

Space Mechanics

Hanno Essén
Department of Mechanics
Royal Institute of Technology
S-100 44 Stockholm, Sweden

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Chapter 1

Motion in a Central Force Field

Before Kepler all men were blind.
Kepler had one eye,
Newton had two.
Voltaire

In this chapter we introduce the concept of angular momentum for a particle and show that it is conserved for a particle in a central force field. We then show how the two-particle problem can be reduced to an effective one-particle problem in the center of mass system. After that we use conservation of energy to find out things about central motion. The Kepler problem for motion in a $1/r^2$ -force field is solved using the conservation of energy and angular momentum. Some properties of the resulting solutions, the ellipse, the parabola, and the hyperbola are presented. We then discuss the problem of the force-field from an extended body, in particular the spherically symmetric case.

1.1 Angular Momentum

First we must now remind ourselves of the concept of *moment* (or torque) of a force with respect to a point. This quantity is a vector quantity, \mathbf{M} , which, intuitively, corresponds to the rotating ability of the force around the point. Its direction gives the axis around which the force tends to rotate and its magnitude measures how large the tendency is. When the base point is chosen as the origin one has by definition

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}, \quad (1.1)$$

where \mathbf{r} thus is the position vector of the particle on which the force \mathbf{F} acts.

We'll now try to rewrite the equation of motion of a particle, $\dot{\mathbf{p}} = \mathbf{F}$, so that it describes rotation and contains \mathbf{M} instead of the force. To do this it turns out to be advantageous to first define the vector quantity

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad (1.2)$$

called the *angular momentum*, with respect to the origin, of the particle having momentum $\mathbf{p} = m\mathbf{v}$. The time derivative of this vector gives

$$\dot{\mathbf{L}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \dot{\mathbf{p}} \quad (1.3)$$

and this is seen to give us

$$\dot{\mathbf{L}} = \mathbf{M}. \quad (1.4)$$

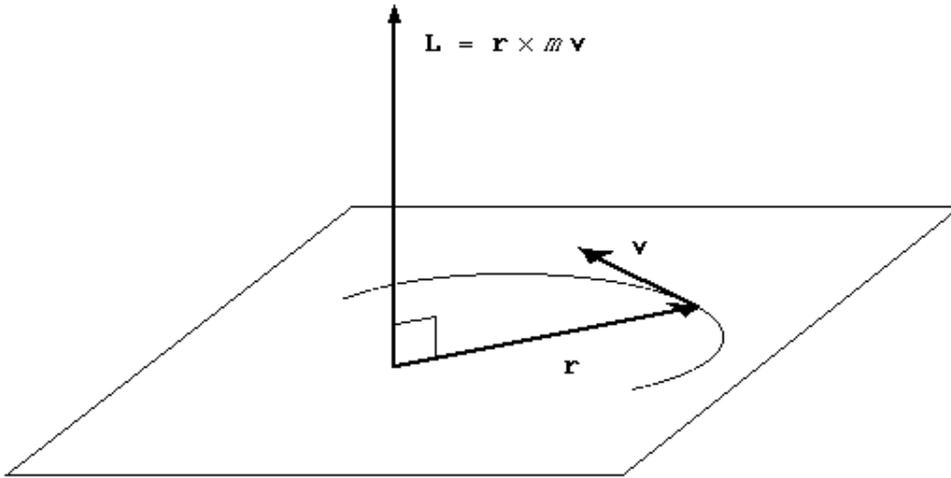


Figure 1.1: The plane in which central motion takes place is perpendicular to the (constant) angular momentum vector \mathbf{L} .

This equation, which tells us that the time derivative of the angular momentum is given by the moment, is sometimes called the principle of angular momentum. Just as the ordinary equation of motion $\dot{\mathbf{p}} = \mathbf{F}$ tells us that $\mathbf{p} = \text{const.}$ when $\mathbf{F} = \mathbf{0}$, this equation gives the conservation law

$$\mathbf{L} = \text{const. when } \mathbf{M} = \mathbf{0}. \quad (1.5)$$

When $\mathbf{F} = \mathbf{0}$ then, trivially, $\mathbf{M} = \mathbf{0}$ and this does not give anything very interesting. But there *are* situations when $\mathbf{M} = \mathbf{0}$ even though the force is non-zero and it is under these circumstances that this is a useful result.

Assume that a particle is affected by a *central* force field, i.e. a force field such that the vectors $\mathbf{F}(\mathbf{r})$, for all values of \mathbf{r} , point along lines going through a single common point, the center of the force field. If this center is chosen as the origin, the force field can be expressed as follows using spherical coordinates

$$\mathbf{F}(\mathbf{r}) = F_r(\mathbf{r}) \mathbf{e}_r. \quad (1.6)$$

For such a field the moment with respect to the origin will always be zero:

$$\mathbf{M} = \mathbf{r} \times F_r(\mathbf{r}) \mathbf{e}_r = r \mathbf{e}_r \times F_r \mathbf{e}_r = \mathbf{0}. \quad (1.7)$$

A particle moving in such a field will thus have a constant angular momentum vector \mathbf{L} . The value of this vector can be found from initial values of position vector and momentum: $\mathbf{L} = \mathbf{r}(0) \times m\mathbf{v}(0)$ and one important conclusion that we immediately can draw from the equation

$$\mathbf{r}(t) \times m\mathbf{v}(t) = \text{const.} = \mathbf{L} \quad (1.8)$$

is that both the position vector $\mathbf{r}(t)$ and the velocity $\mathbf{v}(t)$ always *lie in a plane* (see Figure (1.1)) perpendicular to \mathbf{L} . A particle in a central force field thus moves in a plane.

The *direction* of the angular momentum vector \mathbf{L} gives the orientation of the plane in which the motion takes place, but how can one interpret the constancy of the *length* $L = |\mathbf{L}|$ of this vector? Since the motion is planar we can choose this plane as the xy -plane and study the motion using cylindrical coordinates ρ, φ (in the plane these coincide with the spherical coordinates r, φ and therefore one often sees r used instead of ρ in other treatments). The angular momentum vector expanded along the basis vectors belonging to cylindrical coordinates becomes

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = \rho \mathbf{e}_\rho \times m(\dot{\rho} \mathbf{e}_\rho + \rho \dot{\varphi} \mathbf{e}_\varphi) = m\rho^2 \dot{\varphi} \mathbf{e}_z \quad (1.9)$$

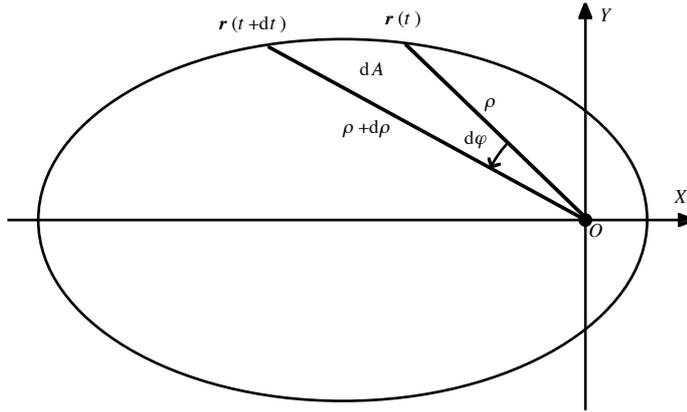


Figure 1.2: The area swept over by the position vector \mathbf{r} (with origin at the center of force) in time dt is given by $dA = \frac{1}{2}\rho^2 d\varphi$ when dt is small. The sectorial (or area) velocity $\dot{A} = \frac{dA}{dt}$ = constant for motion in a central force field.

so the magnitude of \mathbf{L} is

$$L = m\rho^2\dot{\varphi}. \quad (1.10)$$

In a central force field, which can be written $\mathbf{F} = F_\rho \mathbf{e}_\rho$ when restricted to the plane of motion, this quantity, $m\rho^2\dot{\varphi}$ is thus constant. This has a geometric interpretation: the area dA traversed by the position vector \mathbf{r} in time dt is given by $dA = \frac{1}{2}\rho^2 d\varphi$, see Figure (1.2). and thus the area traversed (swept over) in unit time, the *sectorial velocity* \dot{A} , is given by

$$\dot{A} = \frac{1}{2}\rho^2\dot{\varphi} \quad (1.11)$$

When we compare we see that we have

$$L = 2m\dot{A} = \text{const.}, \quad (1.12)$$

and for central motion thus the constancy of the magnitude of the angular momentum is equivalent to the constancy of the sectorial velocity.

To study the motion in a central force field one can write down the equations of motion in terms of cylindrical coordinates in the plane of the motion. These become

$$m(\ddot{\rho} - \rho\dot{\varphi}^2) = F_\rho \quad (1.13)$$

$$m(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) = 0. \quad (1.14)$$

The second of these equations, which results from the fact that $F_\varphi = 0$ for a central force, is now easily seen to be equivalent to the equation $\dot{L} = m(\rho^2\ddot{\varphi} + 2\rho\dot{\rho}\dot{\varphi}) = 0$ after multiplication with ρ . This means that the conservation of angular momentum L is a *first integral* to the equation (1.14).

Example 1 A particle of mass m is confined to move on a smooth horizontal plane. It is attached to an inelastic string of negligible mass which goes through a small smooth hole in the plane. Under the plane it is kept still by the application of a suitable force F , see Figure 1.3. Assume that the particle moves in a circle of radius r_1 with angular velocity ω_1 . Find the angular velocity ω if the radius is changed to r .

Solution: Since the force from the string on the particle is directed towards the hole it is a central force. The angular momentum L of the particle with respect to the hole is thus conserved. This gives us

$$r_1 m r_1 \omega_1 = r m r \omega \quad (1.15)$$

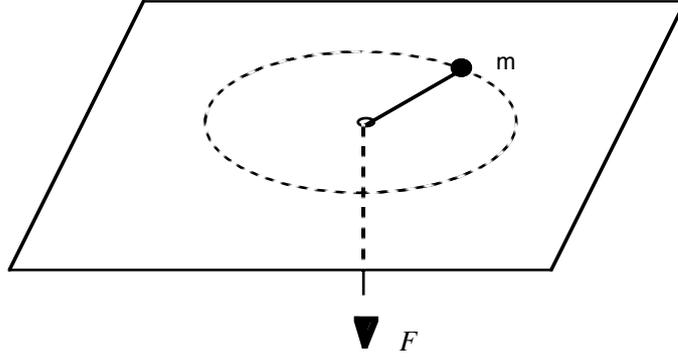


Figure 1.3: This figure refers to Example 1 where the angular velocity of the particle in its circular motion as a function of the radius of the circle is calculated.

since the momentum, \mathbf{p} , of the particle has magnitude $p = mr\omega$, and is perpendicular to the position vector of length r . This gives the answer

$$\omega = \omega_1 \left(\frac{r_1}{r} \right)^2 \quad (1.16)$$

so that the angular velocity goes as the inverse square of the radius. \triangle

Up till now we have only assumed that $\mathbf{F} = F_r(\mathbf{r}) \mathbf{e}_r$. In what follows we shall consider the more restricted case that $F_r(\mathbf{r}) = f(r)$, i.e. the case when the force only depends on the distance from the origin. This is the type of force that acts on a particle from a gravitating mass or an electric charge at the origin. One problem is that the particle at the origin won't be at rest unless fixed by external forces and therefore it is not obvious that the force field really is a central force field in practice. This problem is solved in the next section.

1.2 The Two-particle Problem

Consider two particles with masses m_1 and m_2 and position vectors \mathbf{r}_1 and \mathbf{r}_2 respectively. We assume that the only forces affecting the particles are the forces with which they act upon each other: \mathbf{F}_{21} from particle number 2 on number 1, and \mathbf{F}_{12} from 1 on 2. The equations of motion are then

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad (1.17)$$

$$m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12}. \quad (1.18)$$

According to Newton's third law we have $\mathbf{F}_{21} = -\mathbf{F}_{12}$ so if we add the two equations of motion we get

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{0}. \quad (1.19)$$

With the help of the center of mass vector $\mathbf{R} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) / (m_1 + m_2)$ and the notation $M = m_1 + m_2$ this is easily seen to give

$$M \ddot{\mathbf{R}} = \mathbf{0}. \quad (1.20)$$

This tells us that there is no force on the center of mass which thus stays at rest or moves with constant velocity depending on the inertial reference frame.

