = \frac{1}{2}\left[\left(1 - \frac{2}{3}\frac{b}{y}\right) - \frac{1}{3}\frac{h}{d}\frac{b}{y}\right]\left(m\frac{b}{2}\omega\right).$$

b) S_A cannot be made zero for positive values of y but S_B is zero if

$$y = \frac{b}{3}\left(2 + \frac{h}{d}\right) = \frac{0.8}{3}\left(2 + \frac{2}{0.75}\right) = 1.24\,\mathrm{m}$$

c) It turns out that $S_A = S_B$ has no solution but that $S_A = -S_B$ when $[1-(2/3)(b/y)] = 0$ i.e. when

$$y = \frac{2}{3}b = \frac{2}{3}0.8 = 0.53\,\mathrm{m}.$$

This is thus a good place to put the door stop since then the two hinges will share the reaction impulse evenly.

Answer 6.6

The reaction impulse should be $S_B = (S/5)\cos\alpha$.

Chapter 7

Open Systems

This chapter discusses the equations of motion for systems which gain or loose matter through flow through the bounding surface. Such mechanical systems are called open systems. Rockets, jet-planes and turbines are examples of such systems. The theoretical basis for this type of problem is given by the principles of linear and angular momentum. Mechanical energy, on the other hand, is rarely conserved since internal non-conservative forces usually are at work in these systems.

7.1 The Momentum Principle for Open Systems

We shall consider the motion of a system of particles $\mathrm{S}(t) = \{m_i, \mathbf{r}_i;\ i = 1, \ldots, N(t)\}$ which is defined as the set of $N(t)$ particles that, at a given time t, lie in a region of space $\Omega(t)$ i.e.

$$\mathrm{S}(t) = \{m_i, \mathbf{r}_i;\ \mathbf{r}_i(t) \in \Omega(t)\}. \tag{7.1}$$

The region Ω may move so that it follows the majority of the particles in their motion, but the system $\mathrm{S}(t)$ is by definition an *open* system if it loses or gains particles which flow in and/or out through the bounding surface of Ω. The centre of mass position $\mathbf{r}_{\mathcal{G}}$ of the system is given by

$$m(t)\mathbf{r}_{\mathcal{G}}(t) \equiv \sum_i^{N(t)} m_i \mathbf{r}_i(t). \tag{7.2}$$

and can be regarded as the mass weighted average position of the $N(t)$ particles in the system at time t.

The momentum of this open system S is defined as

$$\mathbf{p}_\mathrm{o}(t) \equiv \sum_i^{N(t)} m_i \mathbf{v}_i(t) \ = \ m(t)\mathbf{v}_\mathrm{o}(t), \tag{7.3}$$

where the sum is over the particles inside the system at time t and where $\mathbf{v}_\mathrm{o}$ can be regarded as the centre of mass velocity of the open system. We use the notation $\mathbf{v}_\mathrm{o}$ here since this vector is *not* the time derivative of $\mathbf{r}_{\mathcal{G}}$.

To find the equation of motion for the system S we must study the time rate of change of the momentum $\mathbf{p}_\mathrm{o}$. When we do this we must remember that the particles in S are not the same all the time. If we assume that in the time Δt the open system S has gained ΔN_+ particles and lost ΔN_- then

$$N(t + \Delta t) = N(t) + \Delta N = N(t) + (\Delta N_+ - \Delta N_-) \tag{7.4}$$

is the number of particles of S at time $t + \Delta t$. We can thus write

$$\mathbf{p}_\mathrm{o}(t + \Delta t) \equiv \sum_i^{N(t+\Delta t)} m_i \mathbf{v}_i(t + \Delta t) \tag{7.5}$$

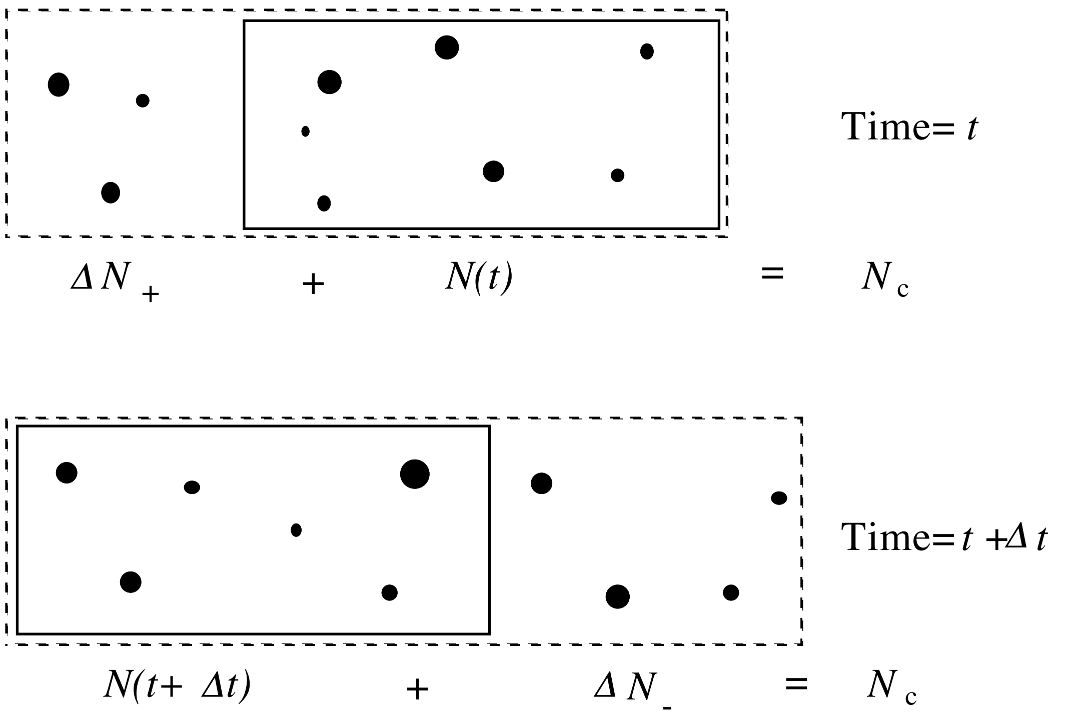


Figure 7.1: This figure illustrates the relationships in equation 7.6. The open system is represented by the solid line box and moves to the left. The closed system with a fixed number of particles N_c, corresponding to the time Δt, is indicated by the dashed line box.

for the momentum of S at time $t + \Delta t$ where the sum now is over the particles in S at this time.

