Magnetism of Matter and Phase Space Energy of Charged Particle Systems

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Abstract

The (Darwin) Lagrangian, and energy, valid for systems of charged particles when radiation is negligible, are derived in a new way that avoids the usual v/c-expansion. This shows more clearly their range of validity. Expressing the energy in terms of canonical momenta gives the corresponding Hamiltonian. When there are many particles it is intractable but useful approximations are given and general conclusions about magnetism of matter are drawn from these. Macroscopic energy extremising self-consistent vortex solutions are presented which can be interpreted as corresponding to superconductivity and ferromagnetism. There is a discussion of the quantum mechanics of the Hamiltonian for conduction electrons in a metal and a phase transition is predicted at low temperature.

1 Introduction

In 1920 C. G. Darwin [1] (a grandson of the famous Darwin) derived the Lagrangian of a system of charged particles from the full Lagrangian of particles and fields, under the assumption that radiation can be neglected. He also found an approximate Hamiltonian corresponding to this Lagrangian. Considering the ubiquity of charged particle systems and the importance of Hamiltonians in statistical and quantum mechanics, one would assume that there is a large literature on the subject by now. This is not at all the case. The literature on the Darwin approach is rather small and, when it comes to the Hamiltonian it is even smaller and quite confused.

It is a well known fact that parallel currents attract each other, and that anti-parallel repel. This implies a magnetic interaction force between moving charges which is correctly described by the Darwin approach. The author has, for some time, been investigating the relevance and consequences of this force to statistical and quantum dynamics of systems of moving charged particles i.e. plasmas and metals [2, 3, 4]. General results have been found implying the presence of currents, as well as magnetic structures and activity, in thermal equilibrium of these systems. In particular it is clear that the interaction must play a role in low temperature superconductivity. A lot, however, remains to be done. The entire approach is unfamiliar and thus difficult to most physicists. Also, reliable quantitative techniques that can be used to derive stringent quantitative results need to be developed by statistical, solid state, and plasma physicists. The present article attempts to begin to address these problems.

We start with a somewhat novel rederivation of the Darwin Lagrangian and energy expression for a system of charged particles. Since we will be interested in macroscopic systems we can not assume that magnetic effects are small relativistic corrections, as they are in the few-body systems that Darwin originally had in mind. The derivation presented here requires no expansion in v/c; all that is required for its validity is that acceleration terms, i.e. $c^{-2}\boldsymbol{A}\cdot(\partial^2\boldsymbol{A}/\partial t^2)$ and $c^{-2}(\partial\boldsymbol{A}/\partial t)^2$ can be neglected, compared to other contributions to the energy density. $\boldsymbol{A}$ itself need not be small.

We then discuss the physical significance of the resulting (Darwin) energy expression. A Hamiltonian is obtained when the velocities in the energy expression are replaced by expressions in terms of canonical coordinates and momenta. For a many-body system this exact Darwin Hamiltonian does not have a meaningful simple closed form but relevant useful approximations are presented and motivated.

Finally we consider some physical consequences of these Hamiltonians. It is pointed out that a London type relation between the electric momentum current density and the (internal, Coulomb gauge) vector potential must hold if magnetic energy is to be minimized. Using this we then study self-consistent vortex solutions and find one which corresponds closely to the vortices of type II superconductors. Finally we study the quantum mechanical two-body problem

and discuss how the attractive interaction might cause a phase transition among metallic conduction electrons at low temperature.

It is *important* to be aware of the fact that the vector potential $\boldsymbol{A}$ throughout this article is that produced by the particles of the system itself. It is thus not an external field and as a result some relations may seem unfamiliar.

2 Lagrangian and Energy of Charged Particle Systems with Negligible Radiation

The full Lagrangian for a system of charged particles and the electromagnetic fields that they produce is given by

$$L = E_{\mathrm{k}} + L_{\mathrm{int}} + L_{\mathrm{em}} \tag{1}$$

where E_{k} is the kinetic energy of the particles,

$$L_{\mathrm{int}} = \sum_i \left(\frac{q_i}{c} \boldsymbol{v}_i \cdot \boldsymbol{A}(\boldsymbol{r}_i) - q_i \phi(\boldsymbol{r}_i) \right), \tag{2}$$

and L_{em} is the Lagrangian (3) of the electromagnetic field. From (1), using the principle of least action, one can derive equations of motion for the particles (the Lorentz force) assuming a given field and Maxwell's equations for the electromagnetic field assuming a given charge and current distribution.

The electromagnetic field that arises from a system of charges can be decomposed into bound and radiation fields (for a recent discussion, see Comay [5]). The radiation field is small, arises only if there is acceleration, and decreases slowly with distance. If acceleration terms are neglected this field vanishes and only the bound electromagnetic field, determined entirely by the positions and velocities of the charged particles, remains. When this is the case the electromagnetic field does not have independent degrees of freedom and can be eliminated from the Lagrangian. This will be done below and the result is the Darwin Lagrangian containing only particle degrees of freedom. In a relativistic context Wheeler and Feynman [6] showed that one can get rid of the field degrees of freedom without approximation but the result is then a non-local theory.

2.1 The Electromagnetic Lagrangian

The electromagnetic part of the Lagrangian (1) is

$$L_{\mathrm{em}} = \frac{1}{8\pi} \int (\boldsymbol{E}^2 - \boldsymbol{H}^2) \mathrm{d}V. \tag{3}$$

Use of Maxwell's equations, various vector identities, and the Coulomb gauge,

$$\nabla \cdot \boldsymbol{A} = 0, \tag{4}$$

allows us to rewrite this as

$$L_{\mathrm{em}} = \frac{1}{2}\int(\phi\varrho - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V + \frac{1}{8\pi c^2}\int\left[\left(\frac{\partial\boldsymbol{A}}{\partial t}\right)^2 + \boldsymbol{A}\cdot\frac{\partial^2\boldsymbol{A}}{\partial t^2}\right]\mathrm{d}V$$
$$+\frac{1}{8\pi c}\oint\left(\phi\frac{\partial\boldsymbol{A}}{\partial t} + \frac{\partial\phi}{\partial t}\boldsymbol{A}\right)\cdot\mathrm{d}\boldsymbol{s} - \frac{1}{8\pi}\oint(\phi\boldsymbol{E} + \boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{s} \tag{5}$$

as is shown in appendix A. Note that the second and the third integral in this expression are both total time derivatives.

Normally a total time derivative can be dropped from the Lagrangian. If radiation from the system is negligible, the surface integral, goes to zero as the distance to the control surface surrounding the system goes to infinity. In this, negligible radiation approximation, therefore only the first term survives, and

$$L_{\mathrm{em}} = \frac{1}{2}\int(\phi\varrho - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V. \tag{6}$$

We see that, in this approximation, it can be expressed entirely in terms of the potentials and the sources.