Since we obtained something interesting from the sum of the equations, let us try the difference. We first divide equation (1.17) with m_1 and equation eq.two.part.two.eq.of.motion

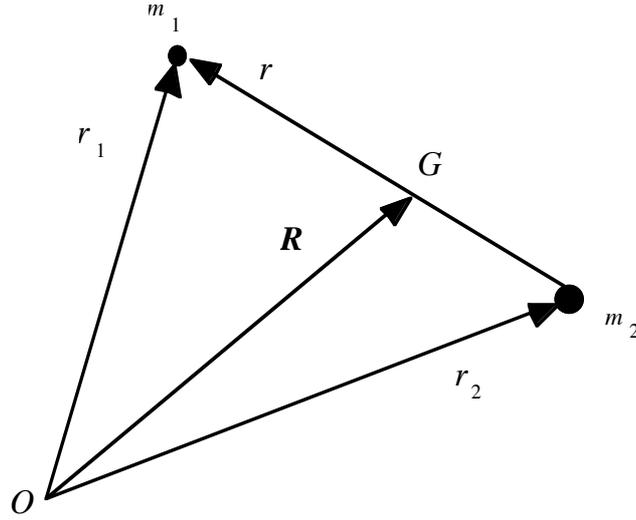


Figure 1.4: This figure shows geometrically the coordinate transformation from the position vectors \mathbf{r}_1 and \mathbf{r}_2 to the center of mass vector \mathbf{R} and the relative vector \mathbf{r} .

with m_2 and then subtract the resulting equation for particle 2 from that for 1. This gives

$$(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{F}_{21} \quad (1.21)$$

so, if we put $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ and $\mu \equiv \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1}$, we can write

$$\mu \ddot{\mathbf{r}} = \mathbf{F}_{21}. \quad (1.22)$$

This now tells us that the position vector of particle 1 relative to particle 2 obeys an equation of motion in which the so called *reduced mass* μ plays the role of (inertial) mass and the force is the force on particle 1 from particle 2.

In summary we have thus found that the coordinate transformation, see Figure 1.4,

$$\begin{aligned} \mathbf{R} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} & \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M} \mathbf{r} & M &= m_1 + m_2 \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 & \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M} \mathbf{r} & \mu &= \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1} \end{aligned} \quad (1.23)$$

turns the ordinary equations of motion for the particles 1 and 2 into the two equations

$$M \ddot{\mathbf{R}} = \mathbf{0}, \quad (1.24)$$

$$\mu \ddot{\mathbf{r}} = \mathbf{F}_{21}. \quad (1.25)$$

Here the force is simply the force from particle 2 on particle 1 but the mass is the so called reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$.

This shows that one formally can treat the two particle problem as a one particle problem. The fact that it is the center of mass which is at rest and not particle 2 is taken care of by the introduction of the reduced mass.

Example 2 Calculate the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ for the two-particle system in terms of the center of mass and relative position vectors and the corresponding time derivatives $\dot{\mathbf{R}}$ and $\dot{\mathbf{r}}$.

Solution: We have

$$\mathbf{L} = \mathbf{r}_1 \times m_1 \mathbf{v}_1 + \mathbf{r}_2 \times m_2 \mathbf{v}_2 \quad (1.26)$$

so by means of equations (1.23) we get

$$\mathbf{L} = \left(\mathbf{R} + \frac{m_2}{M}\mathbf{r}\right) \times m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right) + \left(\mathbf{R} - \frac{m_1}{M}\mathbf{r}\right) \times m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right). \quad (1.27)$$

Applying the distributive law to the above vector products gives eight terms. Four of these terms cancel each other pairwise while the remaining four can be added together pairwise so that one gets

$$\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{r} \times \mu\dot{\mathbf{r}}. \quad (1.28)$$

In the center of mass coordinate system $\dot{\mathbf{R}} = \mathbf{0}$ and thus $\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}$ so that $\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F}_{21}$. This means that, if $\mathbf{r} \parallel \mathbf{F}_{21}$, then $\dot{\mathbf{L}} = \mathbf{0}$ and therefore $\mathbf{L} = \text{const.}$ also in this case. \triangle

1.3 Energy Conservation for a Central Force

In case the magnitude of the central force only depends on the distance from the center: $\mathbf{F}(\mathbf{r}) = F_r(r)\mathbf{e}_r$, one can show that the force is conservative and that the potential energy can be taken to be

$$\Phi(r) = - \int^r F_r(r) dr. \quad (1.29)$$

We'll now study the motion of a particle in this force field and since we know that the motion will be confined to a plane we choose it as the xy -plane so that $r = \rho$ and we use cylindrical coordinates in this plane. The law of conservation of energy can then be written

$$\frac{1}{2}m\mathbf{v}^2 + \Phi(\rho) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) + \Phi(\rho) = E. \quad (1.30)$$

If we now use equation (1.10), i. e. $L = m\rho^2\dot{\varphi}$, and the fact that L is constant, we can rewrite the conservation of energy equation entirely in terms of ρ and its time-derivative as follows

$$\frac{1}{2}m\left(\dot{\rho}^2 + \frac{L^2}{m^2\rho^2}\right) + \Phi(\rho) = E. \quad (1.31)$$

From this equation one can draw many useful conclusions about the radial motion of the particle in the central force field. If one formally introduces the so called *effective potential* energy function

$$\Phi_{\text{eff}}(\rho) = \frac{L^2}{2m\rho^2} + \Phi(\rho) \quad (1.32)$$

energy conservation in terms of the radial ρ -motion will look like a one-dimensional energy conservation:

$$\frac{1}{2}m\dot{\rho}^2 + \Phi_{\text{eff}}(\rho) = E. \quad (1.33)$$

Now assume that the force is of the form (in the plane of the motion)

$$F_\rho(\rho) = -\frac{K}{\rho^2}. \quad (1.34)$$

In the case of gravity, $K = Gm_1m_2$ and is always positive. In the case of charged particles, $K = -Q_1Q_2/(4\pi\epsilon_0)$ and is positive (attraction) when the charges have opposite sign, otherwise it is negative and the force is repulsive. The potential energy is then

$$\Phi(\rho) = -\frac{K}{\rho} \quad (1.35)$$

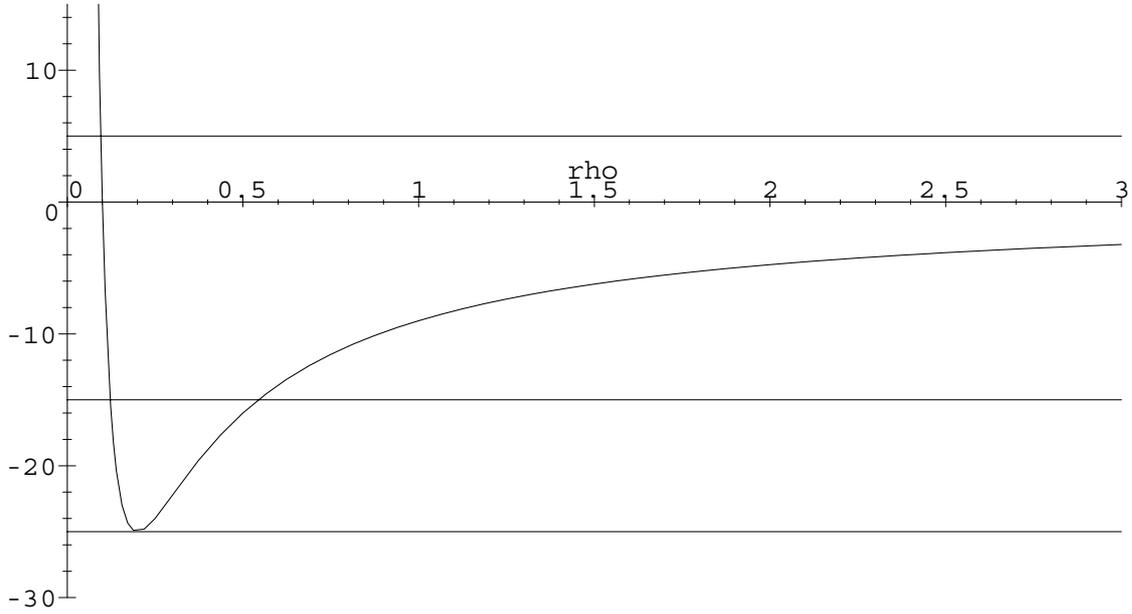


Figure 1.5: Plot of the function $\Phi_{\text{eff}} = \frac{L^2}{2m} \frac{1}{\rho^2} - \frac{K}{\rho} = \frac{1}{\bar{\rho}^2} - \frac{\alpha}{\bar{\rho}}$ where $\bar{\rho} \equiv \frac{\sqrt{2m}}{L} \rho$ and $\alpha \equiv K \frac{\sqrt{2m}}{L}$ for the value $\alpha = 10$. Also shown are four horizontal lines at $E = -25, -15, 0, 5$. The minimum of the curve, which corresponds to circular orbits, is at $\bar{\rho} = \frac{2}{\alpha}$ and the value of Φ_{eff} at that minimum is $E_{\text{min}} = -\frac{\alpha^2}{4} = -\frac{K^2 m}{2L^2}$. With $\alpha = 10$ this gives a minimum at -25 . For higher, negative, energies E there are two roots of the equation $\Phi_{\text{eff}}(\rho) = E$ which correspond to the maximum (ρ_-) and minimum (ρ_+) values of ρ , the ‘turning’-points, in the elliptic trajectory. $E \geq 0$ gives only one turningpoint, $E = 0$ corresponding to a parabola and $E > 0$ to hyperbolas.

and the effective potential energy

$$\Phi_{\text{eff}}(\rho) = \frac{L^2}{2m\rho^2} - \frac{K}{\rho}, \quad (1.36)$$

see Figure 1.5.

One useful application of this equation is to find the *turning points* in the radial motion. These correspond to the maximum and minimum values of ρ and they thus correspond to zero radial velocity $\dot{\rho} = 0$. Equation (1.33) now gives the following equation for the ρ -values of these turning-points:

$$\frac{L^2}{2m\rho^2} - \frac{K}{\rho} = E. \quad (1.37)$$

This quadratic equation is easily solved and the roots are given by

$$\frac{1}{\rho_{\pm}} = \frac{mK}{L^2} \pm \sqrt{\left(\frac{mK}{L^2}\right)^2 + \frac{2mE}{L^2}}. \quad (1.38)$$

Note that the energy E may be negative here. The smallest physically allowed value of the energy is that which makes the expression under the square root zero:

$$E_{\min} = - \left(\frac{mK}{L^2} \right)^2 \frac{L^2}{2m} = - \frac{mK^2}{2L^2} \quad (1.39)$$

For this minimum energy the ρ -value of the trajectory must be given by the constant $\rho = \frac{L^2}{mK}$ and the trajectory is thus a circle with this radius. For smaller energies the turning-points become imaginary and this is unphysical. Note also that since by definition $\rho \geq 0$ any negative turning-points must be discarded. One easily sees from equation (1.38) that if the energy E is positive, or if $K < 0$ (repulsive force), then the ‘-’-root is negative and must be discarded. The radial motion then only has a minimum ρ -value and the trajectory extends to infinity. This type of motion is called unbound. When the motion lies between two turning-points, it is called bound and this can only happen when the force is attractive $K > 0$ and the energy is negative $E < 0$.

Example 3 (Solution of Central Motion by Direct Integration)

Eliminate the time differential dt from the two conservation laws $L = \text{const.}$ and $E = \text{const.}$ which hold for motion of a particle in a conservative central force field, and thus find a relationship between the differentials $d\varphi$ and $d\rho$ on the form $d\varphi = h(\rho)d\rho$. Integrate this to get $\varphi - \varphi_0 = \int_{\rho_0}^{\rho} h(\rho)d\rho$. Do the integral in the (Kepler) case $\Phi(\rho) = -K/\rho$.

Solution: From equation (1.33) we get

$$\frac{d\rho}{dt} = \sqrt{\frac{2}{m}[E - \Phi_{\text{eff}}(\rho)]} \quad (1.40)$$

so

$$dt = \frac{d\rho}{\sqrt{\frac{2}{m}[E - \Phi_{\text{eff}}(\rho)]}}. \quad (1.41)$$

From $\dot{\varphi} = \frac{L}{m\rho^2}$ we get

$$dt = \frac{m\rho^2}{L}d\varphi \quad (1.42)$$

and equating these expressions for dt results in

$$d\varphi = \frac{L d\rho/\rho^2}{\sqrt{2m[E - \Phi_{\text{eff}}(\rho)]}} \quad (1.43)$$

and this is the desired relationship.