The equation of motion for S follows from the momentum principle in the general form $\dot{\mathbf{p}} = \mathbf{F}^{\mathrm{e}}$ but it must be remembered that the expression $\mathbf{p} = m\mathbf{v}_{\mathcal{G}}$ cannot now simply be differentiated as was the case for a closed system of particles. To get around this problem we define a *closed* system for the time interval t to $t + \Delta t$ as the set of all N_{c} particles that have been in the system S in this time interval. This number is clearly

$$N_{\mathrm{c}} = N(t) + \Delta N_{+} = N(t + \Delta t) + \Delta N_{-} \tag{7.6}$$

i.e. the number of particles in the system at t plus those gained at $t+\Delta t$, or the number of particles in the system at time $t + \Delta t$ plus those lost since t. This is illustrated in figure 7.1.

The total momentum of this closed system is by definition

$$\mathbf{p}_{\mathrm{c}}(t, \Delta t) = \sum_{i}^{N_{\mathrm{c}}} m_i \mathbf{v}_i(t), \tag{7.7}$$

and it is on this system that some total external force $\mathbf{F}^{\mathrm{e}}$ may act so that

$$\dot{\mathbf{p}}_{\mathrm{c}} = \mathbf{F}^{\mathrm{e}}. \tag{7.8}$$

When the time interval $\Delta t \to 0$ the momentum of the closed system we have defined becomes identical to that of the open system,

$$\lim_{\Delta t \to 0} \mathbf{p}_{\mathrm{c}}(t, \Delta t) = \mathbf{p}_{\mathrm{o}}(t), \tag{7.9}$$

but the time derivative of the two momenta are *not* the same as we'll now see.

The change in $\mathbf{p}_c$ in time Δt is

$$\Delta \mathbf{p}_c = \mathbf{p}_c(t+\Delta t, \Delta t) - \mathbf{p}_c(t, \Delta t) = \sum_i^{N_c} m_i \mathbf{v}_i(t+\Delta t) - \sum_i^{N_c} m_i \mathbf{v}_i(t) = \tag{7.10}$$

$$\left[\sum_i^{N(t+\Delta t)} m_i \mathbf{v}_i(t+\Delta t) + \sum_j^{\Delta N_-} m_j \mathbf{v}_j(t+\Delta t)\right] - \left[\sum_i^{N(t)} m_i \mathbf{v}_i(t) + \sum_k^{\Delta N_+} m_k \mathbf{v}_k(t)\right] \tag{7.11}$$

$$= \left[\mathbf{p}_o(t+\Delta t) + \sum_j^{\Delta N_-} m_j \mathbf{v}_j(t+\Delta t)\right] - \left[\mathbf{p}_o(t) + \sum_k^{\Delta N_+} m_k \mathbf{v}_k(t)\right], \tag{7.12}$$

where we have used equation 7.6. We have thus found that the changes in the momenta of the closed and the open systems are related as follows:

$$\Delta \mathbf{p}_c = \Delta \mathbf{p}_o + \sum_j^{\Delta N_-} m_j \mathbf{v}_j(t+\Delta t) - \sum_k^{\Delta N_+} m_k \mathbf{v}_k(t). \tag{7.13}$$

Here $\Delta \mathbf{p}_o$ is the change of the momentum of the open system, the first sum is over the particles that have been lost by the system in the time from t to $t+\Delta t$, and the second sum is over the particles that have been gained in the same time.

If the system consists of a lot of particles with different velocities this is all one can say about the change in its momentum. We now assume that we can make a continuum approximation of the matter that flows in and out of the system. We also assume that there is only a finite number n_- of velocities $\mathbf{v}_a^-$ with which the masses Δm_a^- flow out in time Δt and a finite number n_+ of velocities $\mathbf{v}_b^+$ with which the masses Δm_b^+ flow in. We can then rewrite the above equation in the form

$$\Delta \mathbf{p}_c = \Delta \mathbf{p}_o + \sum_{a=1}^{n_-} \Delta m_a^- \mathbf{v}_a^- - \sum_{b=1}^{n_+} \Delta m_b^+ \mathbf{v}_b^+. \tag{7.14}$$

The first term on the right hand side is the momentum change of the open system S which we can write

$$\Delta \mathbf{p}_o = m(t)\, \Delta \mathbf{v}_o(t) + \Delta m(t)\, \mathbf{v}_o(t). \tag{7.15}$$

One should note that the net mass change

$$\Delta m = \sum_{b=1}^{n_+} \Delta m_b^+ - \sum_{a=1}^{n_-} \Delta m_a^- \tag{7.16}$$

contributes to this change in the momentum $\mathbf{p}_o$ but that $\Delta \mathbf{p}_c \neq \Delta \mathbf{p}_o$ even if $\Delta m = 0$ since the velocities of the in and out flows also contribute.

We now divide equation 7.14 by Δt and take the limit $\Delta t \to 0$. If we denote the *mass flows* by

$$q_a^- \equiv \lim_{\Delta t \to 0} \frac{\Delta m_a^-}{\Delta t} \quad \text{and} \quad q_b^+ \equiv \lim_{\Delta t \to 0} \frac{\Delta m_b^+}{\Delta t} \tag{7.17}$$

we get

$$\dot{\mathbf{p}}_c = \dot{\mathbf{p}}_o + \sum_{a=1}^{n_-} q_a^- \mathbf{v}_a^- - \sum_{b=1}^{n_+} q_b^+ \mathbf{v}_b^+. \tag{7.18}$$

Since $\dot{\mathbf{p}}_c = \mathbf{F}^e$ rearrangement now gives

equation of motion for open system

$$\dot{\mathbf{p}}_o = \mathbf{F}^e - \sum_{a=1}^{n_-} q_a^- \mathbf{v}_a^- + \sum_{b=1}^{n_+} q_b^+ \mathbf{v}_b^+. \tag{7.19}$$

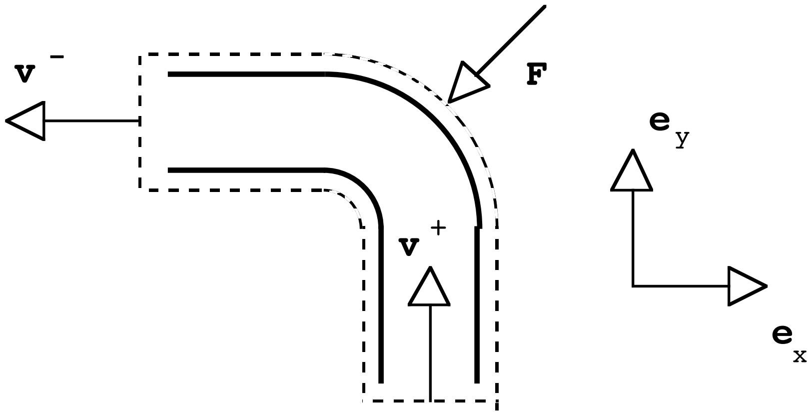


Figure 7.2: This figure illustrates the situation in example 7.1. Water flows through the 90 degree bend in the pipe. What is the force needed to keep the pipe at rest? The dashed line shows the boundary of the open system of pipe with moving water that one must consider.