2.2 The Total Lagrangian

The total Lagrangian of a system of charged particles $L = E_{\mathrm{k}} + L_{\mathrm{int}} + L_{\mathrm{em}}$ becomes using (6), instead of (3),

$$L = \sum_i\left(\frac{1}{2}m_i\boldsymbol{v}_i^2 + \frac{q_i}{c}\boldsymbol{v}_i\cdot\boldsymbol{A}(\boldsymbol{r}_i) - q_i\phi(\boldsymbol{r}_i)\right) + \frac{1}{2}\int(\phi\varrho - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V. \tag{7}$$

If we put

$$\varrho(\boldsymbol{r},t) = \sum_i q_i\delta(\boldsymbol{r}-\boldsymbol{r}_i), \tag{8}$$

$$\boldsymbol{j}(\boldsymbol{r},t) = \sum_i q_i\boldsymbol{v}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i), \tag{9}$$

into the volume integral we now get

$$L = \sum_i\left(\frac{1}{2}m_i\boldsymbol{v}_i^2 + \frac{q_i}{2c}\boldsymbol{v}_i\cdot\boldsymbol{A}(\boldsymbol{r}_i) - \frac{q_i}{2}\phi(\boldsymbol{r}_i)\right). \tag{10}$$

To appreciate this expression it is important to recall that, here, the fields, $\boldsymbol{A}$ and ϕ, are not external fields but are, in fact, the fields that are produced by the charged particles of the system itself.

To complete the removal of the electromagnetic degrees of freedom we must now calculate $\boldsymbol{A}$ and ϕ in terms of the particle variables. In the Coulomb gauge, the Coulomb potential is simply the solution of Poisson's equation

$$\nabla^2\phi = -4\pi\varrho, \tag{11}$$

and this solution is

$$\phi(\boldsymbol{r},t) = \int \frac{\varrho(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = \sum_i \frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}, \tag{12}$$

assuming the charge density (8). The equation for the vector potential, in the Coulomb gauge, is

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{j}. \tag{13}$$

No approximation has been made here so the solution will contain the radiation field. If we neglect the second time derivative $c^{-2}\partial^2\boldsymbol{A}/\partial t^2$, we get rid of the radiation part, and get

$$\nabla^2\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}_t, \tag{14}$$

where $\boldsymbol{j}_t$ is the, transverse, divergenceless current density

$$\boldsymbol{j}_t = \boldsymbol{j} - \frac{1}{4\pi}\nabla\frac{\partial\phi}{\partial t}. \tag{15}$$

One sees, using the Poisson equation (11), that the divergence of this is zero because of charge conservation as expressed through the continuity equation. Interesting aspects of the Coulomb gauge have been discussed by Brill and Goodman [7].

The solution of equation (14) is treated in Jackson [8], section 12.7. Assuming a current density (9) one finds that

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_i \frac{q_i[\boldsymbol{v}_i + (\boldsymbol{v}_i\cdot\boldsymbol{e}_i)\boldsymbol{e}_i]}{2c|\boldsymbol{r}-\boldsymbol{r}_i|}. \tag{16}$$

Here $\boldsymbol{e}_i = (\boldsymbol{r}-\boldsymbol{r}_i)/|\boldsymbol{r}-\boldsymbol{r}_i|$. Time dependence here, as in $\phi(\boldsymbol{r},t)$, enters only through the time dependence of the position and velocity vectors of the particles, $\boldsymbol{r}_i(t), \boldsymbol{v}_i(t)$.

When the expressions (12) and (16) are inserted into equation (10) one finds infinite contributions from self-interactions. When these are discarded, so that each particle only interacts with the field from the others, one obtains

$$L = \sum_i \frac{1}{2}m_i\boldsymbol{v}_i^2 + \sum_{i<j}\frac{q_iq_j}{r_{ij}}\frac{[\boldsymbol{v}_i\cdot\boldsymbol{v}_j + (\boldsymbol{v}_i\cdot\boldsymbol{e}_{ij})(\boldsymbol{v}_j\cdot\boldsymbol{e}_{ij})]}{2c^2} - \sum_{i<j}\frac{q_iq_j}{r_{ij}}, \tag{17}$$

where now r_{ij} is the distance between particles i and j and $\boldsymbol{e}_{ij}$ is the unit vector pointing from i to j. This is the so called Darwin Lagrangian [1] for the system. It includes both electric and magnetic interactions between the particles and is valid in when radiation can be neglected. The present derivation is only slightly different from some that can be found in the literature. Note that we have not assumed that v/c is small. Some textbooks that present standard derivations are [8, 9, 10, 11, 12]. Other interesting discussions can be found in [13, 14, 15, 16, 17]. The fact that the Darwin approach arises from the neglect of the second time derivative in (13), thereby turning a hyperbolic problem into an elliptic has been stressed by Nielson and Lewis [18].

Note that we could have kept a relativistic expression for the kinetic energy from the beginning. When applying the Darwin Lagrangian to few-particle systems one usually keep terms up to order $(v/c)^2$ in the kinetic energy since, in the standard derivations, this is the order to which the exact $\boldsymbol{A}$ has been approximated. In such systems magnetic effects are always relativistic so it would also be inconsistent not to keep such terms in the kinetic energy. For macroscopic systems, that we mainly will have in mind here, however, magnetism need not be relativistic. The long range nature of the (magnetic) interaction and the largeness of Avogadro's number more than compensate for the smallness of $(v/c)^2$. It is therefore perfectly consistent to keep only the classical kinetic energy expression in the above Lagrangian, for these systems. The assumption of no radiation is, of course, also much more reliable, in the non-relativistic case, so actually, the Darwin Lagrangian should be ideally suited for non-relativistic macroscopic systems where magnetism may be strong but radiation, and velocities, are small.

2.3 The Electromagnetic Energy

The energy of the electromagnetic field is

$$E_{\rm em} = \frac{1}{8\pi}\int (\boldsymbol{E}^2 + \boldsymbol{H}^2)\mathrm{d}V. \tag{18}$$

Calculations similar to those for the Lagrangian above and given in Appendix B, show that, in the Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$), the electromagnetic energy can be written

$$\begin{aligned} E_{\rm em} = \frac{1}{2}\int (\phi\varrho + \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V + \frac{1}{8\pi c^2}\int \left[\left(\frac{\partial \boldsymbol{A}}{\partial t}\right)^2 - \boldsymbol{A}\cdot\frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right]\mathrm{d}V \\ + \frac{1}{8\pi c}\oint \left(\phi\frac{\partial \boldsymbol{A}}{\partial t} - \frac{\partial \phi}{\partial t}\boldsymbol{A}\right)\cdot \mathrm{d}\boldsymbol{s} - \frac{1}{8\pi}\oint (\phi\boldsymbol{E} - \boldsymbol{A}\times\boldsymbol{H})\cdot \mathrm{d}\boldsymbol{s}. \end{aligned} \tag{19}$$

It is interesting to note that the terms of this expression are exactly the same as those in formula (5) except that the second term in each integral has opposite sign.

In the case of the Lagrangian (5) two integrals were ignored because they were total time derivatives. When we then tried to find $\boldsymbol{A}$ in terms of positions and velocities we found that we had to neglect the second derivative term in the wave-equation. Here the second integral is no longer a total time-derivative but the terms in it depend on acceleration (divided by c^2) and should thus be negligible, just as the term in the wave-equation, when radiation is negligible. As for the Lagrangian one can motivate that the surface integrals should go to zero for increasing radius of the surrounding surface, provided radiation can be neglected. We thus get that the energy of the electromagnetic field of a system can be written

$$E_{\mathrm{em}} = \frac{1}{2}\int (\phi\varrho + \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V. \tag{20}$$

when radiation is negligible. This expression should be as correct as the expression (6) for the electromagnetic Lagrangian.