If we insert $\Phi_{\text{eff}}(\rho) = -\frac{K}{\rho} + \frac{L^2}{2m^2\rho^2}$ in this we get

$$d\varphi = \frac{L d\rho/\rho^2}{\sqrt{2m[E + \frac{K}{\rho}] - \frac{L^2}{\rho^2}}} = \frac{d\rho/\rho^2}{\sqrt{\frac{2mE}{L^2} + \frac{2mK}{L^2\rho} - \frac{1}{\rho^2}}} = \frac{d\rho/\rho^2}{\sqrt{-\left(\frac{1}{\rho_+} - \frac{1}{\rho}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)}}. \quad (1.44)$$

Here we have rewritten the expression under the square root using the roots given in equation (1.38). If we now introduce the new integration variable $u = 1/\rho$, so that $du = -d\rho/\rho^2$, this becomes

$$d\varphi = \frac{-du}{\sqrt{-(u_+ - u)(u - u_-)}}. \quad (1.45)$$

where, of course, u_{\pm} correspond to ρ_{\pm} . Finally we change the integration variable to $\chi = u - \frac{1}{2}(u_- + u_+)$ and get

$$d\varphi = \frac{-d\chi}{\sqrt{\chi_+^2 - \chi^2}}. \quad (1.46)$$

We have now turned our integrand into a form that makes the integration elementary. If we choose the direction of the X -axis so that $\varphi = 0$ for $\rho = \rho_+$ (i.e. when the particle is closest to the center of force) we get

$$\varphi = \int_{\rho_+}^{\rho} \frac{L d\rho/\rho^2}{\sqrt{2m[E + \frac{K}{\rho}] - \frac{L^2}{\rho^2}}} = \int_{\chi_+}^{\chi} \frac{-d\chi}{\sqrt{\chi_+^2 - \chi^2}}. \quad (1.47)$$

The integrand is the derivative of $\arccos \frac{\chi}{\chi_+}$ so, since $\arccos 1 = 0$, we get

$$\varphi = \arccos \frac{\chi}{\chi_+} = \arccos \left(\frac{\frac{1}{\rho} - \frac{1}{2}(\frac{1}{\rho_-} + \frac{1}{\rho_+})}{\frac{1}{2}(\frac{1}{\rho_+} - \frac{1}{\rho_-})} \right) = \arccos \left(\frac{\frac{L}{\rho} - \frac{mK}{L}}{\sqrt{2mE + \frac{m^2 K^2}{L^2}}} \right). \quad (1.48)$$

If $K > 0$ (attractive force) and we put

$$p = \frac{L^2}{mK} \quad \text{and} \quad e = \sqrt{1 + \frac{2EL^2}{mK^2}} \quad (1.49)$$

this can be rewritten on the form

$$\rho = \frac{p}{1 + e \cos \varphi}. \quad (1.50)$$

Should $K < 0$ (repulsive force) one instead puts $p = \frac{L^2}{m|K|}$ and finds

$$\rho = \frac{p}{-1 + e \cos \varphi}. \quad (1.51)$$

In the next section we will derive these results in a different way and discuss their interpretation. \triangle

1.4 The Kepler Problem

We have seen above that the problem of the radial motion, in the inverse square central force field, $F_\rho = -K/\rho^2$, can be solved using conservation of energy, E , and angular momentum, L . Physically this inverse square force model describes the motion of a moon around a planet or a planet around a star or the relative motion of two charged particles interacting via their electrostatic fields. This is thus an important physical problem and we should try to solve it also for the angular motion.

Let us write down the equations of motion in terms of cylindrical coordinates:

$$m(\ddot{\rho} - \rho\dot{\varphi}^2) = F_\rho, \quad (1.52)$$

$$m(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) = 0. \quad (1.53)$$

Our task is now to solve these coupled equations and, preferably, find the trajectory on the parametric form $(\rho(t), \varphi(t))$. Unfortunately this problem is quite difficult and is best approached numerically. It turns out, however, that if we *eliminate the time* and only seek the shape of the trajectory as a relationship between ρ and φ , perhaps on the form $\rho = f(\varphi)$, then the problem can be treated analytically. The problem is often referred to as the Kepler problem since it was Johannes Kepler who empirically found the shape of the planetary trajectories to be ellipses.

We already know that the second of the two equations (1.52) has a first integral: angular momentum $L = m\rho^2\dot{\varphi}$. If we use this to eliminate $\dot{\varphi}$ in the ρ -equation it becomes

$$m\ddot{\rho} = F_\rho + \frac{L^2}{m\rho^3}. \quad (1.54)$$

In order to eliminate the time-derivatives in this equation we can again use the conservation of angular momentum on the form

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{L}{m\rho^2} \quad (1.55)$$

since this makes it possible to transform from time-derivatives to φ -derivatives using

$$\frac{d}{dt} = \frac{d\varphi}{dt} \frac{d}{d\varphi} = \frac{L}{m\rho^2} \frac{d}{d\varphi}. \quad (1.56)$$

We now get

$$\dot{\rho} = \frac{L}{m\rho^2} \frac{d\rho}{d\varphi} \quad (1.57)$$

and, by applying the same operator again,

$$\ddot{\rho} = \frac{L}{m\rho^2} \frac{d}{d\varphi} \left(\frac{L}{m\rho^2} \frac{d\rho}{d\varphi} \right) = \frac{L^2}{m^2} \frac{1}{\rho^2} \frac{d}{d\varphi} \left(\frac{1}{\rho^2} \frac{d\rho}{d\varphi} \right). \quad (1.58)$$

Instead of differentiating the parenthesis in the last equation directly we now take advantage of the fact that

$$\frac{1}{\rho^2} \frac{d\rho}{d\varphi} = -\frac{d}{d\varphi} \frac{1}{\rho}. \quad (1.59)$$

If we therefore introduce the new variable

$$u = \frac{1}{\rho} \quad (1.60)$$

we simplify things considerably. We get

$$\ddot{\rho} = -\frac{L^2}{m^2} u^2 \frac{d^2 u}{d\varphi^2}. \quad (1.61)$$

Our equation of motion (1.54) now becomes (this result is usually referred to as *Binet's formula*)

$$-m \frac{L^2}{m^2} u^2 \left(\frac{d^2 u}{d\varphi^2} + u \right) = F_\rho. \quad (1.62)$$

We now see that for the case of the inverse square force the left hand side, $F_\rho = -K/\rho^2 = -Ku^2$, so after division by u^2 it becomes a constant. The resulting differential equation is

$$\frac{d^2 u}{d\varphi^2} + u = \frac{mK}{L^2}. \quad (1.63)$$

We can solve this differential equation using the methods from the chapter on linear oscillations. The general solution is given by

$$u(\varphi) = A \cos(\varphi + \alpha) + \frac{mK}{L^2}. \quad (1.64)$$

The value of the angle φ will depend on the choice of the direction of the x -axis in the plane of the motion. We can therefore decide that we choose this direction so that $\alpha = 0$. If we now introduce $\rho = 1/u$ again and assume $K > 0$ (attractive force) we get from this

$$\rho = \frac{p}{1 + e \cos \varphi} \quad (1.65)$$

where

$$p = \frac{L^2}{mK} \quad \text{and,} \quad e = Ap. \quad (1.66)$$

$2p$ is called the *latus rectum* of the curve and e the *eccentricity*. Should $K < 0$ (repulsive force), p and e as defined here become negative. They are then redefined to be positive, i.e. one puts

$$p = \frac{L^2}{m|K|} \quad \text{when} \quad K < 0. \quad (1.67)$$

and the expression for the trajectory is changed to

$$\rho = \frac{p}{-1 + e \cos \varphi}. \quad (1.68)$$

(Compare with the results in Example 3.) These equations thus constitute the desired relationship between ρ and φ which gives the shape of the trajectory. It turns out that this shape is a so called ‘conic section’ (quadratic curve) and these are, for different values of the *eccentricity* $e(> 0)$,

$e = 0$	circle	$E = -\frac{mK^2}{2L^2}$	(1.69)
$0 \leq e < 1$	ellipse	$E < 0$	
$e = 1$	parabola	$E = 0$	
$1 < e$	hyperbola	$E > 0$	

An ellipse is shown in Figure 1.6 and some of the quantities introduced above are indicated.

1.4.1 Energy in the Kepler Problem

In equations (1.69) we have also indicated the corresponding value for the energy E . Starting from equation (1.49) it is easy to show that

$$E = \frac{1}{2}m \frac{K^2}{L^2}(e^2 - 1). \quad (1.70)$$

This equation can, however, also be found directly by inserting the trajectory, equation (1.65), into the energy expression as we will now show.

The energy, E , of the Kepler problem can be written on one of the following forms

$$E = \frac{1}{2}mv^2 - \frac{K}{\rho} = \frac{1}{2}m\dot{\rho}^2 + \Phi_{\text{eff}}(\rho) = \frac{1}{2}m\dot{\rho}^2 + \frac{L^2}{2m\rho^2} - \frac{K}{\rho}. \quad (1.71)$$

In the last of these forms we now insert the result (1.65). In order to do this we need the expression (1.57) which using (1.65) gives

$$\dot{\rho} = \frac{L}{m\rho^2} \frac{d\rho}{d\varphi} = \frac{L}{m\rho^2} \frac{pe \sin \varphi}{(1 + e \cos \varphi)^2} = \frac{L}{m} \frac{e}{p} \sin \varphi. \quad (1.72)$$

Using this our energy expression now gives

$$E = \frac{1}{2}m \left(\frac{L}{m} \frac{e}{p} \sin \varphi \right)^2 + \frac{L^2}{2m} \frac{(1 + e \cos \varphi)^2}{p^2} - \frac{K(1 + e \cos \varphi)}{p}. \quad (1.73)$$

Some algebra and use of $p = \frac{L^2}{mK}$ shows that this reduces to equation (1.70) as was to be shown. We have done the derivation assuming $K > 0$ but the same formula is obtained also in the case $K < 0$ in a similar way.

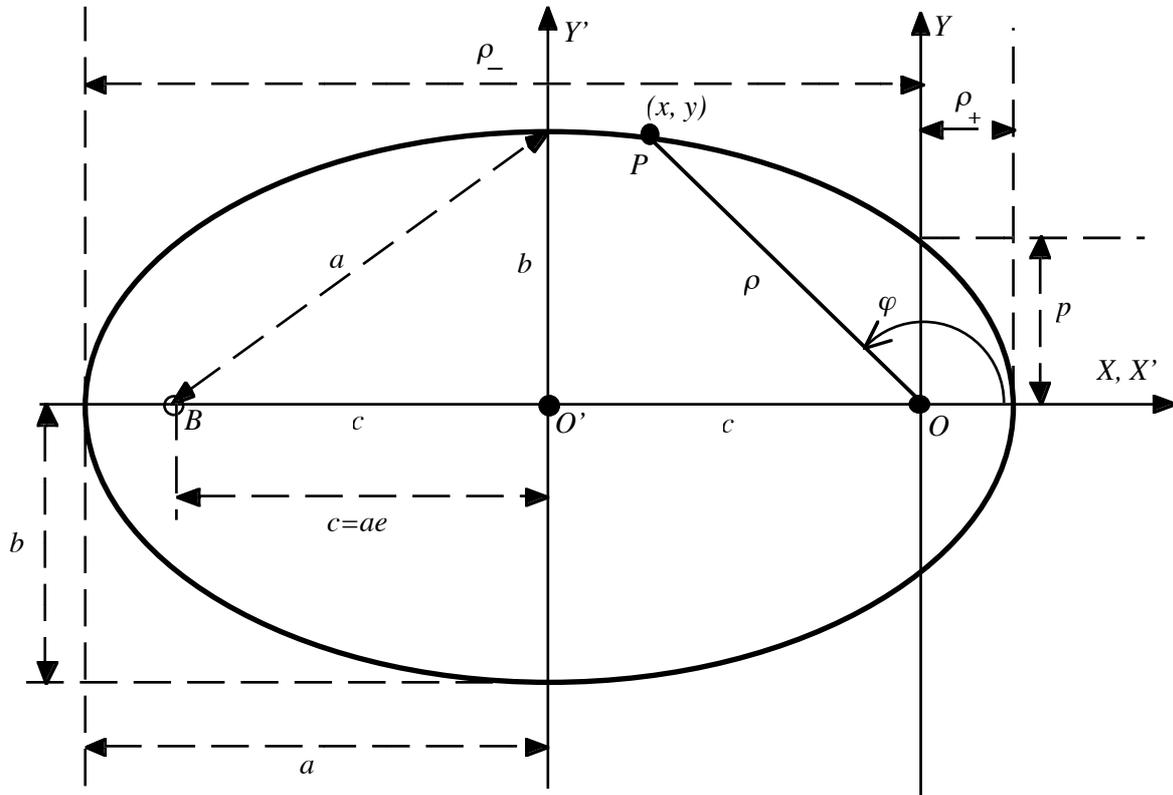


Figure 1.6: **Notation and relationships for the ellipse**

This figure shows the ellipse

$$(x'/a)^2 + (y'/b)^2 = 1,$$

with center at the origin \mathcal{O}' . The quantities a and b are called the major and minor semi-axis respectively. The equation for the ellipse can also be written

$$\rho = p/(1 + e \cos \varphi)$$

in terms of cylindrical coordinates ($x = \rho \cos \varphi, y = \rho \sin \varphi$) when the origin is chosen at \mathcal{O} , one of the foci of the ellipse. The other focus is at \mathcal{B} . The distance between the focus \mathcal{O} and the center of the ellipse \mathcal{O}' is denoted c and one has $c = ae$ where e is the eccentricity. In terms of the Cartesian coordinates x, y , with $\rho = \sqrt{x^2 + y^2}$, one has

$$[(x + c)/a]^2 + (y/b)^2 = 1.$$

The 'semi-latus rectum' p , which is the y -coordinate when $x = 0$, obeys $a = p/(1 - e^2)$ and $b = p/\sqrt{1 - e^2}$. Note also the relations $2a = \rho_- + \rho_+$, $2c = \rho_- - \rho_+$ and $b^2 = a^2 - c^2 = \rho_- \rho_+$.