This is thus the basic equation of motion for the open system S. Note that the mass flows q are defined to be positive ($q \geq 0$) and that all velocities in this equation are with respect to an inertial system. In the limit $\Delta t \to 0$ equations 7.15 and 7.16 give

$$\dot{\mathbf{p}}_{\mathrm{o}}(t) = m(t)\,\dot{\mathbf{v}}_{\mathrm{o}}(t) + \dot{m}(t)\,\mathbf{v}_{\mathrm{o}}(t), \tag{7.20}$$

$$\dot{m}(t) = \sum_{b=1}^{n_+} q_b^+ - \sum_{a=1}^{n_-} q_a^-. \tag{7.21}$$

These together with the equation of motion 7.19 provide sufficient information for solving many open system problems.

Example 7.1 Water flows at a rate of $q = 1$ kg/s through a pipe of constant cross-section A in which there is a 90 degree bend. The speed of the water, which is assumed uniform in the pipe, is $v = 1$m/s. Calculate the external force F needed to keep one meter of the pipe around the bend at rest. See figure 7.2.

Solution: Since in this case the centre of mass velocity $\mathbf{v}_{\mathrm{o}}$ of the open system is constant, and since the loss of mass is exactly balanced by the gain its mass $m(t)$ is constant, and we get that $\dot{\mathbf{p}}_{\mathrm{o}} = m\,\dot{\mathbf{v}}_{\mathrm{o}} + \dot{m}\,\mathbf{v}_{\mathrm{o}} = \mathbf{0}$. Equation 7.19 now gives

$$\mathbf{0} = \mathbf{F} - q\mathbf{v}^- + q\mathbf{v}^+. \tag{7.22}$$

With suitably directed basis vectors this gives

$$0\,\mathbf{e}_x + 0\,\mathbf{e}_y = F_x\,\mathbf{e}_x + F_y\,\mathbf{e}_y - (-qv)\,\mathbf{e}_x + qv\,\mathbf{e}_y \tag{7.23}$$

and thus $F_x = -qv$ and $F_y = -qv$. The magnitude of the force is thus $F = \sqrt{2}qv = \sqrt{2}$N, and this is the answer. $\square$

Equation 7.19 can also be written in an alternative form if we note that $\dot{\mathbf{p}}_{\mathrm{o}} = m\,\mathbf{a}_{\mathrm{o}} + \dot{m}\,\mathbf{v}_{\mathrm{o}}$ so that

$$m\mathbf{a}_{\mathrm{o}} = \mathbf{F}^{\mathrm{e}} - \dot{m}\mathbf{v}_{\mathrm{o}} - \sum_{a=1}^{n_-} q_a^-\mathbf{v}_a^- + \sum_{b=1}^{n_+} q_b^+\mathbf{v}_b^+, \tag{7.24}$$

and that $\dot{m} = \sum q^+ - \sum q^-$. Combining these we get

$$m\mathbf{a}_{\mathrm{o}} = \mathbf{F}^{\mathrm{e}} - \sum_{a=1}^{n_-} q_a^-\mathbf{u}_a^- + \sum_{b=1}^{n_+} q_b^+\mathbf{u}_b^+, \tag{7.25}$$

where the $\mathbf{u}^- = \mathbf{v}^- - \mathbf{v}_{\mathrm{o}}$ stand for the velocities of the out-flowing matter relative to the system itself and correspondingly for the $\mathbf{u}^+$.

Example 7.2 A rocket has a motor which ejects mass with a constant rate q with a constant speed u relative to the rocket. It rises vertically starting with zero velocity at time $t = 0$. Calculate its velocity as a function of time.
Solution: Equation 7.25 gives

$$m(t)\,\frac{d\mathbf{v}_\mathrm{o}}{dt} = m(t)\,\mathbf{g} - q\,\mathbf{u}^-. \tag{7.26}$$

With the Z-axis vertically upwards we have $\mathbf{v}_\mathrm{o} = v_\mathrm{o}(t)\,\mathbf{e}_z$, $\mathbf{g} = -g\,\mathbf{e}_z$, and $\mathbf{u}^- = -u\,\mathbf{e}_z$. The mass of the rocket is

$$m(t) = m_0 - qt \tag{7.27}$$

so that $q = -\frac{dm}{dt}$. All this gives us

$$m(t)\,\frac{dv_\mathrm{o}}{dt} = -m(t)\,g - \frac{dm}{dt}\,u. \tag{7.28}$$

Multiply by dt, divide by m and integrate from $t = 0$ to t to get

$$\int_0^t dv_\mathrm{o} = \int_0^t (-g\,dt - u\frac{dm}{m}). \tag{7.29}$$

This gives

$$v_\mathrm{o}(t) = -g\,t - u\,\ln\frac{m(t)}{m_0} = -g\,t + u\,\ln\frac{m_0}{m_0 - qt}. \tag{7.30}$$

This solution is of course only valid until the time $t = T$ at which all the fuel $m_\mathrm{f} = qT$ is gone.
□

7.2 Angular Momentum of Open Systems

In some problems, usually involving rotation, it is of interest to find the time rate of change of the angular momentum of an open system. To find this we can use the same method as in the previous section. We use the origin as base point throughout and define the angular momentum of the open system S to be

$$\mathbf{L}_\mathrm{o}(t) \equiv \sum_i^{N(t)} \mathbf{r}_i(t) \times m_i \mathbf{v}_i(t), \tag{7.31}$$

where, again, the sum is over the particles in the system at time t. We also use the closed system corresponding to the time interval t to $t + \Delta t$. It will have the angular momentum

$$\mathbf{L}_\mathrm{c}(t, \Delta t) \equiv \sum_i^{N_\mathrm{c}} \mathbf{r}_i(t) \times m_i \mathbf{v}_i(t). \tag{7.32}$$