2.4 The Kinetic Energy

A system of charged particles in an electromagnetic field has kinetic energy

$$E_{\mathrm{k}} = \sum_i \frac{1}{2} m_i \boldsymbol{v}_i^2 = \sum_i \frac{1}{2m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2. \tag{21}$$

The appearance of the vector potential in this expression is nicely explained in Appendix G of Kittel [19]. It is due to the fact that the generalized momentum, $\boldsymbol{p}_i = \partial L/\partial \boldsymbol{v}_i$, as derived from the Lagrangian (1), or (17), has two contributions

$$\boldsymbol{p}_i = m_i\boldsymbol{v}_i + \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i). \tag{22}$$

The first of these contributions, $m_i\boldsymbol{v}_i$, might be called the kinetic momentum and the second, $\frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i)$, the field momentum. Kittel [19] shows that, in the non-relativistic approximation and assuming the Coulomb gauge, $\nabla\cdot\boldsymbol{A} = 0$, one has

$$\frac{1}{4\pi c}\int \boldsymbol{E}\times\boldsymbol{H}\,\mathrm{d}V = \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i), \tag{23}$$

when the integration is over a delta function density particle at $\boldsymbol{r}_i$. The integral here is essentially the integral of the Poynting vector (28).

Let us denote the kinetic momentum $\boldsymbol{p}_v = m\boldsymbol{v}$. An expression for the time derivative of the kinetic energy of a charged particle in an electromagnetic field will be useful later. From the Lorentz force law

$$\frac{\mathrm{d}\boldsymbol{p}_v}{\mathrm{d}t} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v}\times\boldsymbol{H} \tag{24}$$

and the relation (see Landau and Lifshitz [9] §17)

$$\frac{\mathrm{d}E_{\mathrm{k}}}{\mathrm{d}t} = \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{p}_v}{\mathrm{d}t} \tag{25}$$

one finds

$$\frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}t} = q\boldsymbol{v}\cdot\boldsymbol{E}. \tag{26}$$

In these expressions both E_k and $\boldsymbol{p}_v$ can be replaced by relativistic expressions, if desired.

2.5 The Total Energy

It can be shown that (Landau and Lifshitz [9] §31) that the time derivative of the total electromagnetic field energy density is given by

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}^2+\boldsymbol{H}^2}{8\pi}\right) = -\boldsymbol{j}\cdot\boldsymbol{E} - \nabla\cdot\boldsymbol{S}. \tag{27}$$

Here $\boldsymbol{S}$ is the Poynting vector

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H}. \tag{28}$$

The volume integral of this, together with Gauss' theorem, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{\boldsymbol{E}^2+\boldsymbol{H}^2}{8\pi}\right)\mathrm{d}V = -\int\boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V - \oint\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{s}. \tag{29}$$

If we now use equation (9) for the current density and then formula (26) for the time derivative of the kinetic energy we find

$$\int\boldsymbol{j}\cdot\boldsymbol{E}\mathrm{d}V = \sum_i q_i\boldsymbol{v}_i\cdot\boldsymbol{E}(\boldsymbol{r}_i) = \frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}t}. \tag{30}$$

Use of this, and equation (18), leads to the result that

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_\mathrm{em}+E_\mathrm{k}) = -\oint\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{s}. \tag{31}$$

This tells us that the change in electromagnetic field energy plus particle kinetic energy is due to flux of energy through a surface surrounding the system.

If there is no energy flux through, some sufficiently large surrounding surface, i.e. no radiation, we thus find that

$$E_\mathrm{k} + E_\mathrm{em} = E = \text{constant}. \tag{32}$$

For the no radiation case we can use formula (20) for E_em. If we also use formula (21) for the kinetic energy we find that the total energy, E, of the system of charged particles can be written

$$E = \sum_i \frac{1}{2m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2 + \frac{1}{2}\int(\phi\varrho + \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V, \tag{33}$$

and is constant, provided radiation can be neglected.

If we now use equation (9) for the current density and insert the velocity, expressed in terms of momentum and vector potential, as obtained from (22), we get

$$\boldsymbol{j}(\boldsymbol{r}) = \sum_i \left(\frac{q_i}{m_i} \boldsymbol{p}_i - \frac{q_i^2}{m_i c} \boldsymbol{A}(\boldsymbol{r}_i) \right) \delta(\boldsymbol{r} - \boldsymbol{r}_i). \tag{34}$$

Calculations making use this equation, and of equation (8) for the charge density of point particles, now give us the expression

$$E = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}(\boldsymbol{r}_i) + \frac{q_i}{2} \phi(\boldsymbol{r}_i) \right) \tag{35}$$

for the total energy of the particles and the fields that they produce. Note that few approximations have been made; essentially only those following from the assumption of no radiation and the assumption that the non-relativistic form of the kinetic energy is valid. These two assumptions are not logically equivalent but definitely have a large range of common validity. Note in particular that $\boldsymbol{A}$ itself need not be small, only its time derivative. This is in accord with the fact that magnetism can be very strong without accompanying radiation.

Here it must be remembered that the field $\boldsymbol{A}$ is not an external field; it is the vector potential generated by the particles of the system, in the Coulomb gauge. Hence the absence of the $\boldsymbol{A}^2$-term, see below, however. The expression (35) has been derived before by the present author (see formula (37) of Essén [3]) directly from the Darwin Lagrangian, as the non-relativistic limit of a relativistic expression.

2.6 Significance of the Energy Expression

The result (35), together with the expression (16), provides us with a very accurate result for the energy of a system of charged particles. Normally when such an energy is written down only the Coulomb electrostatic part of the interaction energy is included but this is clearly not good enough for a large number of applications. The negative sign of the term containing the vector potential shows that energy is lowered by parallel currents in the system, or more precisely, energy is lowered when the $q_i \boldsymbol{p}_i$ are parallel to the $q_j \boldsymbol{v}_j$ contributing to $\boldsymbol{A}$. Since $\boldsymbol{A}$ falls off like r^{-1} it decreases slowly and consequently, if many particles have parallel (correlated) contributions to $\boldsymbol{A}$ it can become very large (and even diverge, it seems, in the thermodynamic limit).

There is ample evidence for this predicted spontaneous occurrence of current in various systems. The, so called, bootstrap current [20] in tokamaks is one example from plasma physics. Another is the phenomenon of runaway electrons. Superconducting persistent currents are well known. Somewhat less well known is the occurrence of persistent currents in mesoscopic metal rings in their normal

state [21]. These turn out to be two orders of magnitude larger experimentally [22] than predicted by conventional theory (without the Darwin term).

These, seemingly, elementary facts are, however, completely at odds with a number of, so called, established results in the literature. One of these is that the magnetic energy always is positive. A large number of papers, for example [23, 24, 25], calculate the magnetic energy density of plasmas as a positive quantity. This is in accord with what has been called the Chandrasekhar-Fermi-Schmidt virial theorem [26]. It should be clear from the result (35) that this must be wrong, an opinion previously expressed and motivated by Witalis [27]. When self-interactions are correctly removed and only interparticle interactions are included the magnetic energy, just like the electrostatic, can be both positive and negative. Nature is then known to chose the latter alternative. The present author has used the virial theorem to show this rigorously [4].

Another established result that seems irrelevant in the light of the energy expression (35) is the so called Bohr-van Leeuwen theorem (for discussions see Alfvén [28] or Van Vleck [29]). According to Pippard [30] this theorem entails "... a general demonstration that classical mechanics does not allow an assembly of charged particles to possess any magnetic properties". To the extent that the Darwin approach is part of classical mechanics this is clearly wrong. Currents in a system will clearly be correlated so as to minimize the energy and this will result in magnetic properties. Normally these can be predicted to be diamagnetic since an external field will interfere with the spontaneous energy minimizing correlations of a system and raise its energy.