Example 4 Show that the speed v of a particle moving in an ellipse with major semi axis a in the field from a central body with mass m_c is given by

$$v = \sqrt{Gm_c \left(\frac{2}{\rho} - \frac{1}{a} \right)} \quad (1.74)$$

where ρ is the distance from the center of force.

Solution: We start from formula (1.70) and take advantage of the result (1.66) to eliminate L :

$$E = \frac{1}{2} \frac{K}{p} (e^2 - 1) = -\frac{K}{2} \frac{(1 - e^2)}{p} = -\frac{K}{2a}. \quad (1.75)$$

Here we also have used $a = p/(1 - e^2)$ which is shown under Figure 1.6. We can now write the energy conservation law as follows

$$-\frac{K}{2a} = \frac{1}{2}mv^2 - \frac{K}{\rho} \quad (1.76)$$

If this is solved for v and use is made of $K = Gmm_c$ we obtain the desired result. \triangle

1.5 The Simplest Solution to the Kepler Problem

Usually the solution to the Kepler problem of planetary motion is found by means of either direct integration starting from the constants of motion L and E (angular momentum and energy) as in Example 3 or by eliminating the time derivative from the problem using the so called Binet's formula as in the previous section. Both methods lead to fairly cumbersome calculations.

Here I wish to draw attention to a recently published (González-Villanueva et al. 1996) very short and simple path to the solution. The derivation is based on cylindrical coordinates. It starts from the equation for angular momentum,

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\rho^2\dot{\varphi}\mathbf{e}_z \quad \Rightarrow \quad \dot{\varphi} = \frac{L}{m\rho^2}, \quad (1.77)$$

the equation of motion,

$$m\dot{\mathbf{v}} = -\frac{K}{\rho^2}\mathbf{e}_\rho, \quad (1.78)$$

and the relation for the (moving) cylindrical basis vectors,

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi}\mathbf{e}_\rho. \quad (1.79)$$

Using (1.77) to eliminate $\dot{\varphi}$ in this equation we get

$$\dot{\mathbf{e}}_\varphi = -\frac{L}{m\rho^2}\mathbf{e}_\rho. \quad (1.80)$$

and thus the equation of motion (1.78) immediately gives

$$\dot{\mathbf{v}} = \frac{K}{L}\dot{\mathbf{e}}_\varphi. \quad (1.81)$$

This equation can be integrated directly and if we denote by \mathbf{w} the vector of integration constants we find

$$\mathbf{v} = \mathbf{w} + \frac{K}{L}\mathbf{e}_\varphi. \quad (1.82)$$

This equation shows that the hodograph (trajectory in velocity space) is a circle about the tip of \mathbf{w} with radius K/L .

We now take the scalar product of this equation with \mathbf{e}_φ and use that fact that $\mathbf{v} \cdot \mathbf{e}_\varphi = \rho\dot{\varphi}$ is the φ -component of the velocity. If we choose the y -axis to point along the vector \mathbf{w} we get

$$\rho\dot{\varphi} = w \cos \varphi + \frac{K}{L}. \quad (1.83)$$

Again using (1.77) to eliminate $\dot{\varphi}$ in this equation we get

$$\frac{L}{m\rho} = w \cos \varphi + \frac{K}{L} \quad (1.84)$$

and this can be rewritten

$$\frac{1}{\rho} = \frac{1}{p}(1 + e \cos \varphi), \quad (1.85)$$

where

$$p = \frac{L^2}{mK} \quad \text{and} \quad e = \frac{wL}{K} \quad (1.86)$$

are constants. This is the equation for an ellipse of eccentricity e expressed in cylindrical (polar) coordinates. p is the value of ρ , i.e. the length of the radius vector from one focus, at $\varphi = \pi/2$, and is called the semi-latus rectum.

One also easily gets the energy out of the constancy of the length of the vector \mathbf{w} . Equation (1.82) gives

$$\mathbf{w} \cdot \mathbf{w} = v^2 + \left(\frac{K}{L}\right)^2 - 2\frac{K}{L}\rho\dot{\varphi} \quad (1.87)$$

which, using (1.77), gives

$$v^2 - \frac{2K}{m\rho} = w^2 - \left(\frac{K}{L}\right)^2. \quad (1.88)$$

Thus, eliminating w using equation (1.86), we find

$$E = \frac{m}{2}v^2 - \frac{K}{\rho} = \frac{m}{2} \left[w^2 - \left(\frac{K}{L}\right)^2 \right] = \frac{m}{2} \left(\frac{K}{L}\right)^2 (e^2 - 1) = \text{constant}. \quad (1.89)$$

This thus also gives the energy in terms of the eccentricity fairly easily. Compare with the derivation of equation (1.70) in the previous section.

In conclusion I find that this method for treating the Kepler problem gives the hodograph, the ellipse in polar coordinates, and the energy, easily and with a minimum of algebra. It also does this without introducing new concepts or mathematics. Therefore this method should be the preferred method in undergraduate teaching.

1.6 The Field from Extended Bodies

The potential energy of a particle of mass m at \mathbf{r} in the gravitational field from a point mass m' at the origin has been shown to be $\Phi(r) = -m\frac{Gm'}{r}$ where $r = |\mathbf{r}|$. Should the mass m' be at some other point than the origin, at \mathbf{r}' say, the potential energy is instead

$$\Phi(\mathbf{r}) = -m\frac{Gm'}{|\mathbf{r} - \mathbf{r}'|} \quad (1.90)$$

and the force on m can be obtained in the usual way as the negative gradient of this.

Now assume that we have N particles with masses m_i and position vectors \mathbf{r}_i where $i = 1, \dots, N$. The potential energy of the force field, acting on m , from these will then be the sum of the potential energies from the individual sources

$$\Phi(\mathbf{r}) = -m \sum_{i=1}^N \frac{Gm_i}{|\mathbf{r} - \mathbf{r}_i|}. \quad (1.91)$$

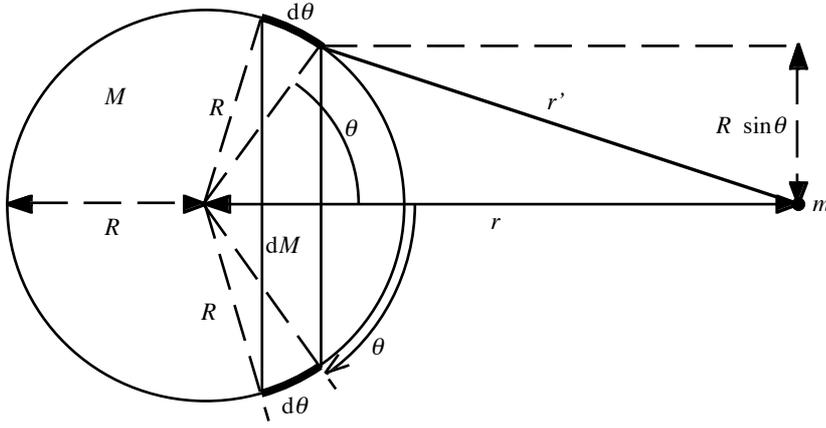


Figure 1.7: This figure shows the integration variables used when finding the potential energy due to a spherical shell of mass M for a particle of mass m at a distance r from the center of the shell.

The total force will then be the vector sum of the forces from the individual particles. Should the number N be too large for it to be meaningful to keep track of the individual particles one instead considers a mass density $\rho_m(\mathbf{r})$ which, as usual, gives the mass dm in the volume element dV at \mathbf{r} as

$$dm = \rho_m(\mathbf{r})dV. \quad (1.92)$$

The potential energy sum will then have to be replaced by the volume integral

$$\Phi(\mathbf{r}) = -m \int \frac{G\rho_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (1.93)$$

For arbitrary mass distributions this integral must be calculated on computer with numerical techniques. Only when the mass distribution has some high symmetry can the integral be calculated analytically. The gravitational force, \mathbf{F} , on the particle of mass m at \mathbf{r} is

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \quad (1.94)$$

from the mass distribution ρ_m of equation (1.93).

1.6.1 The Field Outside a Spherically Symmetric Body

An important case in which the potential energy of a mass distribution ρ_m can be calculated analytically is the case of a homogeneous spherical shell. Let us do this.

Assume that the spherical shell has radius R and total mass M . The mass per unit area, σ , on the shell then satisfies

$$M = 4\pi R^2\sigma. \quad (1.95)$$

We now calculate the potential energy at a distance r from the center of the sphere. To do this we choose coordinates according to Figure 1.7. The potential energy $d\Phi$ from the ring of radius $R\sin\theta$ of angular thickness $d\theta$ at the distance r' from the mass m for which we determine the potential energy, is

$$d\Phi = -m \frac{GdM}{r'} = -m \frac{G\sigma dA}{r'} = -m \frac{G\sigma(2\pi R \sin\theta)R d\theta}{r'}. \quad (1.96)$$

Here $dA = (2\pi R \sin\theta)R d\theta$ is the area of the ring. The total potential energy is now obtained by integrating this expression over all values of the angle θ , i. e. over the whole shell, and this gives

$$\Phi(r) = -m \int_0^\pi \frac{G\sigma(2\pi R \sin\theta)R d\theta}{r'}. \quad (1.97)$$

If we now use that the distance $r' = \sqrt{r^2 + R^2 - 2Rr \cos \theta}$ we find

$$\Phi(r) = -m \frac{GM}{2} \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{r^2 + R^2 - 2Rr \cos \theta}}. \quad (1.98)$$

The primitive function of the integrand here is $\frac{\sqrt{r^2 + R^2 - 2Rr \cos \theta}}{Rr}$ and we finally find

$$\Phi(r) = -m \frac{GM}{2Rr} [(r + R) - |r - R|]. \quad (1.99)$$

The absolute value around the difference $r - R$ gives two cases:

$$\Phi(r) = \begin{cases} -m \frac{GM}{r} & , \quad r > R \\ -m \frac{GM}{R} & , \quad r \leq R. \end{cases} \quad (1.100)$$

Note that the potential is constant inside the shell and that outside the shell it is the same as for a mass M concentrated at the center of the sphere.

From this we can find the potential from a spherically symmetric mass distribution by integrating over thin shells. To find the gravitational force, however, there is no need to do any calculations. It should be clear that the force at a point a distance r from the center inside a spherically symmetric mass distribution is entirely determined by the mass $M(r)$ closer to the center than the given point. The shells outside cannot give rise to any force since the potential from these is a constant. Therefore we have

$$F_r(r) = -m \frac{GM(r)}{r^2} \quad (1.101)$$

If the spherically symmetric mass distribution is given by $\rho_m(r)$, then the function $M(r)$ is given by

$$M(r) = \int_0^r \rho_m(r') 4\pi r'^2 dr' \quad (1.102)$$

where $4\pi r^2 dr$ is the volume of a spherical shell of radius r and thickness dr .

Example 5 Assume that a planet has radius R and surface acceleration of gravity g and assume that it is homogeneous. Calculate the force on a particle of mass m as function of the distance r from the center of the planet for $r \leq R$.