According to the principle of angular momentum we have $\dot{\mathbf{L}}_\mathrm{c} = \mathbf{M}^\mathrm{e}$ where $\mathbf{M}^\mathrm{e}$ is the total moment of the external forces on the system. We can now find the relationship between the changes in the two angular momenta in time Δt in the same way as we did for the linear momenta in the previous section. The result found for momentum changes in equation 7.13 translates directly to

$$\Delta\mathbf{L}_\mathrm{c} = \Delta\mathbf{L}_\mathrm{o} + \sum_j^{\Delta N_-} \mathbf{r}_j(t + \Delta t) \times m_j \mathbf{v}_j(t + \Delta t) - \sum_k^{\Delta N_+} \mathbf{r}_k(t) \times m_k \mathbf{v}_k(t) \tag{7.33}$$

in the angular momentum case. We assume, as before, that the mass flows through the system boundary can be considered as continuous with mass per unit time $q^\pm$ flowing in/out with a finite number of different velocities $\mathbf{v}^\pm$. We now also assume that these mass flows take place at well defined positions $\mathbf{r}^\pm$. The above equation then gives

$$\Delta\mathbf{L}_\mathrm{c} = \Delta\mathbf{L}_\mathrm{o} + \sum_{a=1}^{n_-} \mathbf{r}_a^- \times q_a^- \Delta t\, \mathbf{v}_a^- - \sum_{b=1}^{n_+} \mathbf{r}_b^+ \times q_b^+ \Delta t\, \mathbf{v}_b^+. \tag{7.34}$$

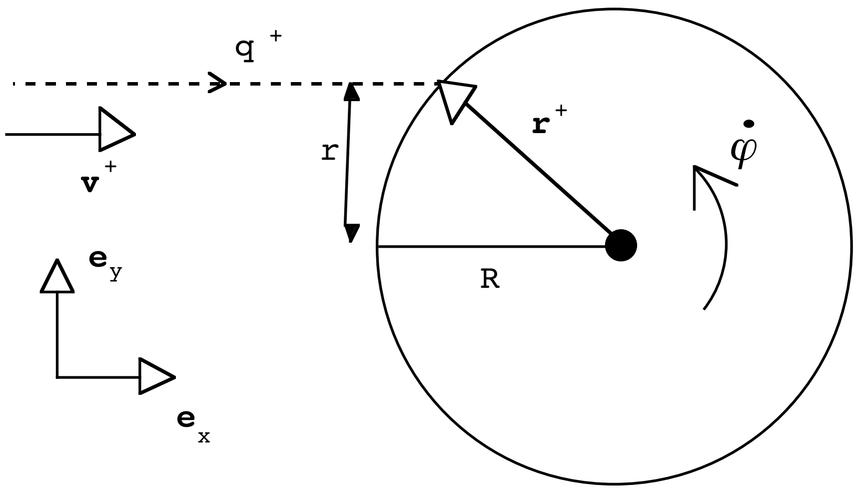


Figure 7.3: This figure illustrates the situation in example 7.3. The dashed line is the trajectory along which the bullets move. They hit the wooden carousel and come to rest in it near the circumference.

Dividing by Δt and taking the limit now gives the final result

angular momentum principle for open system

$$\dot{\mathbf{L}}_o = \mathbf{M}^e - \sum_{a=1}^{n_-} \mathbf{r}_a^- \times q_a^- \mathbf{v}_a^- + \sum_{b=1}^{n_+} \mathbf{r}_b^+ \times q_b^+ \mathbf{v}_b^+ \tag{7.35}$$

for the angular momentum principle for an open system.

Example 7.3 A carousel (merry-go-round) of radius R can rotate freely around a vertical axis. The moment of inertia with respect to this axis is J_0. Find the expression for the angular acceleration $\ddot{\varphi}$ of the carousel if it is shot at by a machine gun which produces a horizontal mass flow q^+ of bullets with speed v^+. Also find the limiting angular velocity if the shooting goes on for long. The bullets hit the carousel at a perpendicular distance r from its centre, see figure 7.3, and are embedded in it near the circumference.
Solution: If we assume that the shooting started at time $t = 0$ the moment of inertia of the carousel will be

$$J(t) = J_0 + q^+ t R^2 \tag{7.36}$$

since the bullets increase the mass at radius R. The angular momentum of this open system (carousel under fire) is

$$\mathbf{L}_o = J(t)\dot{\varphi}(t)\,\mathbf{e}_z, \tag{7.37}$$

with respect to an origin at its centre. Equation 7.35 now gives

$$\frac{d}{dt}(J(t)\dot{\varphi}(t)\,\mathbf{e}_z) \;=\; \mathbf{r}^+ \times q^+ \mathbf{v}^+ = -rq^+v^+\,\mathbf{e}_z. \tag{7.38}$$

This gives

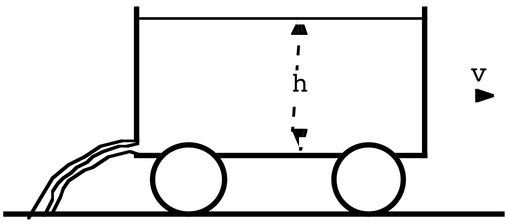
$$(J_0 + q^+ t R^2)\ddot{\varphi} = -rq^+v^+ - q^+R^2\dot{\varphi} \tag{7.39}$$

and this is the desired differential equation for $\ddot{\varphi}$.