3 Hamiltonians of Charged Particle Systems with Negligible Radiation

The expression (35) for the total energy of a non-radiative system of charged particles will result in a Hamiltonian for the system provided we can express $\boldsymbol{A}(\boldsymbol{r}_i)$ in terms of positions and momenta of the other particles. Since a Hamiltonian is of the utmost importance for investigating the statistical mechanics and quantum mechanics of such systems this is highly desirable.

The straightforward approach to an expression for $\boldsymbol{A}$ in terms of the momenta is to insert

$$\boldsymbol{v}_i = \frac{1}{m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right), \tag{36}$$

obtained from (22), into the expression (16) for $\boldsymbol{A}$. This is seen to lead to the implicit equation

$$\boldsymbol{A}(\boldsymbol{r}_i) = \sum_{j(\neq i)}\left(\frac{q_j[\boldsymbol{p}_j + (\boldsymbol{p}_j\cdot\boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]}{2m_jc|\boldsymbol{r}_i-\boldsymbol{r}_j|} - \frac{q_j^2\{\boldsymbol{A}(\boldsymbol{r}_j) + [\boldsymbol{A}(\boldsymbol{r}_j)\cdot\boldsymbol{e}_{ij}]\boldsymbol{e}_{ij}\}}{2m_jc^2|\boldsymbol{r}_i-\boldsymbol{r}_j|}\right) \tag{37}$$

for $\boldsymbol{A}(\boldsymbol{r}_i)$ in terms of $\boldsymbol{p}_j$ and $\boldsymbol{r}_j$. Here $\boldsymbol{e}_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|$ and self interactions are excluded. If we denote the solution of this equation by $\boldsymbol{A}_p$, (35) gives us the exact Darwin Hamiltonian

$$H = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_p(\boldsymbol{r}_i) + \frac{q_i}{2} \phi(\boldsymbol{r}_i) \right). \tag{38}$$

Only for a two-body system, however, does this exact solution lead to a tractable explicit expression [3]. See subsection 3.3 below, however, for a solution in a continuum approximation.

3.1 Approximate Hamiltonian in the Few-body or Low Density Limit

If magnetic effects can be assumed small one might keep only the approximation $\boldsymbol{v}_i \approx \boldsymbol{p}_i/m_i$. This approximation, inserted into (16) leads, via (35) or (38), to Darwin's original Hamiltonian (see Darwin [1], or Landau and Lifshitz [9] §65). It represents an excellent approximation for few-body systems. The first order relativistic correction to the kinetic energy should then also be retained, for consistency. In atomic physics the Darwin term in the Hamiltonian is often called the Breit term [31, 32] (to add to the confusion a completely different term is called the Darwin term). Interesting recent studies of the relevance of the Darwin approach in atomic physics are by Crisp [33] and De Luca [34].

For macroscopic systems, on the other hand, one can not assume that magnetic effects are small (relativistic corrections). This problem was first pointed out and addressed by Trubnikov and Kosachev [24], who pessimistically concluded that there was no reasonable closed expression for the Hamiltonian in the case of macroscopic numbers of charged particles, see also Alastuey and Appel [35, 36]. The present author does not agree with that conclusion. When one tries to solve the implicit equation for $\boldsymbol{A}$ mentioned above, one finds a solution in terms of a series expansion [3, 4]. Keeping the first and second order term in this expansion gives the Hamiltonian

$$\mathcal{H} = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m_i} - \frac{q_i}{2m_i c} \boldsymbol{p}_i \cdot \boldsymbol{A}_1(\boldsymbol{r}_i) + \frac{q_i^2}{2m_i c^2} \boldsymbol{A}_1^2(\boldsymbol{r}_i) + \frac{q_i}{2} \phi(\boldsymbol{r}_i) \right), \tag{39}$$

where

$$\boldsymbol{A}_1(\boldsymbol{r}_i) = \sum_{j(\neq i)} \frac{q_j [\boldsymbol{p}_j + (\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij}) \boldsymbol{e}_{ij}]}{2m_j c |\boldsymbol{r}_i - \boldsymbol{r}_j|}. \tag{40}$$

The expansion parameter, in the relevant expansion, is $N r_{\mathrm{e}}/R$, where R is the typical length of a spatial region in which momenta (currents) are correlated, and N is the number of particles in this region, and, finally, r_{e} is the classical electron radius, $e^2/(mc^2)$. The author has presented thermodynamic and other arguments that this parameter should be small enough for the relevant expansion

to converge. It can also be shown that the full expansion will give terms that all are of the same qualitative type as the two retained in (39). That is, they are all either paramagnetic and lead to a lowering of the energy for parallel currents, like the second term of (39), or diamagnetic and positive definite, like the third term.

Krizan, in a debate with R. D. Jones, has criticised all improvements on the traditional Darwin Hamiltonian as being inconsistent since they go beyond the $(v/c)^2$ order to which the approximation is valid to start with [37, 38, 39, 40]. Our point of view is that this is irrelevant for two reasons. First, since the derivation of the present work makes it clear that velocity is not really relevant; only acceleration terms are neglected. Second, because, the correction term given here *is* of order $(v/c)^2$. When the speed of light appears to higher powers it is only in combinations not involving the velocities of the particles but involving Nr_{e}/R, the number of particles, the classical electron radius, and the size of the system.

A further strong argument in favor of (39) is obtained if one calculates the effective one-particle Hamiltonian that it leads to. The effective one-particle Hamiltonian is simply the Hamiltonian that determines the motion of particle i when one assumes that the positions and momenta of all the other particles of the system are given. If things work out correctly one should find an effective Hamiltonian such that particle i moves in the (external) electric and magnetic fields produced by the other particles. This Hamiltonian is well known to be

$$H_i = \frac{1}{2m_i}\left(\boldsymbol{p}_i - \frac{q_i}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2 + q_i\phi(\boldsymbol{r}_i). \tag{41}$$

The calculation, proceeding from (39), is presented in Essén [4]. It turns out that one finds an effective one-particle Hamiltonian which differs from the expected one only in terms that can be expected to be very small, except possibly under astronomical circumstances.

3.2 Finding the Vector Potential in Terms of Momenta in the Continuum Limit

Using (8) and (9) we can express the transverse current density (15) in the form [8]

$$\boldsymbol{j}_t(\boldsymbol{r}) = \sum_i q_i\boldsymbol{v}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i) - \frac{q_i}{4\pi}\nabla\left(\frac{\boldsymbol{v}_i\cdot(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right) = \sum_i q_i\boldsymbol{v}_i\,\mathbf{P}(\boldsymbol{r}-\boldsymbol{r}_i). \tag{42}$$