Solution: We can use equations (1.101) and (1.102) to calculate this force. Assume that the constant density is given by ρ_0 . The function $M(r)$ becomes $M(r) = 4\pi \rho_0 r^3 / 3$ so the force is

$$F_r(r) = -mG \frac{4\pi}{3} \rho_0 r. \quad (1.103)$$

At the surface of the planet, $r = R$, we must have

$$F_r(R) = -mg \quad (1.104)$$

This means that $G \frac{4\pi}{3} \rho_0 = \frac{g}{R}$ and this gives us the expression

$$F_r(r) = -mg \frac{r}{R}, \quad 0 \leq r \leq R, \quad (1.105)$$

as the answer to our problem. \triangle

The main conclusion of this section is that the gravitational force and potential energy from a spherical body is the same outside the body as if they originated in a point mass (particle) at the center of the sphere. This means that one can study the motion of planets and stars under the influence of their mutual gravitational forces as if these bodies were particles, as long as one can ignore their departure from spherical shape.

1.6.2 The Field Outside a Rotating Body

In the general case one can expand the potential energy, $\Phi(\mathbf{r})$, from an arbitrary mass distribution (1.93) in powers of $1/|\mathbf{r} - \mathbf{r}_0|$. Thus

$$\Phi(\mathbf{r}) = \Phi_0 + \Phi_1 + \Phi_2 + \dots \quad (1.106)$$

where Φ_n is proportional to $1/|\mathbf{r} - \mathbf{r}_0|^{n+1}$ and \mathbf{r}_0 is some point inside the distribution. These first three terms are called the monopole, the dipole, and the quadrupole terms respectively.

Rotating bodies will in general be flattened at the poles and bulging at the equator. One can show that their form, for moderate angular velocities, will be spheroids (ellipsoids with two semi-axes equal) of small ellipticity. The gravitational potential energy outside the Earth turns out to be well described by

$$\Phi(\mathbf{r}) = -m \frac{GM_e}{r} \left[1 - J_2 \left(\frac{R_e}{r} \right)^2 \frac{(3 \cos^2 \theta - 1)}{2} \right]. \quad (1.107)$$

Here θ is the angle between the Earth's axis and the position vector \mathbf{r} of the particle of mass m relative to an origin at the center of the Earth. R_e is the mean radius of the Earth. The first order correction to the monopole is a quadrupole for the well known reason that a dipole contribution always can be made to vanish by an appropriate choice of origin (at the center of mass). The strength of this quadrupole contribution is given by the dimensionless number

$$J_2 = 1.083 \cdot 10^{-3}. \quad (1.108)$$

It is a measure of the flattening of the Earth due to rotation.

Note that here we are not dealing with a central force any more. The vector $\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$ does not point to the center of the Earth unless the θ -dependence of (1.107) can be neglected. Only for orbits in the equatorial plane, $\theta = \pi/2$, does one get a central force. It is still not of the simple (Kepler) form $F \propto 1/r^2$. For larger distances the higher order multipole terms rapidly become negligible. The effects of these corrections are thus mainly important for low-lying satellite orbits.

1.7 Planetary Motion

Johannes Kepler (1571-1630) discovered the following three basic laws of planetary motion empirically:

1. The orbits of the planets are ellipses with the sun at one focus.
2. The vector from the sun to a planet sweeps over equal areas in equal times.
3. The square of the period of a planet is proportional to the cube of the major semi-axis of the elliptic orbit.

We have already shown that the first of these is a consequence of the inverse square gravitational force field. The second is nothing but the constancy of the sectorial (area) velocity, equation (1.12), which is valid for any central force field. Here we will show that the third result also is a consequence of the inverse square gravitational force field.

From equation (1.12) we get $\frac{dA}{dt} = \frac{L}{2m}$ and thus

$$dt = \frac{2m}{L} dA. \quad (1.109)$$

Integration of this around the ellipse now gives

$$T = \frac{2m}{L}A = \frac{2m}{L}\pi ab \quad (1.110)$$

where T is the period, i.e. the total time needed for traversing the whole orbit once, and $A = \pi ab$ is the area of the ellipse. We now use equation (1.66) to get $L = \sqrt{pmK}$ and the formulae presented in the text to Figure 1.6 to get $b = \frac{p}{\sqrt{1-e^2}} = \sqrt{pa}$. Combining these we end up with

$$T = 2\pi\sqrt{\frac{m}{K}}a^{\frac{3}{2}} \quad (1.111)$$

so $T^2 \propto a^3$. Here the constant of proportionality contains the mass m of the planet so, according to this formula, it seems as if it is different for different planets, and this is not what Kepler found. He could not vary the distance for a given planet, not even theoretically as we have done. Kepler found empirically that the different planets of the solar system obey the law $T^2 \propto a^3$ with the *same* constant of proportionality. If we assume that the sun, with mass $m_\odot \gg m$ is fixed at the center of force, then we have $K = Gm m_\odot$ and equation (1.111) gives

$$T = 2\pi\sqrt{\frac{1}{Gm_\odot}}a^{\frac{3}{2}}. \quad (1.112)$$

Here the constant of proportionality is the same for all planets and we have proved Kepler's third law.

Example 6 Use the result of Example 2 to find the correct version of equation (1.111) when the two body system is treated exactly.

Solution: In Example 2 we found that, in the center of mass system, the constant angular momentum is given by $\mathbf{L} = \mathbf{r} \times \mu \dot{\mathbf{r}}$ where \mathbf{r} is the vector between the two bodies and where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. If we introduce cylindrical coordinates for \mathbf{r} this gives $L = \mu \rho^2 \dot{\varphi}$. This leads to $dt = \frac{2\mu}{L} dA$. Instead of equation (1.111) we now get

$$T = 2\pi\sqrt{\frac{\mu}{K}}a^{\frac{3}{2}}. \quad (1.113)$$

The force in the two body problem is still given by $F_\rho = -K/\rho^2$ with $K = Gm_1 m_2$. If we use this we get for the period of elliptic motion in the two body system

$$T = 2\pi\sqrt{\frac{1}{G(m_1 + m_2)}}a^{\frac{3}{2}}. \quad (1.114)$$

If one of the masses is a planet, $m_1 = m$, and the other the sun, $m_2 = m_\odot$, then the above equation can be written

$$T = 2\pi\sqrt{\frac{1}{Gm_\odot(1 + m/m_\odot)}}a^{\frac{3}{2}}. \quad (1.115)$$

This should be compared with equation (1.112). One sees that when the two body problem is treated exactly the factor is, in fact, not the same for all planets but the discrepancy is proportional to the small number m/m_\odot . For Jupiter this number is approximately 10^{-3} and that is the largest it can get in the solar system. Other perturbations than neglect of two-body effects naturally also affect the periods of the planets but they are all limited by this small number. \triangle

The point on the elliptic orbit of a planet that is nearest to the sun ($\varphi = 0$) is called the *perihelion* and the most distant point ($\varphi = \pi$) is called the *aphelion*. The corresponding points for a satellite in an elliptic orbit round the Earth are called *perigeum* and *apogeum* respectively. Due to perturbations the orbits are not perfect ellipses. The planetary orbits are

perturbed mainly by the influence of other planets. The main effect of these perturbations is the fact that the ellipse no longer has a fixed direction in space. The major axis of the ellipse rotates slowly, one says that it *precesses*. These problems are treated in courses on *celestial mechanics*.

For satellites in orbit near the Earth, the main perturbations are due to the non-spherical, flattened, shape of the Earth, drag from the upper atmosphere, and influence from the sun and the moon. The main effect of the flattening is precession of the ellipse and a rotation of the plane of the orbit (if it is not in the Earth's equatorial plane). The main effect of the drag is a reduction of eccentricity (e) and an eventual loss of energy which makes the satellite spiral down. The relative importance of the perturbations on satellite orbits depend very much on the height of the orbit above the Earth.

Example 7 Find the speed of a planet in its elliptic orbit as a function of the azimuthal angle φ .

Solution: In Example 4 we found an equation for the speed, v , as a function of the distance, ρ , from the center of force, formula (1.74). To find a formula for the speed as a function of the angle, $v(\varphi)$, all we have to do is insert $\rho(\varphi) = p/(1 + e \cos \varphi)$ into this formula. Simple calculations give

$$v(\varphi) = \sqrt{\frac{Gm_c}{p}(1 + 2e \cos \varphi + e^2)^{1/2}} \quad (1.116)$$

where we have used that $a = p/(1 - e^2)$ (see Figure 1.6). The constant in front of the root, $\sqrt{Gm_c/p}$, can also be written

$$\sqrt{\frac{Gm_c}{p}} = \frac{K}{L}. \quad (1.117)$$

By means of these results one finds the following expressions:

$$v(0) = \frac{K}{L}(1 + e), \quad v(\pi) = \frac{K}{L}(1 - e), \quad (1.118)$$

for the speeds at perihelium and aphelium respectively. \triangle

1.8 Problems

Problem 1 A particle \mathcal{P} of mass m is suspended in a light string of length ℓ which is fixed at a point \mathcal{O} . With the string \mathcal{OP} stretched the particle is brought to a position such that the string is horizontal (i.e. \mathcal{P} is at the same height as \mathcal{O}). The particle is now released and given a horizontal velocity of magnitude v_0 and direction perpendicular to the string. How far below \mathcal{O} will the particle \mathcal{P} get in its subsequent motion?

Problem 2 A particle moves on a trajectory with the shape of a circle, under the influence of a force which always is directed towards a fixed point on the circumference the circle. Show that the magnitude of the force is inversely proportional to the fifth power of the distance between particle and center of force. I.e. show that $F(\rho) \propto \rho^{-5}$.

Problem 3 A particle moves along a trajectory given by $\rho(\varphi) = e^{-\varphi}$. Show that the central force is $F(\rho) = k/\rho^3$. Conversely, what are the possible trajectories for a particle in this force field?

Problem 4 Consider the elliptic orbit of a satellite. Assume that it has smallest distance from the center of the Earth ρ_{\min} (perigeum) and greatest distance ρ_{\max} (apogeum). Find the semi-axes, a and b , the eccentricity e , the period T , and the sectorial velocity for the satellite.

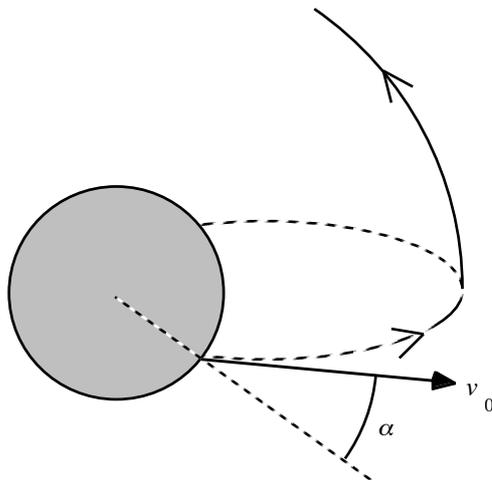


Figure 1.8: This figure shows the orbits described in Problem 7.

Problem 5 A comet has a perihelium distance (smallest distance from the Sun) of 10^6 km and a speed at this distance of 500 km/s. What is the speed of the comet at the greatest distance from the Sun (aphelium)? Use the following numerical data: The universal constant of gravity: $G = 6.67 \cdot 10^{-11}$ N(m/kg)², and the mass of the Sun: $m_{\odot} = 1.99 \cdot 10^{30}$ kg.

Problem 6 For each point \mathcal{P} outside a spherical planet one can define two characteristic speeds of a satellite. One, which we denote v_c , is the speed of the satellite in its circular orbit around the planet through \mathcal{P} , and one, which we denote v_e , is the smallest speed that the satellite must have if it is to be able to escape from the planet for good. Find expressions for these speeds and show that they, independently of the point \mathcal{P} , obey $v_e = \sqrt{2}v_c$.

Problem 7 One wishes to place a satellite in a circular orbit around the Earth. A rocket has taken the satellite to a point just outside the atmosphere when it has burned out. It has then given the satellite a velocity of magnitude v_0 and direction making an angle α with the vertical. The satellite now continues in the elliptic orbit defined by these initial data. When it reaches apogee one fires a small rocket on the satellite so that the magnitude of its velocity increases by Δv without change of direction. (The distance traveled by the satellite while the rocket is fired can be neglected.) See Figure 1.8. How large should the velocity increase be if the new orbit is to be a circle (with a radius corresponding to the apogee distance of the elliptic orbit)?