If $\dot{\varphi}(0) = 0$ the angular acceleration is negative for small $t > 0$. This gives a negative angular velocity $\dot{\varphi}$ as the sign conventions of the figure 7.3 demands. However, when the angular velocity becomes increasingly negative, the right hand side of the equation for $\ddot{\varphi}$ decreases. Eventually the acceleration will approach zero. When $\ddot{\varphi} = 0$ the above equation gives the limiting angular velocity

$$\dot{\varphi}_{\lim} = -\frac{rv^+}{R^2} \tag{7.40}$$

and this concludes our example. $\square$



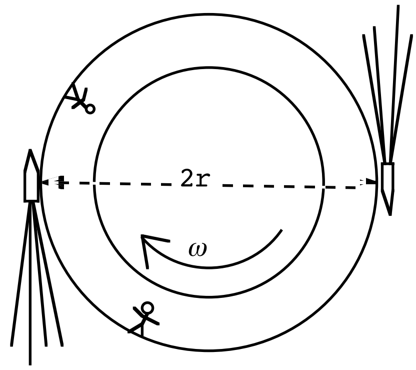


Figure 7.4: The figure on the left refers to problem 7.2. The water flows out through the hole of cross sectional area a. The height of the water surface above the hole is h.

Figure 7.5: The figure on the right refers to problem 7.3. A toroidal space station is given an angular acceleration from zero angular velocity up to a value that corresponds to the acceleration of gravity on the perimeter.

7.3 Problems

Problem 7.1 A jet plane of mass m is propelled by maintaining an air flow of q mass units per unit time through its engine. The speed of the air relative to the engine is $w =$ const.. Calculate the speed $v(t)$ of the plane as a function of time assuming the initial condition $v(0) = v_0$. Neglect the effects of fuel consumption and air resistance.

Problem 7.2 A vehicle is propelled by letting water flow out backwards from a tank of rectangular horizontal cross sectional area A that has collected rain water, see figure 7.4. The cross sectional area of the hole through which the water flows out is a and we assume that $a \ll A$. One can assume that the water flows out with the speed $u = \sqrt{2gh}$ (see example 8.1) relative to the tank, where h is the height of the water surface above the hole. Show that the acceleration of the vehicle must be less than $(a/A)2g$.

Problem 7.3 A toroidal space station has (outer) radius r, initial mass m, and radius of gyration d. To achieve an angular velocity ω, corresponding to an artificial gravity g, the space station, which initially is at rest, is accelerated by means of two identical rockets on the outer rim. The two rockets are placed at opposite ends of a diameter, see figure 7.5, and are directed along the tangent of the circumference. Each rocket has mass flow q (mass per unit time) and exhaust velocity u (relative to the rocket). Assume that all expelled mass was contained in the rockets (i.e. at radius r). How long must the rockets be turned on? (Note that rotation with angular velocity ω gives an artificial 'gravity' corresponding to the centripetal acceleration $r\omega^2$.)

Problem 7.4 A chain has constant mass per unit length λ. It is held at one end so that it hangs vertically with its lower end at a height h above a horizontal table surface. It is released from rest and starts to fall onto the table. Find the force $F(x)$ from the chain against the table as a function of the length x of the chain that has reached the table.

7.4 Hints and Answers

Answer 7.1 One finds the equation of motion $\dot{p} = m\dot{v} = (-q)(-w + v)$ which gives $\dot{v} = -\frac{q}{m}(v - w)$. Integration of this gives

$$v(t) = w + (v_0 - w)\exp\left(-\frac{q}{m}t\right).$$

As $t \to \infty$ this is seen to go to w, a reasonable result.

Answer 7.2 Denote the density of the water by ϱ and the speed of the vehicle by v. Use of equation 7.25 then gives $m\dot{v} = qu$ where $q = \varrho au$ so that $\dot{v} = \varrho au^2/m$. The mass of the vehicle is $m = m_0 + \varrho Ah$ where m_0 is the remaining mass when the height of the water surface above the hole has become zero. Since $u^2 = 2gh$ we find

$$\dot{v} = \frac{\varrho a 2gh}{m_0 + \varrho Ah}.$$

Since $m_0 \geq 0$ we find that $\dot{v} \leq \frac{a}{A}2g$ and this is what we wanted to show.

Answer 7.3 Use equation 7.35 with $\mathbf{M}^{\mathrm{e}} = \mathbf{0}$ and with $\mathbf{L}_{\mathrm{o}} = J(t)\omega\,\mathbf{e}_z$. One finds that $J(t) = md^2 - 2(qt)r^2$ and thus that the z-component of the equation of motion is

$$\frac{d}{dt}[J(t)\omega(t)] = -2rq(r\omega - u).$$

Some calculations then give

$$\dot{\omega} = \frac{2rqu}{md^2 - 2qtr^2}$$

and integration yields

$$\omega(t) = \frac{u}{r}\ln\left(\frac{md^2}{md^2 - 2qrt^2}\right).$$

If one now puts $\omega = \sqrt{\frac{g}{r}}$ and solves for t one finds that

$$t = \frac{md^2}{2qr^2}\left[1 - \exp\left(-\frac{\sqrt{gr}}{u}\right)\right]$$

is the required burning time of the rockets.

Answer 7.4 Use energy conservation to get the speed of the chain. Consider the following forces on the heap of chain on the table: the normal force $N(x)$ from the table, the weight of the heap of chain, $g\lambda x$, and the the 'force' due to the fact that the heap of chain gains mass. Use of equation 7.25 now gives

$$(ma = 0) = N(x) - g\lambda x - q^{+}u$$

where $u = \sqrt{2g(x + h)}$ and $q^{+} = \lambda u$. This gives us

$$N(x) = \lambda g(3x + 2h)$$

for the normal force from the table on the chain heap. The force from the chain heap on the table $F(x)$ is then of equal magnitude according to the law of action and reaction.

Chapter 8

The Mechanics of Fluids

The mechanics of fluids (gases and liquids) belongs to *continuum* mechanics, that is, the number of degrees of freedom can be regarded as infinite. This means that fluids must be described by fields. The most important kinematic object is the velocity field $\mathbf{v}(\mathbf{r}, t)$ which gives the velocity vector at each point $\mathbf{r}$ of the fluid at time t. The mathematical techniques required to handle fluids are those of vector analysis. This chapter presents derivations of some equations which govern the behavior of the velocity field. The scalar fields of (mass) density $\varrho(\mathbf{r}, t)$ and pressure $p(\mathbf{r}, t)$ are introduced. The distinction between laminar and turbulent flow is explained as well as the concepts of ideal contra viscous fluids.

8.1 The Velocity Field

A substance is called a *fluid* if arbitrarily small (tangential) forces lead to flow, that is, motion on a macroscopic scale as long as the force acts. A substance which is not a fluid is called a *solid*. A small force on a solid only leads to a small deformation because it is counteracted by internal forces in the material. A fluid may thus be seen as a substance which does not have internal forces that try to maintain any particular form or shape of the substance.