Here $\mathbf{P}(\boldsymbol{r}-\boldsymbol{r}_i)$ is a tensor that operates on $\boldsymbol{v}_i$ to ensure that the current density $\boldsymbol{j}_t$ is transverse. From (22) we get that

$$q_i\boldsymbol{v}_i = \frac{q_i}{m_i}\boldsymbol{p}_i - \frac{q_i^2}{m_i c}\boldsymbol{A}(\boldsymbol{r}_i) \tag{43}$$

and this results in

$$\boldsymbol{j}_t(\boldsymbol{r}) = \boldsymbol{j}_{pt}(\boldsymbol{r}) - \sum_i \frac{q_i^2}{m_i c}\boldsymbol{A}(\boldsymbol{r}_i)\mathbf{P}(\boldsymbol{r}-\boldsymbol{r}_i) \tag{44}$$

where $\boldsymbol{j}_{pt}$ is, what might be called the transverse momentum current. The $\boldsymbol{A}$ that we are working with here is in the Coulomb gauge and therefore already is transverse. The only effect of the $\mathbf{P}$-operator should then be that of the delta-function, i.e. to represent the particle density. Also since $\boldsymbol{A}(\boldsymbol{r}_i)$ arises, as a slowly decreasing function, from correlated velocities of the other particles of the system it should vary slowly on the scale of individual particles. These considerations indicate that the last term of (44) can be approximated as follows

$$\sum_i \frac{q_i^2}{m_i c}\boldsymbol{A}(\boldsymbol{r}_i)\mathbf{P}(\boldsymbol{r}-\boldsymbol{r}_i) \approx \boldsymbol{A}(\boldsymbol{r})\sum_a \frac{q_a^2 n_a(\boldsymbol{r})}{m_a c}. \tag{45}$$

The last sum here is over the types of particles that occur in the system and $n_a(\boldsymbol{r})$ are their respective number densities.

In most systems electrons, because of their small mass, will provide the main contribution to the last sum in (45) so we might approximate it quite accurately with $e^2 n/(mc)$. If we do this equation (44) gives

$$\boldsymbol{j}_t(\boldsymbol{r}) = \boldsymbol{j}_{pt}(\boldsymbol{r}) - \frac{e^2 n(\boldsymbol{r})}{mc}\boldsymbol{A}(\boldsymbol{r}). \tag{46}$$

When this is inserted in equation (14) one gets

$$\nabla^2\boldsymbol{A} - 4\pi\frac{e^2 n}{mc^2}\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}_{pt}, \tag{47}$$

an equation that might be regarded as a continuum version of the algebraic equation (37), or, more precisely, its Laplacian. This can then be rewritten in the form

$$\left(\nabla^2 - \lambda_m^{-2}\right)\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}_{pt}, \tag{48}$$

where (recall that r_e is the classical electron radius)

$$\lambda_m \equiv 1/\sqrt{4\pi r_{\mathrm{e}} n} \tag{49}$$

has dimension length. Assuming n, and thus λ_m, constant equation (48) has the well known (Yukawa) solution

$$\boldsymbol{A}_p(\boldsymbol{r}) = \frac{1}{c}\int \frac{\exp(-|\boldsymbol{r}-\boldsymbol{r}'|/\lambda_m)\boldsymbol{j}_{pt}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V', \tag{50}$$

so when the source is the momentum current there is an exponential damping.

3.3 Hamiltonian for a Plasma with Constant Charge Density

Considering that $\boldsymbol{j}_{pt}$ is obtained from (42) by replacing $q_i\boldsymbol{v}_i$ by $q_i\boldsymbol{p}_i/m_i$ we find that this solution can be written

$$\boldsymbol{A}_p(\boldsymbol{r}) = \sum_i \frac{q_i[\boldsymbol{p}_i + \boldsymbol{p}_i\mathbf{R}(\lambda_m, \boldsymbol{r} - \boldsymbol{r}_i)]\exp(-|\boldsymbol{r} - \boldsymbol{r}_i|/\lambda_m)}{2cm_i|\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{51}$$

Compare with equation (16) which this equation must have as limit (assuming $\boldsymbol{v}_i = \boldsymbol{p}_i/m_i$) when the length $\lambda_m \to \infty$. Here, thus $\mathbf{R}$ is a suitably defined tensor which in the $\lambda_m \to \infty$ limit should become the dyad $\boldsymbol{e}_i\boldsymbol{e}_i$,

$$\lim_{\lambda_m \to \infty} \mathbf{R}(\lambda_m, \boldsymbol{r} - \boldsymbol{r}_i) = \boldsymbol{e}_i\boldsymbol{e}_i, \tag{52}$$

where $\boldsymbol{e}_i = (\boldsymbol{r} - \boldsymbol{r}_i)/|\boldsymbol{r} - \boldsymbol{r}_i|$. The author is not aware of an explicit expression for this tensor for finite λ_m.

The role of the tensor $\mathbf{R}$ is to make $\boldsymbol{A}_p$ transverse. The precise form of this tensor, however, affects neither the strength nor the general nature of the magnetic interaction between charged particles. It only affects the precise angular dependence of this interaction. For practical purposes we can therefore normally put $\mathbf{R} \approx \boldsymbol{e}_i\boldsymbol{e}_i$. We will do this in what follows. We thus suggest that the expression

$$\boldsymbol{A}_p(\boldsymbol{r}_i) = \sum_{j(\neq i)} \frac{q_i[\boldsymbol{p}_j + (\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]}{2m_i c} \frac{\exp(-r_{ij}/\lambda_m)}{r_{ij}}, \tag{53}$$

is used in the Hamiltonian (38) to yield a Hamiltonian for the case of a constant density n of charged particles (electrons). In fact this result was derived in 1980 by Jones and Pytte [37], in Fourier transformed form, for a homogeneous one component plasma. It has also been suggested previously by the present author on a more heuristic basis [4].

A crucial question when it comes to applications is how the density, n, of charged particles, in formulas (46) and (49), is to be interpreted when quantum mechanical effects are taken into account. Since the derivation above is entirely classical, and assumes that the particles have well defined positions as well as momenta, it seems reasonable to assume that only classically behaving particles will contribute.

One might wonder what happened to the long range magnetic interaction of the Lagrangian formalism. How did it turn into an interaction of the short range λ_m in this Hamiltonian (phase space) formalism? Though there is no precise answer to this question at the moment a guess is that the passage to the continuum introduce macroscopic elements that result in a mechanism related to that which limits the range of Coulomb interaction to the Debye length when polarizability of the medium is considered.

4 Applications

We will now discuss some applications of the above theory. We first point out that our Hamiltonians predict that energy extremising solutions must have current and vector potential parallel. We then use this to find macroscopic self-consistent vortex solutions to the equations of motion. Finally we apply the theory to the conduction electrons of a metal.

4.1 A London Type Relation

Some years ago a paper by Edwards [41] containing a classical derivation of the London phenomenological equation of superconductivity, caused a heated debate. The main argument of the opponents was that superconductivity is a quantum mechanical phenomenon and thus a classical derivation must be wrong.

It does not require much insight to see that the energy expressions of equations (35) or (38) predict a lowering of the energy when nearby currents are parallel and that an energy minimizing state must have its $q_i\boldsymbol{p}_i$ parallel to the internal $\boldsymbol{A}(\boldsymbol{r}_i)$. In a continuum picture this clearly means that the momentum current must be parallel to the vector potential:

$$\boldsymbol{j}_p(\boldsymbol{r}) = C\boldsymbol{A}(\boldsymbol{r}), \tag{54}$$

where C is a positive scalar. This is also seen from equation (50) since it gives $\boldsymbol{A}_p(\boldsymbol{r})$ as a superposition of the $\boldsymbol{j}_{pt}(\boldsymbol{r}')$ essentially within a ball of radius λ_m. As long as λ_m is small compared to macroscopic length scales this implies that the two vector fields are parallel, to the extent that macroscopic continuum averages are meaningful. Now that we have established that the momentum current is parallel to $\boldsymbol{A}$, equation (46) immediately tells us that the ordinary current density is

$$\boldsymbol{j}(\boldsymbol{r}) = \left(C - \frac{e^2 n}{mc}\right)\boldsymbol{A}(\boldsymbol{r}). \tag{55}$$

Clearly it also must be parallel, or anti-parallel, to $\boldsymbol{A}$, depending on the sign of the quantity in the parenthesis.