Problem 8 A space vehicle moves in a circular orbit of radius r_1 around the Earth. The crew wishes to go to a new orbit which is a circular orbit, in the same plane as the original, but with larger radius $r_2 > r_1$. In order to get to the new orbit the rocket of the vehicle is fired for a short time so that so that the speed increases with no change in the direction of the velocity vector. The orbit then changes from being circular to elliptic. If the increase in speed Δv_1 is chosen correctly, the apogee of the elliptic orbit is at the right distance, r_2 , from the Earth. When the vehicle is at this apogee another speed increase, Δv_2 , will change the orbit to the desired circular orbit. Calculate the necessary speed changes Δv_1 and Δv_2 .

Problem 9 Consider a homogeneous spherical planet of radius R . The acceleration due to gravity on the surface of the planet is g . A straight tunnel has been drilled through the planet

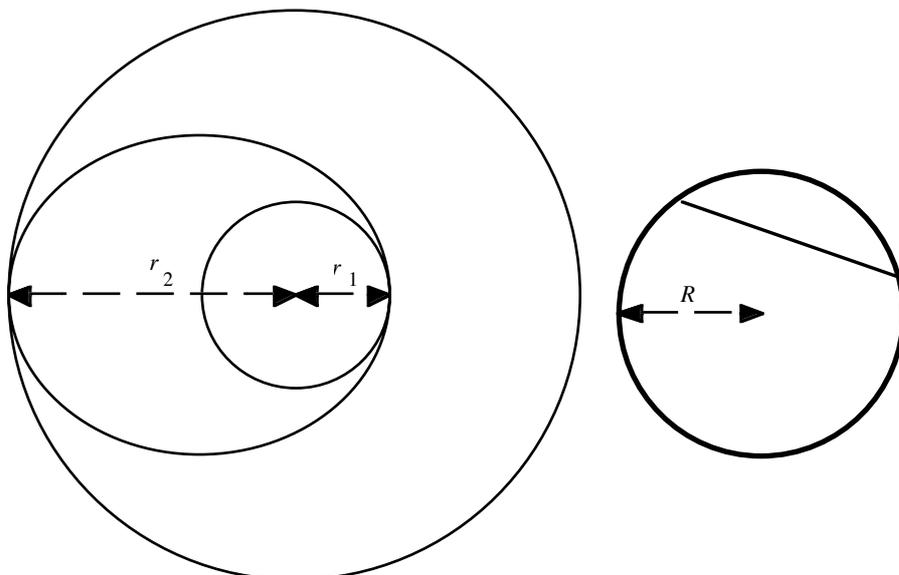


Figure 1.9: The figure on the left shows the two circular orbits and the elliptic transfer orbit described in Problem 8. Such a maneuver is called a ‘Hohmann transfer’.

Figure 1.10: The figure on the right shows a cross section of the planet of radius R with the drilled tunnel described in Problem 9.

from one point on its surface to another. Assume that the walls of the tunnel are smooth and study the motion of a particle which is released from rest at one end of the tunnel at the surface of the planet (neglect air resistance). Show that the motion is periodic and calculate the period T . Find the numerical value of T for the case when $g = 9.81 \text{ m/s}^2$ and $R = 6.4 \cdot 10^6 \text{ m}$. (These values correspond to the Earth which, however, is not homogeneous).

1.9 Hints and Answers

Answer 1 Use the conservation of energy, E , and of the component of angular momentum, L_z , along a vertical Z -axis through the point of suspension of the string. The particle is found to get

$$\frac{v_0^2}{4g} \left(\sqrt{1 + \left(\frac{4g\ell}{v_0^2} \right)^2} - 1 \right)$$

below the initial position.

Answer 2 The equations of motion are

$$\mathbf{e}_\rho : m(\ddot{\rho} - \rho\dot{\varphi}^2) = F,$$

$$\mathbf{e}_\varphi : m(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}) = 0.$$

The second equation can, as we know, be integrated directly and gives

$$m \frac{1}{\rho} \frac{d}{dt} (\rho^2 \dot{\varphi}) = 0$$

so that we have $\dot{\varphi} = \frac{d\varphi}{dt} = \frac{h}{\rho^2}$. We put this into the first of the equations of motion and use the fact that the shape of the trajectory is given by the curve (circle of radius R)

$$\rho(\varphi) = 2R \cos \varphi$$

Some calculation gives

$$\dot{\rho} = -2R \sin \varphi \dot{\varphi} = -\frac{2Rh \sin \varphi}{\rho^2} = -\frac{2Rh \sin \varphi}{4R^2 \cos^2 \varphi}.$$

In a similar way one gets

$$\ddot{\rho} = -h^2(8R^2 - \rho^2)/\rho^5.$$

Insertion of these results into the first of the equations of motion finally gives us

$$F = m(\ddot{\rho} - \frac{h^2}{\rho^3}) = -8mh^2R^2/\rho^5$$

and this is what we wanted to show.

Answer 3 Use Binet's formula (1.62) and insert $u = 1/\rho = \exp \varphi$. Calculations then give the result $F(\rho) = -2mh^2/\rho^3$.

If we conversely assume that $F(\rho) = k/\rho^3$ Binet's formula gives a differential equation for $u = 1/\rho$ that looks as follows:

$$ku^3 = -mh^2u^2 \left(\frac{d^2u}{d\varphi^2} + u \right).$$

We can rewrite this in the form

$$\frac{d^2u}{d\varphi^2} + \left(1 + \frac{k}{mh^2}\right)u = 0$$

The general solution is thus

$$u(\varphi) = A \exp(iw\varphi) + B \exp(-iw\varphi)$$

where $w = \sqrt{1 + k/(mh^2)}$. In the special case $k = -2mh^2$ one can get the given exponential solution.

Answer 4 See Figure 1.6. There it is shown that $2a = \rho_+ + \rho_- = \rho_{\min} + \rho_{\max}$. With this and other equations given there (make sure you understand how to derive them) one finds that:

$$a = \frac{1}{2}(\rho_{\min} + \rho_{\max}), \quad b = \sqrt{\rho_{\min}\rho_{\max}}, \quad e = \frac{\rho_{\max} - \rho_{\min}}{\rho_{\min} + \rho_{\max}}$$

To find the period T one can use formula (1.111). One finds that

$$T = 2\pi \sqrt{\frac{1}{Gm_e} \left[\frac{1}{2}(\rho_{\min} + \rho_{\max}) \right]^{\frac{3}{2}}}$$

The sectorial velocity can be calculated from $\dot{A} = A/T = \pi ab/T$.

Answer 5 Hint:

Use conservation of energy and angular momentum (sectorial velocity).

Solution:

At perihelion and aphelion the conservation of angular momentum simply gives $L = m\rho_{\pm}v_{\pm}$ since the velocity vector is perpendicular to the radius vector (position vector) at both these points. If we now put $\rho_+ = 10^6$ km, and $v_+ = 500$ km/s we get $L/m = \rho_+v_+$. The energy E is given by

$$E/m = \frac{1}{2}v_+^2 - \frac{Gm_{\odot}}{\rho_+}$$

and the numerical value can be found from the data given in the problem. For the quantities at aphelion we now get the system of equations

$$\begin{aligned}\rho_- v_- &= L/m, \\ \frac{1}{2}v_-^2 - \frac{Gm_\odot}{\rho_-} &= E/m.\end{aligned}$$

Solving these gives the answer: $v_- = 29.1$ km/s.

Answer 6 v_c is obtained by identifying the centripetal force in a circular orbit mv_c^2/R , a kinematic result, with the (physical) gravitational force GMm/R^2 . This gives us the equation $mv_c^2/R = GMm/R^2$ and thus,

$$v_c = \sqrt{\frac{GM}{R}},$$

for the *speed in a circular orbit*.

Energy conservation for a particle of mass m that starts at $r = R$ with $v = v_e$, and ends up at $r = \infty$ with $v = 0$, gives $E = T + \Phi = \frac{1}{2}mv_e^2 - GMm/R = \frac{1}{2}m0^2 + GMm/\infty = 0$. The solution of this equation then gives

$$v_e = \sqrt{\frac{2GM}{R}},$$

for the *escape velocity*. Combining these two results gives the desired formula $v_e = \sqrt{2}v_c$.

Answer 7 If we put M and R for the mass and radius of the Earth respectively we get

$$\Delta v = \sqrt{\frac{GM}{r_a}} - \frac{Rv_0 \sin \alpha}{r_a}$$

where

$$r_a = \frac{1}{2} \frac{GM}{(GM/R - v_0^2/2)} \left[1 + \sqrt{1 - 2 \left(\frac{Rv_0 \sin \alpha}{GM} \right)^2 (GM/R - v_0^2/2)} \right].$$

Here G is the gravitational constant.

Answer 8 The semi-major axis of the transfer ellipse must be $a = (r_1 + r_2)/2$. Use of this and of the energy relations gives

$$\Delta v_1 = \sqrt{\frac{Gm_e}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right)$$

and

$$\Delta v_2 = \sqrt{\frac{Gm_e}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right)$$

for the speed changes. This kind of transfer between the two orbits leads to minimum energy consumption. It is called a ‘Hohmann transfer’ after the german W. Hohmann who published a paper on this in 1925.

Answer 9 One finds harmonic oscillations about the midpoint of the tunnel and their period is

$$T = \sqrt{\frac{3\pi}{G\rho}},$$

where ρ is the constant (mass) density. Note that the period is independent of the length and depth of the tunnel.

If one inserts $\rho = M/(4\pi R^3/3)$ and uses $g = GM/R^2$ one finds

$$T = 2\pi\sqrt{\frac{R}{g}} \approx 84 \text{ min}$$

so that the time for a single trip through the Earth is approximately 42 minutes, assuming (erroneously) that the Earth has constant density.

Chapter 2

Rockets and Free Fall

In this chapter we first study, as an application of the momentum principle, the idea that a system of particles can be accelerated forwards by expelling part of its mass backwards. This is the principle behind jet-engines and rockets. We then discuss forces on a space traveler when the rockets are not burning. The concepts of weightlessness, free fall, microgravity, and tidal forces are explained.

2.1 Rockets

We approach the subject of rockets via the following primitive example of rocket propulsion.

Example 8 A person of mass m_p in a boat of mass m_b has a heap of N stones of mass m in the boat. The person can throw these backwards with speed v_e (relative to the boat). The boat is initially at rest and friction can be neglected.

- What is the final speed v_N of the boat if the stones are thrown one by one?
- What is this final speed if they are thrown all at once?
- Insert the numbers $m_p = 50$ kg, $m_b = 100$ kg, $m = 15$ kg, $N = 10$, $v_e = 5$ m/s, and compare the two results above.

Solution: a) Note that v_e is the speed of the stones relative to the boat so that if the boat has velocity v_b the velocity of the stone in the fixed system is $-(v_e - v_b)$. Denote the forward velocity of the boat after k stones have been thrown by v_k , so that $v_0 = 0$, and denote the total mass of the system after k stones have been thrown by

$$M_k = m_p + m_b + (N - k)m = M - km. \quad (2.1)$$

This means that $M_0 = M$ is the total initial mass of the system and we have $M_k + m = M_{k-1}$. The absence of an external force now gives that $p = \text{const.}$ each time a stone is thrown so that we get the system of equations

$$0 = M_0 v_0 = M_1 v_1 - m(v_e - v_1), \quad (2.2)$$

$$M_1 v_1 = M_2 v_2 - m(v_e - v_2), \quad (2.3)$$

$$M_2 v_2 = M_3 v_3 - m(v_e - v_3), \quad (2.4)$$

$$\dots = \dots$$

$$M_{N-1} v_{N-1} = M_N v_N - m(v_e - v_N). \quad (2.5)$$

Solving these consecutively we get

$$v_1 = \frac{m}{M_0} v_e \quad (2.6)$$

$$v_2 = v_1 + \frac{m}{M_1} v_e = \left(\frac{1}{M_0} + \frac{1}{M_1} \right) m v_e \quad (2.7)$$

$$\begin{aligned} v_3 &= v_2 + \frac{m}{M_2} v_e = \left(\frac{1}{M_0} + \frac{1}{M_1} + \frac{1}{M_2} \right) m v_e \\ \dots &= \dots \end{aligned} \quad (2.8)$$

$$v_N = v_{N-1} + \frac{m}{M_{N-1}} v_e = \left(\sum_{k=0}^{N-1} \frac{1}{M_k} \right) m v_e \quad (2.9)$$

We thus have for the final velocity

$$v_N = \left(\sum_{k=0}^{N-1} \frac{1}{M - km} \right) m v_e = \left(\sum_{k=0}^{N-1} \frac{1}{1 - k \frac{m}{M}} \right) \frac{m}{M} v_e \geq N \frac{m}{M} v_e. \quad (2.10)$$

b) If all the stones are thrown at once we get

$$0 = M_N v_a - N m (v_e - v_a) \quad (2.11)$$

and this gives

$$v_a = N \frac{m}{M} v_e \leq v_N. \quad (2.12)$$

We see that for the case of one stone, $N = 1$, the two results agree as they must.

c) When the given numbers are inserted into the formulas found in a) and b) some calculation (use an electronic calculator to check this) leads to:

$$v_{10} = 3.34 \text{ m/s} \quad \text{and} \quad v_a = 2.50 \text{ m/s}. \quad (2.13)$$

So, throwing one stone at the time gives roughly 34% higher final speed, compared to throwing all at once. \triangle

We see in this example that it is advantageous to throw many small stones rather than one big. The rocket can be considered as the limit of the above example when the number of stones $N \rightarrow \infty$ and their mass $m \rightarrow 0$ so that a continuous flux of mass is obtained. To derive an equation for this case of continuous mass flow we denote by v_e the speed of the expelled mass relative to the rocket. We denote the mass of the rocket at time t with $M(t)$ and its speed at that time $v(t)$. Momentum conservation during the short time dt now gives (compare with the equations for the boat in the example above):

$$M(t)v(t) = M(t + dt)v(t + dt) - |dM|[v_e - v(t + dt)] \quad (2.14)$$

Note that $dM < 0$ as the mass $M(t)$ of the rocket decreases, see Figure 2.1.