From a fundamental point of view a fluid consists of a large number of particles (molecules) and predicting its detailed behavior requires the solution of the equations of motion for each of these particles. This is, of course, impossible in practice and it turns out that it is not necessary to carry out this program in order to get an understanding of the behavior of fluids. Instead of studying the individual molecules one considers quantities that are averages over a number of molecules which is large from the microscopic point of view but which is small from the macroscopic point of view. Such average quantities are said to be 'mesoscopic'. Quantities such as density, temperature, and pressure are mesoscopic quantities which become undefined on the microscopic level of individual molecules.

The average velocity $\mathbf{v}$ of a large number of fluid molecules at time t, which nevertheless can be regarded as essentially localized to a point $\mathbf{r}$ from the macroscopic point of view, defines the *velocity field*, $\mathbf{v}(\mathbf{r}, t)$, of the fluid. This quantity is a vector field and its study belongs to the area of vector analysis.

8.1.1 Divergence and Curl of the Velocity Field

Let us consider the fluid inside some region $\Omega(t)$ of space. The volume of the fluid inside this region at time t is then

$$V(t) = \int_{\Omega(t)} dV. \tag{8.1}$$

Let us investigate how this volume changes if we allow the region Ω and its bounding surface $S \equiv \partial\Omega$ to follow the fluid in its motion. Consider a small element of area dA of the surface S and denote by $\mathbf{e}_n$ the outward directed normal to the surface S at dA. One then defines the vector element of area

$$d\mathbf{A} \equiv \mathbf{e}_n \, dA. \tag{8.2}$$

The change in volume of Ω due to the fluid flow during the time interval Δt will be localized at the bounding surface and it is easy to see that, at a given element of area dA of the surface S, the change in volume will be given by

$$d\Delta V = \Delta t \, \mathbf{v} \cdot d\mathbf{A}. \tag{8.3}$$

This volume is the base area dA times the 'height', that is, the component of the displacement $d\mathbf{r} = \Delta t \, \mathbf{v}$ along the unit normal $\mathbf{e}_n$. It is negative if the flow has a component into the region Ω since then $\mathbf{v}$ has a negative projection on $\mathbf{e}_n$. It is zero if the flow is parallel to the surface, and it is positive if the flow has a component out of the region. The total change in volume in time Δt is then the sum of all these changes in the limit when the $dA \to 0$, i.e. the surface integral

$$\Delta V = \int_S \Delta t \, \mathbf{v} \cdot d\mathbf{A}. \tag{8.4}$$

This means that the time derivative of the volume is given by

$$\frac{dV}{dt} = \int_S \mathbf{v} \cdot d\mathbf{A}. \tag{8.5}$$

One of the fundamental results of vector calculus is *Gauss' theorem* which says that $\int_\Omega \nabla \cdot \mathbf{v} \, dV = \int_S \mathbf{v} \cdot d\mathbf{A}$. Using this we find that

$$\frac{dV}{dt} = \int_\Omega \nabla \cdot \mathbf{v} \, dV. \tag{8.6}$$

This shows us that the divergence of the velocity field, $\operatorname{div} \mathbf{v} \equiv \nabla \cdot \mathbf{v}$ is a measure of the *change of volume of the fluid*.

If the fluid moves like a rigid body we know that its velocity field must have the form

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{0}) + \boldsymbol{\omega} \times \mathbf{r} \tag{8.7}$$

(see equation 2.74). The divergence is then identically zero $\nabla \cdot \mathbf{v} = 0$, which is natural since the parts of a rigid body have fixed volumes. Another important quantity in vector calculus if the 'curl' (or rotation) of a vector field: $\operatorname{curl} \mathbf{v} \equiv \nabla \times \mathbf{v}$. For the case of 'rigid' flow one finds that

$$\nabla \times \mathbf{v}(\mathbf{r}) = \nabla \times [\mathbf{v}(\mathbf{0}) + \boldsymbol{\omega} \times \mathbf{r}] = 2\boldsymbol{\omega}. \tag{8.8}$$

From this one concludes that, in general, the curl of the velocity field, which sometimes is called the *vorticity*, is twice the local angular velocity of the fluid. Fluid flow for which $\nabla \times \mathbf{v} = \mathbf{0}$ is said to be 'irrotational'.

8.1.2 The Total Time Derivative

The time derivative of some quantity in the fluid can be measured in two different ways. Either one can study the time derivative by measuring the quantity at some fixed point of space. If the quantity is temperature T for example, the time rate of change of the readings of a thermometer at some fixed point of space corresponds to the partial derivative

$$\lim_{\Delta t \to 0} \left(\frac{\Delta T(\mathbf{r}, t)}{\Delta t} \right)_{\mathbf{r}=\text{const}} = \frac{\partial T}{\partial t} \tag{8.9}$$

of the scalar field $T(\mathbf{r}, t)$ with respect to time.

If, on the other hand, one measures some quantity, such as temperature for example, in some (material) fluid element with trajectory $\mathbf{r}(t)$ and considers the time rate of change of these measurements one finds what is called the *total* time derivative

$$\lim_{\Delta t \to 0} \frac{\Delta T(\mathbf{r}(t), t)}{\Delta t} = \frac{dT(\mathbf{r}(t), t)}{dt} = \tag{8.10}$$

$$= \frac{\partial T}{\partial t} + \frac{dx}{dt}\frac{\partial T}{\partial x} + \frac{dy}{dt}\frac{\partial T}{\partial y} + \frac{dz}{dt}\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T. \tag{8.11}$$

This quantity is sometimes referred to as the 'substantial' or 'material' time derivative. This time rate of change of temperature is measured by a small light thermometer which follows the fluid in its flow. We will write

total time derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{8.12}$$

for the total time derivative operator.