We also immediately see that the kinetic and magnetic energies are both zero when the momenta are zero. According to equation (54) this means that the constant C must be zero ($C = 0$). Equation (55) then gives

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{e^2 n}{mc}\boldsymbol{A}(\boldsymbol{r}). \tag{56}$$

If we identify $\boldsymbol{j}$ with the superconducting current this is one of London's relations [42]. It says that if charges move with zero momentum current, then the current and the vector potential must be related in this way.

Here one must, again, make an important distinction in order not to be confused. According to equation (50) there is no vector potential if the momentum

current is zero. The London relation (56) is thus impossible if $\boldsymbol{A}$ refers to a vector potential produced by the particles of the system. But, as should be well known, it does refer to the vector potential of an *external* magnetic field. This situation has not been treated at all in the present developments.

4.2 Self-Consistent Vortex Solutions

One of the reasons that Edwards [41] thought that a classical system (a plasma) can obey the London relation, and thus resemble a superconductor, was that plasmas sometimes seem to exhibit vortex ropes that resemble the vortices or type II superconductors. We now investigate solutions to the equations of motion for a charged particle, assuming the London-like relation (54). We will look for self-consistent solutions i.e. solutions that give momentum currents that yield vector potentials that give equations of motions that have these momentum currents as solutions.

Consider the Hamiltonian (38). The effective Hamiltonian for one of the particles (considering the positions and momenta of the others as given) is then

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} - \frac{q}{mc}\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r}). \tag{57}$$

Here we assume that the Coulomb interaction can be neglected. The reason for the doubling of the interaction energy as compared to (38), or (35), is that the interaction energy occurs twice in the total energy, see $\boldsymbol{A}_p$ of equation (53).

From this Hamiltonian we find the equations of motion

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{p}}{m} - \frac{q}{mc}\boldsymbol{A}, \tag{58}$$

$$\dot{\boldsymbol{p}} = \frac{q}{mc}[\boldsymbol{p}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{p}\cdot\nabla)\boldsymbol{A}]. \tag{59}$$

We now assume that the London like relation (54) is fulfilled. We also introduce cylindrical coordinates ρ and φ. To ensure the London relation we look for solutions of the form: $\boldsymbol{A} = A(\rho)\boldsymbol{e}_\varphi$ and $\boldsymbol{p} = p_\varphi\,\boldsymbol{e}_\varphi$. Here $\boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y = (-y\boldsymbol{e}_x + x\boldsymbol{e}_y)/\rho$, and $\rho = \sqrt{x^2+y^2}$. This gives

$$\boldsymbol{H} = \nabla\times\boldsymbol{A} = \left(\frac{A}{\rho} + \frac{\mathrm{d}A}{\mathrm{d}\rho}\right)\boldsymbol{e}_z = H(\rho)\boldsymbol{e}_z \tag{60}$$

and $(\boldsymbol{p}\cdot\nabla)\boldsymbol{A} = 0$. The equations (58,59) now give

$$\dot{\rho}\,\boldsymbol{e}_\rho + \rho\dot{\varphi}\,\boldsymbol{e}_\varphi = \left(\frac{p_\varphi}{m} - \frac{q}{mc}A\right)\boldsymbol{e}_\varphi, \tag{61}$$

$$\dot{p}_\varphi\boldsymbol{e}_\varphi - p_\varphi\dot{\varphi}\,\boldsymbol{e}_\rho = \frac{qp_\varphi H}{mc}\boldsymbol{e}_\rho. \tag{62}$$

An immediate consequence is $\dot{\rho} = \dot{p}_\varphi = 0$ so that ρ as well as p_φ are constant in time. Use of these equations and (60) show that

$$\frac{\mathrm{d}A}{\mathrm{d}\rho} = -\frac{c}{q}\frac{p_\varphi}{\rho}. \tag{63}$$

The solution for A depends on the ρ - dependence of p_φ. Unless A and p_φ have the same ρ-dependence one finds that the scalar C of equation (54) will be position dependent. Some solutions are displayed in Table 1.

The assumption $A = \mu/\rho^k$ is seen to give the same ρ-dependence for A and p_φ. In the first row of the Table a general solution is given, for arbitrary integer k. The following three rows give the specific results for $k = 0, 1$, and, -1, respectively. In the Table μ, β, A_0, and H_0 denote constants of suitable physical dimension. Inserting our basic assumption into equation (54) gives

$$\frac{q}{m} p_\varphi(\rho) n \, \boldsymbol{e}_\varphi = CA(\rho) \, \boldsymbol{e}_\varphi. \tag{64}$$

Table 1 shows that only for positive k-values, in $A = \mu/\rho^k$, do we get the positive C-values that are required if the vector potential arises from the momentum current of the system itself.

The solution, with $k = 1$ and $H = 0$, gives $C = q^2 n/(mc)$. Identifying q with $\pm e$ and use of equation (55) then shows that $\boldsymbol{j}(\boldsymbol{r}) = 0$ in this case. This agrees with the result that $v_\varphi = 0$ for this solution. This remarkable, seemingly non-classical, $k = 1$ solution thus has a non-zero circulating momentum current simultaneously caused by, and acting as source of, a non-zero vector potential. The constant C-value means that the solution is relevant to a spatially homogeneous system. The fact that the magnetic field is zero leads, irresistibly, one's thoughts to the Meissner effect and *superconductivity*. Of course, near the z-axis ($\rho = 0$) the solution must break down and a magnetic field go through, just as they do in the vortices of type II superconductors.

For $k = 0$ one gets a solution that obeys the traditional London relation. As discussed above this is only possible for an external magnetic field. The solution with $k = -1$ is of some interest. The vector potential must be external. Since our theory deals with scalar particles magnetic moment arising from spin of the particles is external (to the theory, but not necessarily to the physical system). Such a magnetic field is present in the $k = -1$ solution. Since C is negative we see that magnetic energy is maximized for this solution. This can be understood as a *ferromagnetic* solution. The reason it can exists, in nature, is that maximizing the magnetic energy may minimize the electrostatic energy, and thus the total energy.

4.3 Magnetic Interaction of Metallic Conduction Electrons

Combining (38) with (53) we find that the Hamiltonian for a system of (classical?) electrons (charge e, mass m), including magnetic interaction, is

$$\mathcal{H}(\boldsymbol{r}_j, \boldsymbol{p}_j) = \sum_i \left(\frac{\boldsymbol{p}_i^2}{2m} + \frac{e}{2}\phi_{(i)} - \frac{e}{2mc}\boldsymbol{p}_i \cdot \boldsymbol{A}_{(i)} \right). \tag{65}$$

Here the potentials at particle i are assumed given in terms of positions and momenta of the other particles. In particular we have

$$\boldsymbol{A}_{(i)}(\boldsymbol{r}_j, \boldsymbol{p}_j) = \sum_{j(\neq i)} \frac{e[\boldsymbol{p}_j + (\boldsymbol{p}_j \cdot \boldsymbol{e}_{ij})\boldsymbol{e}_{ij}]}{2mc} \frac{\exp(-r_{ij}/\lambda_m)}{r_{ij}}. \tag{66}$$

Here $\boldsymbol{e}_{ij}$ is a unit vector from j toward i, $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$, and $\lambda_m = 1/\sqrt{4\pi r_e n}$ where $r_e = e^2/mc^2$, the classical particle radius, and n the number density of particles. Note carefully that there are no external fields and no gauge freedom; $\mathcal{H}$, and $\boldsymbol{A}_{(i)}$, are phase space functions. We take the Coulomb interaction to be given by

$$\phi_{(i)}(\boldsymbol{r}_j) = \sum_{j(\neq i)} \frac{e \exp(-r_{ij}/\lambda_D)}{r_{ij}}, \tag{67}$$

where λ_D is the Debye length.