The approximations $M(t + dt) = M(t) + \dot{M}dt$ and $v(t + dt) = v(t) + \dot{v}dt$ and neglect of higher order terms leads after some algebra to the following result

$$M\dot{v} = |\dot{M}|v_e. \quad (2.15)$$

This says that the force, $Ma = F_{\text{rocket}}$, which the rocket engine produces is given by the mass flow $|\dot{M}|$ times the exhaust velocity v_e of this mass relative to the rocket:

$$F_{\text{rocket}} = |\dot{M}|v_e. \quad (2.16)$$

If there is an external force in the direction of motion of the rocket it must, of course, be added to the right hand side of equation (2.15).

Example 9 A rocket starts from rest and is not affected by external forces. The mass of the rocket itself is m_r and the mass of its fuel is m_f when it starts. The exhaust speed v_e of the fuel is assumed constant. Calculate the final velocity of the rocket (when the fuel is finished).

Solution: Equation (2.15) in the form $Mdv/dt = -\dot{M}v_e$ gives

$$dv = -\frac{dM}{M} v_e. \quad (2.17)$$

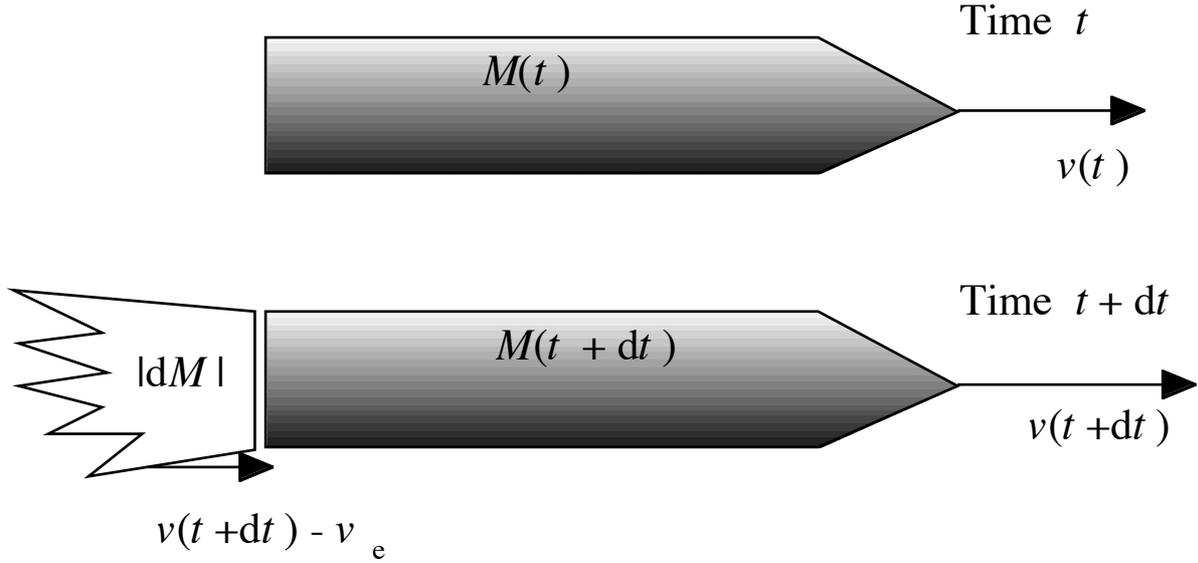


Figure 2.1: A rocket is shown, first at time t , then at time $t + dt$ together with the mass, $|dM|$, that has been exhausted since t . The absolute sign is needed because $dM < 0$ for a rocket that loses mass. Momentum conservation gives $M(t)v(t) = M(t + dt)v(t + dt) + |dM|[v(t + dt) - v_e]$, see equation (2.14).

Integration from $v = 0$ to $v = v_{\max}$ on the left hand side and from $M = m_r + m_f$ to $M = m_r$ on the right gives

$$v_{\max} - 0 = -v_e[\ln(m_r) - \ln(m_r + m_f)] \quad (2.18)$$

and we get for the final speed

$$v_{\max} = -v_e \ln \frac{m_r}{m_r + m_f} = v_e \ln \left(1 + \frac{m_f}{m_r} \right). \quad (2.19)$$

The important quantities that determine the performance of a rocket are thus the exhaust speed of the fuel v_e and the mass ratio of fuel mass to remaining mass m_f/m_r . \triangle

The power, P_{rocket} , (work per unit time) delivered by a rocket engine is of interest. Consider the rest frame of the rocket. In this frame it is seen that, during time dt , the (exhaust) mass $|dM|$ has been accelerated to the speed v_e . This means that in time dt the work $dW = \frac{1}{2}|dM|v_e^2$ has been done. Power P is work per time, $P = dW/dt$, and thus

$$P_{\text{rocket}} = \frac{1}{2}|\dot{M}|v_e^2 = \frac{1}{2}v_e F_{\text{rocket}} \quad (2.20)$$

is the power of the rocket engine. Typical values of v_e , the exhaust velocity (relative to the rocket), are in the range $v_e = (1.6 - 4.4)$ km/s.

2.1.1 General Results and Rotating Systems

A general open system is one that loses mass at n_- points, on its defining surface, with position vectors \mathbf{r}_k^- ($k = 1, \dots, n_-$) and gains mass at n_+ points with position vectors \mathbf{r}_i^+ ($i = 1, \dots, n_+$). We denote (the absolute value of) the mass flows (mass per unit time) at these points by q_k^- and q_i^+ respectively and the velocities of these flows relative to the open system by \mathbf{u}_k^- and \mathbf{u}_i^+ respectively. (This means that $|\dot{M}|$ of the previous section is equal to q^- with corresponding $\mathbf{u}^- = -v_e \mathbf{e}_e$ where \mathbf{e}_e is a unit vector in the forward direction the

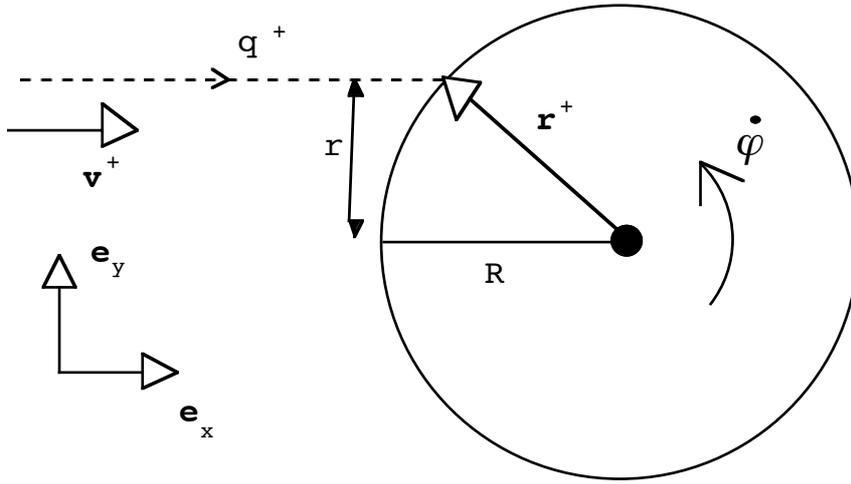


Figure 2.2: This figure illustrates the situation in Example 10. The dashed line is the trajectory along which the bullets move. They hit the wooden carousel and come to rest in it near the circumference.

rocket engine.) With these notations one can show that

$$m\mathbf{a}_o = \mathbf{F}^e - \sum_{k=1}^{n_-} q_k^- \mathbf{u}_k^- + \sum_{i=1}^{n_+} q_i^+ \mathbf{u}_i^+. \quad (2.21)$$

is the equation of motion for the open system. Here \mathbf{F}^e is the total external force on the system (due to other causes than mass flow).

To study rotational open systems one needs the equation

$$\dot{\mathbf{L}}_o = \mathbf{M}^e - \sum_{k=1}^{n_-} \mathbf{r}_k^- \times q_k^- \mathbf{v}_k^- + \sum_{i=1}^{n_+} \mathbf{r}_i^+ \times q_i^+ \mathbf{v}_i^+, \quad (2.22)$$

the angular momentum principle for an open system. Here \mathbf{L}_o is the angular momentum of the open system, \mathbf{M}^e is the total moment (of force) due to external forces), \mathbf{v}_k^- the *absolute* velocity of out flowing mass, and \mathbf{v}_i^+ the absolute velocity of in flowing mass.

Example 10 A carousel (merry-go-round) of radius R can rotate freely around a vertical axis. The moment of inertia with respect to this axis is J_0 . Find the expression for the angular acceleration $\ddot{\phi}$ of the carousel if it is shot at by a machine gun which produces a horizontal mass flow q^+ of bullets with speed v^+ . Also find the limiting angular velocity if the shooting goes on for long. The bullets hit the carousel at a perpendicular distance r from its center, see Figure 2.2, and are embedded in it near the circumference.

Solution: If we assume that the shooting started at time $t = 0$ the moment of inertia of the carousel will be

$$J(t) = J_0 + q^+ t R^2 \quad (2.23)$$

since the bullets increase the mass at radius R . The angular momentum of this open system (carousel under fire) is

$$\mathbf{L}_o = J(t) \dot{\phi}(t) \mathbf{e}_z, \quad (2.24)$$

with respect to an origin at its center. Equation (2.22) now gives

$$\frac{d}{dt}(J(t) \dot{\phi}(t) \mathbf{e}_z) = \mathbf{r}^+ \times q^+ \mathbf{v}^+ = -r q^+ v^+ \mathbf{e}_z. \quad (2.25)$$

This gives

$$(J_0 + q^+ t R^2) \ddot{\phi} = -r q^+ v^+ - q^+ R^2 \dot{\phi} \quad (2.26)$$

and this is the desired differential equation for $\ddot{\phi}$.

If $\dot{\phi}(0) = 0$ the angular acceleration is negative for small $t > 0$. This gives a negative angular velocity $\dot{\phi}$ as the sign conventions of the Figure 2.2 demands. However, when the angular velocity becomes increasingly negative, the right hand side of the equation for $\ddot{\phi}$ decreases. Eventually the acceleration will approach zero. When $\ddot{\phi} = 0$ the above equation gives the limiting angular velocity

$$\dot{\phi}_{\text{lim}} = -\frac{rv^+}{R^2} \quad (2.27)$$

and this concludes our example. \triangle

2.2 Free Fall and Microgravity

Imagine that you are traveling with a rocket from the surface of the Earth. Once in empty space above the atmosphere the rocket engines are switched off. As everyone knows one then becomes "weightless".

If weight is defined as usual, i.e. as the gravitational force from the Earth on a body, this is, of course, not true. This force is still substantial just above Earth's atmosphere. So what happens? Recall that when you are on Earth you feel gravity via contact with the ground or some floor. The fact that you can be at rest is due to a balance between gravitational force (weight) and normal force from the ground (or possibly buoyancy forces from a fluid etc.). In empty space all these other forces vanish so that you and everything around you, including your rocket, are only affected by the gravitational force. This, however, means that you and everything around you fall freely with (almost) the same acceleration. Everything gets the same acceleration in a homogeneous gravitational field because of the equality of the *inertial* and the *gravitational* mass.