8.1.3 The Equation of Continuity

The amount of mass in a fixed region Ω of the fluid is found by integrating the (mass) density ϱ over the volume of that region:

$$m[\Omega] = \int_\Omega \varrho(\mathbf{r}, t)\, dV. \tag{8.13}$$

The rate of increase of the mass in the region is then

$$\frac{\partial}{\partial t} m[\Omega] = \frac{\partial}{\partial t} \int_\Omega \varrho(\mathbf{r}, t)\, dV = \int_\Omega \frac{\partial \varrho}{\partial t}\, dV. \tag{8.14}$$

Since mass is conserved the change (increase) per unit time of the mass in Ω must be due to flow of mass through the surface S bounding the region. The mass per unit time flowing in through the boundary must be given by

$$\frac{\partial}{\partial t} m[\Omega] = -\int_S \varrho \mathbf{v} \cdot d\mathbf{A} \tag{8.15}$$

where the minus sign is due to the convention that the area element vector $d\mathbf{A}$ is directed outwards, so that $\varrho \mathbf{v} \cdot d\mathbf{A}$ is positive where matter flows out. Equating these two expressions and using Gauss' theorem in the form

Gauss' theorem

$$\int_S \varrho \mathbf{v} \cdot d\mathbf{A} = \int_\Omega \nabla \cdot (\varrho \mathbf{v})\, dV, \tag{8.16}$$

gives us

$$\int_\Omega \left[\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) \right] dV = 0. \tag{8.17}$$

Since the region Ω is arbitrary one finds that mass conservation is expressed by the partial differential equation

equation of continuity

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) = 0 \tag{8.18}$$

relating the partial derivatives of the fields ϱ and $\mathbf{v}$. This equation is called the *equation of continuity*.

8.1.4 Laminar contra Turbulent Flow

Empirically one finds that there are two qualitatively different types of fluid flow. At low velocities the flow is often *laminar* which means that the particles of the fluid follow smooth, well defined, trajectories called *streamlines.* Then the direction of the velocity field is given by the tangent vectors to these streamlines. The velocity field at a point is either constant, in which case one has *steady flow* with $\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$, or it varies in a regular fashion (*unsteady flow*). Essentially all cases for which there are analytic solutions to the fluid equations of motion correspond to laminar flow.

If other conditions remain constant an increase in the speed of the flow will eventually lead to a qualitative change in the flow pattern which is called the onset of turbulence. *Turbulent* flow is characterized by irregular fluctuation of the velocity field as well as of other fields such as e.g. pressure. Turbulent flow is also characterized by the simultaneous presence of vortices and eddies of many different length scales. In the turbulent domain the fluid equations of motion, which usually consist of a set of non-linear, coupled differential equations, exhibit, so called sensitive dependence on initial conditions. This means that very small changes in the conditions at the starting time soon will lead to radically different solutions of the differential equations. When this is the case prediction of the flow pattern is limited to some given finite time. After this time no details are known about the flow. One sometimes describes this situation as an example of deterministic *chaos.* Weather prediction entails the prediction of fluid flows in the atmosphere of the Earth and these turn out to be turbulent. There is thus a limit to the scope of such predictions.

8.2 The Equation of Motion for an Ideal Fluid

We now wish to find the equation of motion for the fluid, that is, we wish to find the form that the momentum principle $\dot{\mathbf{p}} = \mathbf{F}$ takes when it is expressed in terms of the relevant fields.

8.2.1 Mass Times Acceleration in Fluids

The momentum $d\mathbf{p} = dm\,\mathbf{v}$ of a fluid particle of volume dV at $\mathbf{r}$ is

$$d\mathbf{p}(\mathbf{r}, t) = [\varrho(\mathbf{r}, t)\, dV]\mathbf{v}(\mathbf{r}, t). \tag{8.19}$$

Here we follow a given fluid particle with constant mass, $dm = \varrho\, dV$=const., so the time derivative is given by

$$d\dot{\mathbf{p}} = \frac{d}{dt}[(\varrho\, dV)\mathbf{v}] = (\varrho\, dV)\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v}, \tag{8.20}$$

where we have used equation 8.12 for the total time derivative. This is thus the time rate of change of the momentum for the fluid particle in dV.

8.2.2 Forces in Fluids

To get an equation of motion for the fluid we must now equate this time rate of change of the momentum with the force $d\mathbf{F}$ on the fluid particle in dV at $\mathbf{r}$. This force will usually include the force of gravity (weight) of the fluid particle. This part is given by

$$d\mathbf{W} = (\varrho\, dV)\,\mathbf{g}, \tag{8.21}$$

where $\mathbf{g}$ is the acceleration due to gravity. In continuum mechanics this type of force is called a *volume (or body) force* since it is proportional to the volume dV. Apart from body forces there are also *surface forces* acting on the elements of the fluid. These are of two basic types: those perpendicular to the surface, that is, parallel to the vector surface element $d\mathbf{A}$, and those tangent to the surface. The first type is due to the *pressure* $p(\mathbf{r}, t)$ and the total force on the fluid in the region Ω due to pressure is

force due to pressure

$$\mathbf{F}_{\mathrm{p}} = \int_S -p(\mathbf{r}, t)\, d\mathbf{A} \tag{8.22}$$

where S is the surface surrounding Ω. The minus sign here is needed because of the convention that the vector surface element points out from the region and we want the force on the fluid inside S from the rest of the fluid. Let $\mathbf{a}$ be a constant vector. Then use of Gauss' theorem gives us

$$\mathbf{a} \cdot \int_S p\, d\mathbf{A} = \int_S (p\mathbf{a}) \cdot d\mathbf{A} = \int_\Omega \nabla \cdot (p\mathbf{a})\, dV = \tag{8.23}$$

$$= \int_\Omega \mathbf{a} \cdot \nabla p\; dV = \mathbf{a} \cdot \int_\Omega \nabla p\; dV. \tag{8.24}$$

Since $\mathbf{a}$ is an arbitrary vector this gives us the result

$$\mathbf{F}_{\mathrm{p}} = -\int_\Omega \nabla p\; dV. \tag{8.25}$$

The surface force due to pressure on the small element of fluid in dV can thus be written

$$d\mathbf{F}_{\mathrm{p}} = -\nabla p\; dV. \tag{8.26}$$

It is seen to be proportional to the gradient ∇p of the pressure.

Tangential surface forces in fluids arise when the component of the velocity parallel to the surface varies along the normal and these forces tend to decrease such 'velocity gradients'. Such forces are said to be due to *viscosity* and are non-conservative dissipative, i.e. decrease the mechanical energy. Pressure, on the other hand, turns out to give rise to conservative forces in the fluid. In some fluids viscosity is small and can be neglected as a first approximation. Such a fluid is called an *ideal* fluid.