We apply this Hamiltonian to conduction electrons of a metal in the Sommerfeld approximation. Its derivation above is classical. In Jones and Pytte [37] it is quantum mechanical in but the random phase approximation is made. It can not be expected to apply to the degenerate electrons in the interior of the Fermi sphere. These should at least not be contributing to the magnetic damping. We thus apply it only to the electrons on and above the Fermi surface and assume that these can be approximately dynamically separated from the rest. These mobile electrons can behave semi-classically since they move among a continuum of quantum states. A quantized version of the above Hamiltonian should thus describe them reasonably well. The positive ions and the conduction electrons in the interior of the Fermi sphere contribute a positively charged polarizable background that determines the constant λ_D. Such a background is well known not to screen magnetic fields much. Therefore the density n determining λ_m of equation (66) can only be the density of the mobile, semi-classical, electrons themselves.

This density will be temperature dependent and can be estimated as the density of one electron levels at the Fermi energy, $\mathcal{E}_{\text{F}}$, times the range $d\mathcal{E} = k_{\text{B}}T$. This gives

$$n(T) = g(\mathcal{E}_{\text{F}})k_{\text{B}}T = \frac{mk_{\text{F}}}{\hbar^2\pi^2}k_{\text{B}}T = T\,\frac{a_0}{r_s}\;4.15 \cdot 10^{24}\,\text{m}^{-3}\text{K}^{-1}, \tag{68}$$

where r_s is the radius of a sphere whose volume is the volume per conduction electron [19] and a_0 is the Bohr radius. This gives

$$\lambda_m(T) = 4.93 \cdot 10^4 \sqrt{\frac{\mathrm{K}}{T}} \sqrt{r_s a_0}. \tag{69}$$

The Debye length in a metal (see Kittel [19], pp.280–2) can be estimated by $\lambda_D = 0.64\sqrt{r_s a_0}$ and is thus many orders of magnitude smaller than λ_m for normal temperatures. We thus neglect the Coulomb interaction in what follows, in agreement with the Sommerfeld free electron gas model.

To understand the dynamics of the Hamiltonian (65) – (66) we now investigate its two particle version

$$\mathcal{H}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) = \sum_{i=1}^{2} \frac{\boldsymbol{p}_i^2}{2m} - \frac{e^2 \exp(-r/\lambda_m)}{2m^2 c^2 r} \left[\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + (\boldsymbol{p}_1 \cdot \boldsymbol{e})(\boldsymbol{p}_2 \cdot \boldsymbol{e})\right]. \tag{70}$$

If we make the usual canonical transformation, $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$, $\boldsymbol{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)$, we get

$$\mathcal{H} = \frac{\boldsymbol{P}^2}{2(2m)} + \frac{\boldsymbol{p}^2}{2(m/2)} - \frac{e^2 \exp(-r/\lambda_m)}{8m^2 c^2 r} \{[\boldsymbol{P}^2 + (\boldsymbol{P} \cdot \boldsymbol{e})^2] - 4[\boldsymbol{p}^2 + (\boldsymbol{p} \cdot \boldsymbol{e})^2]\}. \tag{71}$$

The last term has an ordering problem upon canonical quantization but is easily seen to be negligible for the present application. The remaining Hamiltonian has the peculiar property that center of mass momentum squared, $\boldsymbol{P}^2$, acts as an attractive coupling parameter. Consider two electrons on the Fermi surface with $|\boldsymbol{k}| = k_{\mathrm{F}}$. Clearly the lowest energy is obtained when they have maximum center of mass momentum and this is the case when they have (essentially) the same momentum $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \hbar\boldsymbol{k} = \hbar k_{\mathrm{F}} \boldsymbol{e}_k$. As an ansatz for the wave function we thus use

$$\Psi(\boldsymbol{R}, \boldsymbol{r}) = \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_1) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_2) \Phi(\boldsymbol{r}) = \exp(\mathrm{i}2k_{\mathrm{F}} \boldsymbol{e}_k \cdot \boldsymbol{R}) \Phi(\boldsymbol{r}), \tag{72}$$

where $\Phi(\boldsymbol{r})$ is a symmetric function since the electrons must have opposite spins. This ansatz leads to the Schrödinger equation

$$\left(2\mathcal{E}_{\mathrm{F}} - \frac{\hbar^2}{2(m/2)}\nabla^2 - \frac{\mathcal{E}_{\mathrm{F}}}{mc^2} \frac{e^2 \exp(-r/\lambda_m)}{r} [1 + \cos^2\theta]\right) \Phi(\boldsymbol{r}) = E\Phi(\boldsymbol{r}), \tag{73}$$

for the relative motion. Note that, for example, the ansatz $\Psi(\boldsymbol{R}, \boldsymbol{r}) = \sin(2k_{\mathrm{F}} \boldsymbol{e}_k \cdot \boldsymbol{R})\Phi(\boldsymbol{r})$ leads to the same result. There is thus no need for a net center of mass momentum. A state with a superposition of two opposite center of mass momenta gives the same result.

To roughly estimate the solution of (73) we replace $1 + \cos^2\theta$ by its spherical average: $1 + \overline{\cos^2\theta} = 4/3$. If we further put $\Delta E \equiv E - 2\mathcal{E}_{\mathrm{F}}$ we get

$$\left(-\frac{\hbar^2}{2(m/2)}\nabla^2 - \frac{4\mathcal{E}_{\mathrm{F}}}{3mc^2} \frac{e^2 \exp(-r/\lambda_m)}{r}\right) \Phi(\boldsymbol{r}) = \Delta E\, \Phi(\boldsymbol{r}). \tag{74}$$

For low density ($\lambda_m \to \infty$) this becomes a Hydrogen-like equation. Its Bohr-radius is

$$a_m = \frac{3mc^2}{2\mathcal{E}_{\rm F}} a_0 = 1.52 \cdot 10^4 \left(\frac{r_s}{a_0}\right)^2 a_0. \tag{75}$$

The ground state energy becomes $\Delta E = -4[e^2\mathcal{E}_{\rm F}/(mc^2)]^2/(9\hbar^2)$. In a typical metal with $\mathcal{E}_{\rm F} \approx 10\,\mathrm{eV}$ this gives $\Delta E \approx 4 \cdot 10^{-10} E_{\rm H}$ where $E_{\rm H}$ is the Hydrogen ground state energy.

An estimate for the possibility of a phase transition can be obtained by requiring that the two body problem should lead to bound states. A rough estimate for this is given by $\lambda_m(T) > a_m$, which can be rewritten

$$T < 10.5 \left(\frac{a_0}{r_s}\right)^3 \mathrm{K}. \tag{76}$$

For typical metals this pessimistic estimate gives transition temperatures in the range 0.06 K to 1.6 K.

