Kepler problem energy and Kepler's third law simply derived

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Using cylindrical (polar) coordinates ($\rho = \sqrt{x^2 + y^2},\ \varphi$) the ellipse can be expressed in the form $\rho = p/(1 + e\cos\varphi)$. Using the fact that $\boldsymbol{v} = \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\varphi}\boldsymbol{e}_\varphi$ and that $\dot{\varphi} = L/m\rho^2$, the energy can be written

$$E = \frac{1}{2}m\dot{\rho}^2 + \frac{L^2}{2m\rho^2} - \frac{K}{\rho} \tag{1}$$

At the minimum ($\rho = \rho_+,\ \varphi = 0$) and the maximum ($\rho = \rho_-,\ \varphi = \pi$) distance, $\dot{\rho} = 0$, so there we have

$$E = \frac{L^2}{2m\rho_-^2} - \frac{K}{\rho_-} \tag{2}$$

$$E = \frac{L^2}{2m\rho_+^2} - \frac{K}{\rho_+} \tag{3}$$

This gives

$$\rho_-^2 E = \frac{L^2}{2m} - \rho_- K, \tag{4}$$

$$\rho_+^2 E = \frac{L^2}{2m} - \rho_+ K. \tag{5}$$

Subtraction of the second from the first gives

$$E(\rho_-^2 - \rho_+^2) = -K(\rho_- - \rho_+) \tag{6}$$

Since $\rho_-^2 - \rho_+^2 = (\rho_- + \rho_+)(\rho_- - \rho_+)$ we find

$$E = -\frac{K}{\rho_- + \rho_+} \tag{7}$$

but, $\rho_- + \rho_+ = 2a$, so finally we get,

$$E = -\frac{K}{2a}, \tag{8}$$

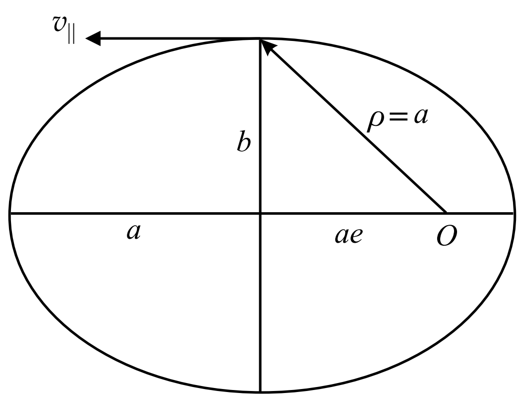

Figure 1: An elliptic orbit with semi major axis a and semi minor axis b. It is indicated that when the distance ρ from the focus O is equal to a the distance of the body from the middle of the ellipse is b. The angular momentum is seen to be given by $L = mbv_\parallel$.

for the energy. It is seen to depend only on the semi major axis a.

We now derive Kepler's third law. In Figure 1 we show that the (constant) angular momentum is $L = mbv_\parallel$. The speed $v_\parallel$ is seen to be the speed at $\rho = a$. Using $K = Gmm_c$ we find from,

$$E = -\frac{K}{2a} = \frac{1}{2}mv^2 - \frac{K}{\rho}, \tag{9}$$

that $v(\rho) = \sqrt{Gm_c(2/\rho - 1/a)}$ and thus $v_\parallel = v(a) = \sqrt{Gm_c/a}$. Thus we find

$$L = mb\sqrt{\frac{Gm_c}{a}}, \tag{10}$$

for the angular momentum. From the expression for the sectorial velocity one finds $\mathrm{d}A = \frac{L}{2m}\mathrm{d}t$. Integration gives $A = \frac{L}{2m}T$ where $A = \pi ab$ is the area of the ellipse and T is the period. Inserting our value for L then gives

$$\pi ab = \frac{1}{2m}mb\sqrt{\frac{Gm_c}{a}}T. \tag{11}$$

Both m and b thus cancel and we find that the period is given by,

$$T = 2\pi\sqrt{\frac{1}{Gm_c}}a\sqrt{a}, \tag{12}$$

where m_c is the mass of the central body. This is Kepler's third law.