8.2.3 Euler's Equation for an Ideal Fluid

If we now collect the three results of equations 8.20, 8.21, and 8.26 we find an equation of motion, $d\dot{\mathbf{p}} = d\mathbf{W} + d\mathbf{F}_{\mathrm{p}}$, for an ideal fluid moving under the influence of gravity, in the form

Euler's equation for fluid

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} - \frac{\nabla p}{\varrho}. \tag{8.27}$$

This equation was first derived by Leonhard Euler in 1755.

Vector analysis provides the identity

$$\frac{1}{2}\nabla v^2 = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}, \tag{8.28}$$

where $v = |\mathbf{v}|$ and this allows us to rewrite Euler's equation above in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{g} - \frac{\nabla p}{\varrho}. \tag{8.29}$$

We will now investigate one of the special cases in which this equation has a first integral.

8.2.4 Bernoulli's Theorem for Steady Incompressible Flow

We now assume that the flow is *steady*, i.e. that $\frac{\partial \mathbf{v}}{\partial t} = 0$, and that the fluid is *incompressible*, which means that it has constant density, $\varrho(\mathbf{r}, t) = \varrho_0$ =constant. Liquids are normally fairly incompressible while gases are more easily compressed. For this case we can write equation 8.29 on the form

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}\nabla v^2 - \mathbf{g} + \frac{\nabla p}{\varrho_0}. \tag{8.30}$$

We now integrate the vector fields on both sides of this equation along a curve $C:\ t \mapsto \mathbf{r}(t)$ which is the solution of the equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}. \tag{8.31}$$

Such a curve C is said to be a 'streamline' and in the case of steady flow it will simply be the trajectory of a fluid particle. If we let 1 stand for some, arbitrary, start point on the streamline and 2 for some end point we get

$$\int_{C(1,2)} [\mathbf{v} \times (\nabla \times \mathbf{v})] \cdot d\mathbf{r} = \int_{C(1,2)} \left[\nabla\left(\frac{1}{2}v^2 - \mathbf{g}\cdot\mathbf{r} + \frac{p}{\varrho_0}\right)\right] \cdot d\mathbf{r}. \tag{8.32}$$

Here we have used that $\nabla(\mathbf{g}\cdot\mathbf{r}) = \mathbf{g}$. The left hand side of this equation is easily seen to be zero since $\mathbf{v}\times(\nabla\times\mathbf{v})$ is perpendicular to $\mathbf{v}$ but $d\mathbf{r}$ is parallel to $\mathbf{v}$ when one integrates along a streamline. The scalar product under the integral sign is thus identically zero for such a curve. On the right hand side we find the differential $d(\frac{1}{2}v^2 - \mathbf{g}\cdot\mathbf{r} + \frac{p}{\varrho_0})$ under the integral sign so we now get

$$0 = \left[\frac{1}{2}v^2(\mathbf{r}_2) - \mathbf{g}\cdot\mathbf{r}_2 + \frac{p(\mathbf{r}_2)}{\varrho_0}\right] - \left[\frac{1}{2}v^2(\mathbf{r}_1) - \mathbf{g}\cdot\mathbf{r}_1 + \frac{p(\mathbf{r}_1)}{\varrho_0}\right]. \tag{8.33}$$

We now choose the coordinate system to have a vertical Z-axis pointing upwards so that $\mathbf{g} = -g\,\mathbf{e}_z$. Using this we can express the above result as follows

Bernoulli's equation

$$\frac{1}{2}\varrho_0\, v^2(\mathbf{r}) + \varrho_0\, gz + p(\mathbf{r}) = \text{const.} \qquad \text{along streamline.} \tag{8.34}$$

Note that the first term in this equation is the kinetic energy per volume and that the second is the potential energy of gravity per volume. The pressure p must consequently also have dimensions of energy per volume and the entire equation can be regarded as an energy integral for the steady flow of the ideal incompressible fluid. The kinetic energy per volume is sometimes called the 'dynamic pressure'. The equation 8.34 is called Bernoulli's equation and can be used to obtain a qualitative understanding of many phenomena associated with the motion of fluids.

Example 8.1 Find the speed u with which an ideal liquid flows out of a tank from a small hole at depth h. The acceleration due to gravity is g. See figure 8.1.
Solution: When the hole is small one can assume that the flow is approximately stationary so that one can use Bernoulli's equation. Consider a stream line from the surface of the liquid $\mathcal{A}$ to the hole $\mathcal{B}$, as in the figure. We can assume that $\mathbf{v}(\mathbf{r}_\mathcal{A}) = \mathbf{0}$, that $z_\mathcal{A} = 0$, and that $p(\mathbf{r}_\mathcal{A}) = p(\mathbf{r}_\mathcal{B}) = p_0$ The constant in Bernoulli's equation 8.34 is thus given by

$$0 + 0 + p_0 = \text{const.} \tag{8.35}$$

At the hole we thus get

$$\frac{1}{2}\varrho_0 u^2 - \varrho_0 gh + p_0 = p_0, \tag{8.36}$$

since $z_\mathcal{B} = -h$. Solving for u we finally find that

$$u = \sqrt{2gh} \tag{8.37}$$

is the speed with which the liquid flows out. □

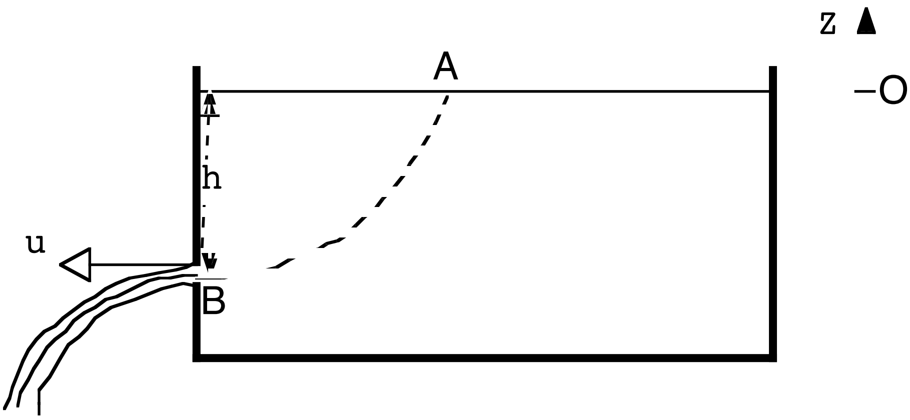


Figure 8.1: This figure illustrates the situation in example 8.1. There is a small hole in the tank at depth h below the liquid surface.