Consider formulas (1.93) and (1.94) for the gravitational potential energy $\Phi(\mathbf{r})$ and force $\mathbf{F}(\mathbf{r})$ from an arbitrary mass distribution on a particle of mass m at \mathbf{r} . If there are no other forces on the particle its equation of motion is $m\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r})$, and m here is the inertial mass. Note however that the potential energy always can be written

$$\Phi(\mathbf{r}) = m\phi(\mathbf{r}) = m \left(- \int \frac{G \rho_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \right). \quad (2.28)$$

The m here is the gravitational mass of the particle and the quantity $\phi(\mathbf{r})$, which is completely determined by the distribution of distant mass, is called the gravitational potential. We define the vector field $\mathbf{g}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ to be the gravitational field. We now see that the assumption that the inertial and gravitational masses are equal leads to the equation of motion

$$\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}). \quad (2.29)$$

This means that all particles at the same point of space get the same acceleration. This result, sometimes referred to as the equivalence principle, has been verified experimentally to very high accuracy. If $\mathbf{g}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ varies slowly near some particle trajectory $\mathbf{r}_0(t)$ we might transform to a reference frame with the acceleration $\ddot{\mathbf{r}}_0(t)$. In this frame we get the equation of motion

$$\ddot{\mathbf{r}}' = \mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r}_0) \quad (2.30)$$

for a particle at \mathbf{r}' . If we can neglect the variation of \mathbf{g} we thus get zero acceleration relative to this frame. This is a good approximation for a small region far away from the masses that produce the field.

It is this state of *free fall* that is sometimes called weightlessness. One may use the accelerated reference frame of the rocket to calculate and study physical phenomena if one introduces a fictitious force that cancels the gravitational force. In this (accelerated) reference frame everything is in equilibrium unless acted on by forces other than gravity.

This force-free state of free fall is of great importance in many experiments that are conducted on board space-craft. It is then of interest to know how accurately the forces are zero in the accelerated reference frame of the space vehicle. The complete removal of forces in this frame depends on two assumptions: 1) there are no non-gravitational forces acting on or inside the vehicle, and 2) the gravitational force from external bodies is homogeneous (the same at all points of space). Neither assumption is ever strictly fulfilled and the small resulting forces are called *micro-gravity*.

Non-gravitational forces come from traces of gas and particles that always are present even in space. There is also radiation pressure from electromagnetic radiation and there may be weak electric and magnetic forces. The space-craft may rotate or have some other relative acceleration of its parts. When it is heavy it may also produce a gravitational field of its own.

The gravitational field from other bodies is of course not homogeneous but varies in strength and direction. Only the average field (roughly the field at the center of mass) can be transformed away by free fall with some given acceleration. The Earth for example is falling freely in the gravitational field from mainly the moon and the sun. We thus do not experience the mean gravity from these bodies. The Earth, however, is not of negligible extension and thus, even if the acceleration of its center cancels the mean gravitational force on Earth as a whole, there remains the difference between the mean field and the extra strong and extra weak fields that arise at the points closest to and farthest away from the moon and the sun. These difference forces are called *tidal* forces since they give rise to the tides (ebb and flow).

2.3 Problems

Problem 10 A rocket has a motor which ejects mass with a constant rate $q = -\dot{m}$ with a constant speed v_e relative to the rocket. It rises vertically starting with zero velocity at time $t = 0$. Calculate its velocity as a function of time. The acceleration due to gravity is g .

Problem 11 A two step rocket has total initial mass $M_t = M_1 + M_2$ where M_1 is the mass of the first step, and M_2 that of the second (both masses include the mass of the fuel). The mass of the fuel of the first step is m_1 and that of the second step is m_2 . Both steps have rocket engines that produce a constant exhaust velocity v_e and a constant flow q of mass per time. When the first step has burned out it is dropped. Find the final velocity of the two step rocket if it starts from rest and if external forces can be neglected.

Problem 12 A vehicle is propelled by letting water flow out backwards from a tank of rectangular horizontal cross sectional area A that has collected rain water, see Figure 2.3. The cross sectional area of the hole through which the water flows out is σ and we assume that $\sigma \ll A$. One can assume that the water flows out with the speed $u = \sqrt{2gh}$ relative to the tank, where h is the height of the water surface above the hole. Show that the acceleration a of the vehicle must be less than $(\sigma/A)2g$.

Problem 13 A toroidal space station has (outer) radius r , initial mass m , and radius of gyration d . To achieve an angular velocity ω , corresponding to an artificial gravity g , the space station, which initially is at rest, is accelerated by means of two identical rockets on the outer rim. The two rockets are placed at opposite ends of a diameter, see Figure 2.4, and are directed along the tangent of the circumference. Each rocket has mass flow q (mass per unit time) and exhaust velocity u (relative to the rocket). Assume that all expelled mass was contained in the rockets (i.e. at radius r). How long must the rockets be turned on? (Note that rotation with angular velocity ω gives an artificial ‘gravity’ corresponding to the centripetal acceleration $r\omega^2$.)

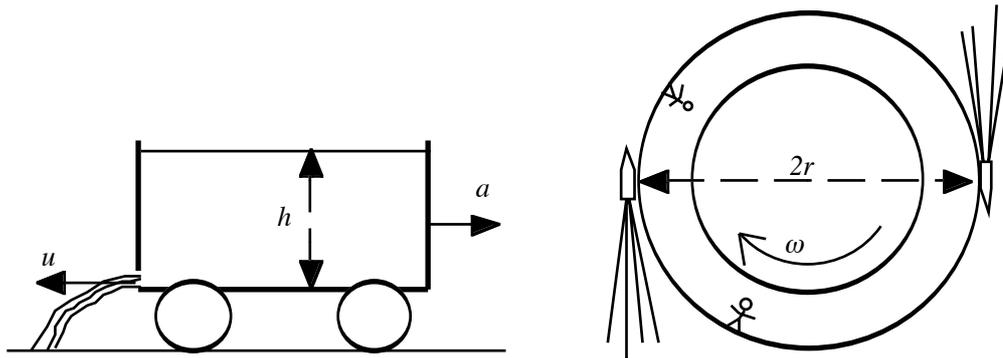


Figure 2.3: The figure on the left refers to Problem 12. The water flows out through the hole of cross sectional area σ . The height of the water surface above the hole is h .

Figure 2.4: The figure on the right refers to Problem 13. A toroidal space station is given an angular acceleration from zero angular velocity up to a value that corresponds to the acceleration of gravity on the perimeter.

2.4 Hints and Answers

Answer 10 In this case one gets

$$M dv = -v_e dM - Mg dt$$

and integration as in Example 9 gives

$$v - v_0 = v_e \ln(1 + m_f/m_r) - gt$$

where v_0 is the speed at time $t = 0$ which was assumed to be zero in this problem.

If the rocket rises to a height h which is of a magnitude comparable to the planet one can no longer assume that the acceleration due to gravity, g , is constant. In such a case the above formula must be replaced by

$$v - v_0 = v_e \ln(1 + m_f/m_r) - \int_0^t g(h) dt.$$

The integral can no longer be done analytically but, as long as the function $g(h)$ is known it can easily be done numerically.

Answer 11 Here one simply does the integration in two steps. After the first step the speed is

$$v_1 = v_e \ln \left(\frac{M_t}{M_t - m_1} \right)$$

and after the second step

$$v_2 = v_1 + v_e \ln \left(\frac{M_t - M_1}{M_t - M_1 - m_2} \right).$$

A typical value of v_e is 2.5 km/s and a typical value of the mass ratio (the argument of the logarithm) is 5. This means that a single step rocket cannot reach the escape velocity of the Earth, which is 11.2 km/s, but a multistage rocket can.

Answer 12 Denote the density of the water by ρ and the acceleration of the vehicle by a . Use of equation (2.21) then gives $ma = qu$ where $q = \rho\sigma u$ so that $a = \rho\sigma u^2/m$. The mass of the vehicle is $m = m_0 + \rho Ah$ where m_0 is the remaining mass when the height of the water surface above the hole has become zero. Since $u^2 = 2gh$ we find

$$a = \frac{\rho\sigma 2gh}{m_0 + \rho Ah}.$$

Since $m_0 \geq 0$ we find that $a \leq \frac{\sigma}{A}2g$ and this is what we wanted to show.

Answer 13 Use equation (2.22) with $\mathbf{M}^e = \mathbf{0}$ and with $\mathbf{L}_o = J(t)\omega \mathbf{e}_z$. One finds that $J(t) = md^2 - 2(qt)r^2$ and thus that the z-component of the equation of motion is

$$\frac{d}{dt}[J(t)\omega(t)] = -2rq(r\omega - u).$$

Some calculations then give

$$\dot{\omega} = \frac{2rqu}{md^2 - 2qtr^2}$$

and integration yields

$$\omega(t) = \frac{u}{r} \ln\left(\frac{md^2}{md^2 - 2qrt^2}\right).$$

If one now puts $\omega = \sqrt{\frac{g}{r}}$ and solves for t one finds that

$$t = \frac{md^2}{2qr^2} \left[1 - \exp\left(-\frac{\sqrt{gr}}{u}\right)\right]$$

is the required burning time of the rockets.

Appendix A

Formulas for Cylindrical Coordinates

Cylindrical or, if $z = 0$, plane polar coordinates, are defined by

$$\mathbf{r} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z = \rho \cos \varphi \mathbf{e}_x + \rho \sin \varphi \mathbf{e}_y + z \mathbf{e}_z. \quad (\text{A.1})$$

If one introduces the position dependent basis vectors

$$\begin{aligned} \mathbf{e}_\rho &= \cos \varphi \mathbf{e}_x + \sin \varphi \mathbf{e}_y \\ \mathbf{e}_\varphi &= -\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y \\ \mathbf{e}_z &= \mathbf{e}_z \end{aligned} \quad (\text{A.2})$$

one can write the *position* vector on the simple form

$$\mathbf{r} = \rho \mathbf{e}_\rho + z \mathbf{e}_z. \quad (\text{A.3})$$

We now assume that this is the position vector of a particle. If it moves ρ, φ , and z depend on time, as well as the basis vectors. If we take the time derivative of this vector we get

$$\dot{\mathbf{r}} = \dot{\rho} \mathbf{e}_\rho + \dot{z} \mathbf{e}_z + \rho \dot{\mathbf{e}}_\rho + z \dot{\mathbf{e}}_z. \quad (\text{A.4})$$

but use of formulas (A.2) gives

$$\dot{\mathbf{e}}_\rho = \dot{\varphi} \mathbf{e}_\varphi \quad (\text{A.5})$$

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi} \mathbf{e}_\rho \quad (\text{A.6})$$

$$\dot{\mathbf{e}}_z = \mathbf{0}. \quad (\text{A.7})$$

Simple calculation now shows that the *velocity* can be written

$$\dot{\mathbf{r}} = \mathbf{v} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\varphi} \mathbf{e}_\varphi + \dot{z} \mathbf{e}_z. \quad (\text{A.8})$$

Taking the time derivative of this gives after similar calculations

$$\dot{\mathbf{v}} = \mathbf{a} = (\ddot{\rho} - \rho \dot{\varphi}^2) \mathbf{e}_\rho + (\rho \ddot{\varphi} + 2\dot{\rho} \dot{\varphi}) \mathbf{e}_\varphi + \ddot{z} \mathbf{e}_z. \quad (\text{A.9})$$

for the *acceleration* in cylindrical coordinates.

Appendix B

Further Reading

Two modern classic texts on celestial mechanics are:

A. E. Roy, *Orbital Motion*, 3rd edition, 1988 (Adam Hilger, Bristol, UK) ISBN 0-85274-229-0.
J. M. A. Danby, *Fundamentals of Celestial Mechanics*, 2nd edition, 1988, (William-Bell, Inc., Richmond, Virginia, USA) ISBN 0-943396-20-4.

An elementary book that deals more specifically with space technology is:

W. E. Wiesel, *Spaceflight Dynamics*, 2nd edition, 1997 (McGrawHill, New York, NY, USA) ISBN 0-07-115631-3.

A couple of more advanced books are:

R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, 1987 (AIAA education series, New York, NY, USA) ISBN 0-930403-25-8.
G. A. Gurzadyan, *Theory of Interplanetary Flight*, 1996, (Gordon and Breach Publishers, Amsterdam, Netherlands) ISBN 2-919875-15-9.

The simplest solution to the Kepler problem can be found in:

A. González-Villaneuva, H. N. Núñez-Ypéz and A. L. Salas-Brito, *In velocity space the Kepler orbits are circular*, European Journal of Physics **17** (1996) pp.168-171.