The energy of our bound pairs correspond to a temperature of only $\Delta E/k_{\rm B} \approx 10^{-4}\,\mathrm{K}$. On the other hand, at 1 K, there are $na_m^3 \propto 10^6$ other semi-classical electrons within its 'Bohr' radius a_m. This makes it clear that any phase transition must be the result of collective effects. When $T \to 0$, $\lambda_m \to \infty$ and (65, 66) become the traditional Darwin Hamiltonian. Its effect on metallic conduction electrons at $T = 0$ was estimated in [2] (the result is summarized in [3]). For a finite density of particles with correlated momenta the magnetic energy would go to minus infinity. The density n, on the other hand, becomes the density of electrons on the surface of the Fermi sphere and this density goes to zero in the thermodynamic limit. The combined result of these two effects is that the maximum Darwin energy per conduction electron (all of them) is $\Delta_D \approx r_e k_{\rm F} \mathcal{E}_{\rm F}$. This value assumes that one hemisphere of the Fermi surface is occupied and the other empty, for maximal momentum correlation. Numerically this number agrees well with the energy gap of low temperature superconductors and this indicates that also collective effects of the Hamiltonian have the correct order of magnitude.

At higher temperatures phonons may, of course, break up the magnetically bound pairs but, as we have seen above, in equation (76), they will vanish at higher temperature independently of phonons due to increased magnetic screening. This may explain why the isotope effect is not universal. The theory also shows that if the density of mobile charges can be reduced without reducing their momentum much higher transition temperatures can be achieved. Such materials can then not be typical metals, also in agreement with current knowledge.

The above discussion indicates that magnetic interaction is a promising candidate for explaining superconductivity. This was originally suggested by Welker in the nineteenthirties. More recently the importance of relativistic effects in superconductivity have been stressed by Capelle and Gross [43].

5 Conclusions

The main conclusion of the studies above is that the subject of the Darwin Hamiltonian deserves to be studied much more intensely. Many promising insights into the nature and physics of magnetism of matter seem to spring naturally from our studies. The precise connection with quantum mechanics deserve to be much better studied. Techniques that extend the usual statistical mechanical results to Hamiltonians with this type of interaction need to be developed. The author is certain, however, that insight into the importance of the concept of the Darwin Hamiltonian and its predictions must, and will, grow in the physics community.

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A Rewriting the Electromagnetic Lagrangian

In §75 (problem 2) of Landau and Lifshitz' The Classical Theory of Fields [9] it is shown that the Lagrangian (3) can be written

$$L_{\rm em} = \frac{1}{2}\int(\phi\varrho - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V$$
$$-\frac{1}{8\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E}\cdot\boldsymbol{A}\mathrm{d}V - \frac{1}{8\pi}\oint(\phi\boldsymbol{E}+\boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{s}. \tag{77}$$

The calculation makes use of the definitions

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi \tag{78}$$

$$\boldsymbol{H} = \nabla\times\boldsymbol{A} \tag{79}$$

and the Maxwell equations

$$\nabla\cdot\boldsymbol{E} = 4\pi\varrho \tag{80}$$

$$\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} - \nabla\times\boldsymbol{H} = -\frac{4\pi}{c}\boldsymbol{j} \tag{81}$$

plus Gauss' theorem, directly, or after partial integration.

If we use equation (78) we get

$$\int\boldsymbol{E}\cdot\boldsymbol{A}\mathrm{d}V = \int\phi(\nabla\cdot\boldsymbol{A})\mathrm{d}V - \frac{1}{c}\int\boldsymbol{A}\cdot\frac{\partial\boldsymbol{A}}{\partial t}\mathrm{d}V - \oint\phi\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}. \tag{82}$$

The time derivative of this formula inserted into (77) leads to equation (5) when the Coulomb gauge is used.

B Rewriting the Electromagnetic Energy

If one uses the same relations as those of Appendix A to rewrite the electromagnetic energy (18) one obtains

$$E_{\rm em} = \frac{1}{2}\int(\phi\varrho - \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V - \frac{1}{4\pi}\int \boldsymbol{A}\cdot[\nabla^2\boldsymbol{A} - \nabla(\nabla\cdot\boldsymbol{A})]\mathrm{d}V$$
$$-\frac{1}{8\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E}\cdot\boldsymbol{A}\mathrm{d}V - \frac{1}{8\pi}\oint(\phi\boldsymbol{E} - \boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{s}. \tag{83}$$

If we now use the (gauge independent) relation

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{j} \tag{84}$$

(see Jackson [8] p.220) we get

$$E_{\rm em} = \frac{1}{2}\int(\phi\varrho + \frac{1}{c}\boldsymbol{A}\cdot\boldsymbol{j})\mathrm{d}V - \frac{1}{4\pi}\int \boldsymbol{A}\cdot\left(\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \frac{1}{c}\nabla\frac{\partial\phi}{\partial t}\right)\mathrm{d}V$$
$$-\frac{1}{8\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E}\cdot\boldsymbol{A}\mathrm{d}V - \frac{1}{8\pi}\oint(\phi\boldsymbol{E} - \boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{s}. \tag{85}$$

Using equation (82) we find that the second and third integrals of equation (85) give us

$$-\frac{1}{4\pi}\int \boldsymbol{A}\cdot\left(\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \frac{1}{c}\nabla\frac{\partial\phi}{\partial t}\right)\mathrm{d}V - \frac{1}{8\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E}\cdot\boldsymbol{A}\mathrm{d}V =$$
$$= \frac{1}{8\pi c^2}\int\left[\left(\frac{\partial\boldsymbol{A}}{\partial t}\right)^2 - \boldsymbol{A}\cdot\frac{\partial^2\boldsymbol{A}}{\partial t^2}\right]\mathrm{d}V \tag{86}$$
$$+\frac{1}{8\pi c}\int\left[\frac{\partial\phi}{\partial t}(\nabla\cdot\boldsymbol{A}) - \phi\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A})\right]\mathrm{d}V - \frac{1}{8\pi c}\oint\left(\frac{\partial\phi}{\partial t}\boldsymbol{A} - \phi\frac{\partial\boldsymbol{A}}{\partial t}\right)\cdot\mathrm{d}\boldsymbol{s}$$

If we finally use the Coulomb gauge we see that this result together with (85) leads to the result (19).

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	$p_\varphi(\rho)$	$v_\varphi(\rho) = \rho\dot{\varphi}$	$A(\rho)$	$H(\rho)$	$C(\rho)$
k	$k\frac{q\mu}{c\rho^k}$	$(k-1)\frac{q\mu}{mc\rho^k}$	$\frac{\mu}{\rho^k}$	$(1-k)\frac{\mu}{\rho^{k+1}}$	$k\frac{q^2n}{mc}$
0	0	$-\frac{qA_0}{mc}$	A_0	$\frac{A_0}{\rho}$	0
1	$\frac{\beta}{\rho}$	0	$\frac{c\beta}{q\rho}$	0	$\frac{q^2n}{mc}$
-1	$-\frac{qH_0}{2c}\rho$	$-\frac{qH_0}{mc}\rho$	$\frac{H_0}{2}\rho$	H_0	$-\frac{q^2n}{mc}$

Table 1: Some solutions of equations (60) - (63). Each row represents one solution. The first row gives a general solution for arbitrary integer k-values. The following rows give special cases for $k = 0, 1$, and, -1, respectively.

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