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Electrodynamic model connecting superconductor response to magnetic field and to rotation

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Abstract
Theory and experiment on the London moment is reviewed. A simple mathematical model is motivated and then used to study the responses of a spherical superconductor to an external field and to rotation. It reveals a connection between perfect diamagnetism (Meissner effect) and the London moment. In the model neither of these are exact but the deviation from $B = 0$ internal field in the former and from $B = (2mc/e)\Omega$ in the latter case is described by the same dimensionless parameter. Apart from its pedagogical value the model might throw some light on the controversy surrounding the correction to the London moment.

1. Introduction

When a superconductor is rotated with angular velocity $\boldsymbol{\Omega} = (\omega \,\text{rad}\,\text{s}^{-1})\boldsymbol{e}_z$ a magnetic field,

$$\boldsymbol{B} = \frac{2mc}{e}\boldsymbol{\Omega}, \tag{1}$$

with $B = 1.137 \times 10^{-11}\,\omega\,\text{T}$, arises inside it. Here $-e$ is electron charge and m electron mass. This is called the London moment since it was predicted by Fritz London [1] on the basis of the London brothers' phenomenological theory of superconductivity, but the formula was, in fact, derived much earlier by Becker *et al* [2] using the non-viscous electronic liquid model. Since then various ways of arriving at this formula have been proposed [3, 4]. The shortest heuristic derivation postulates that effective forces in the rotating system must vanish; the field (1) is then needed to cancel the Coriolis force (Rystephanick [5]).

Formula (1) is remarkable since it gives the electronic charge to mass ratio from macroscopic measurement and its basic correctness has been experimentally verified by Hildebrandt [6]. It has also been verified that it is independent of the type of superconductor [7, 8] and of its initial rotational state [9]. Nowadays, it is used in basic physics experiments [10]. This immediately leads to the question of how accurate it is.

Since replacement of e and m by Ne and Nm leaves formula (1) invariant it may, in fact, refer to the charge to mass ratio of Cooper pairs or of larger groups of electrons such as the entire superconducting condensate. Based on various theoretical assumptions, one can approach the question of corrections to (1) and this has been done by several authors [11–16]. The results do not agree, however; neither with each other nor with experiment [17]. In view of this confusion, it may be worth pointing out that even a very basic classical model of the phenomenon leads to a correction to London's formula.

We will first motivate heuristically that our model should qualitatively describe the physics of a superconducting sphere. After that the model system, and its kinematics, its basic parameters and its dynamics, are presented. Only classical mechanics and electrodynamics are used. Diamagnetism is then studied within the model and it turns out to be perfect only in the limit of infinitely many electrons. We finally turn to the response of the model to rotation and find that the London moment becomes exact in the same limit that achieved perfect diamagnetism.

2. The giant atom idea

After Meissner's [18] discovery in 1933 of the expulsion of a magnetic field from the superconductor at its phase transition it was realized that understanding the perfect diamagnetism might be one clue to a theory of superconductors. This led Welker [19] to the study of superconductors as giant atoms. He was inspired by Langevin's theory of diamagnetism for systems of closed-shell atoms and ions. In this theory, the external field induces a rotation of the atoms and these rotating atoms produce a field that opposes the external field. For an illuminating discussion see Essén [20], see also van Vleck [21]. In ordinary metals, the magnetic susceptibility is nearly zero because, as Welker explained, the diamagnetic effect is exactly balanced by a paramagnetic effect, the ordering of the electron spins along the external field. In this way Welker [19] realized that perfect diamagnetism requires that there is a gap in the spectrum of the conduction electrons which is not present in ordinary metals. With this energy gap the Langevin mechanism can be blown up and the paramagnetism suppressed. In recent years, Hirsch [4, 22] has advocated the giant atom view of superconductors, see also Essén [23].

Since the discovery of the Pauli principle it has been realized that the electrons that participate in conduction of electricity are the electrons at the surface of the Fermi sea of degenerate electrons. Electrons inside the surface are not able to change their state of motion. The relevant electrons are thus those with the largest energies and velocities [24], essentially the Fermi energy and Fermi velocity, v_{F}. In a normal metal, such electrons are scattered and have a short mean free path Λ. The time between collisions is then, on average, $\tau = \Lambda/v_{\mathrm{F}}$. As long as the metal is large compared to Λ the conduction electron gas will thus be *homogeneous* throughout the metal.

In a superconductor, on the other hand, Cooper pairs will form, and at the critical temperature these must be interpreted as having infinite mean free path, $\Lambda \to \infty$. When the mean free path becomes of the same order of magnitude as the container, the gas can no longer be homogeneous. Instead its distribution must be strongly influenced by the shape of the container. In a spherical metal ball of radius R one then gets an even better analogy with a giant atom. The Cooper pairs can move freely in the spherical container. Since their electrons still must have the largest energy and momenta among the electrons according to the Pauli exclusion principle this means that they must spend most of their time near the metal surface. The centrifugal potential for particles with the Fermi momentum will be of the order of magnitude $\sim R^2 p_{\mathrm{F}}^2/2mr^2$, and thus most pairs are pushed to the surface. We will not go deeper into this here; we just note that for a superconductor the Fermi surface and surface of the metal are necessarily close. London [1] has already stated that superconductivity is a *surface phenomenon*, but this, nowadays, sometimes seems to be forgotten. The fact that the superconducting condensate is concentrated near the metal surface is the motivation for the model presented in the next section.

3. The model system

Consider a heavy sphere of radius R with a positive surface charge Q and surface density $\sigma_+ = Q/4\pi R^2$. An oppositely charged thin spherical shell, of mass M, and the same radius R, covers the surface of the sphere but can rotate freely on it. The system is thus electrically neutral but surface currents, corresponding to rigid rotation of the negative surface charge density, $\sigma_- = -\sigma_+$, can flow without dissipation.

We now set up the Lagrangian of this system in an external magnetic field with vector potential $\boldsymbol{A}_\mathrm{e}$. Since we safely can neglect radiation in our problem we can use the Darwin Lagrangian (see Jackson [25], Essén [26, 27]), but we skip the relativistic correction to the kinetic energy as discussed by Essén [27]. We have

$$L(\boldsymbol{r}_k, \boldsymbol{v}_k) = \frac{1}{2}\sum_{k=1}^{N} m_k \boldsymbol{v}_k^2 + \frac{1}{2}\sum_{k=1}^{N} \frac{q_k}{c}\boldsymbol{v}_k \cdot \boldsymbol{A}_\mathrm{i}(\boldsymbol{r}_k) + \sum_{k=1}^{N} \frac{q_k}{c}\boldsymbol{v}_k \cdot \boldsymbol{A}_\mathrm{e}(\boldsymbol{r}_k), \tag{2}$$

where $\boldsymbol{A}_\mathrm{i}(\boldsymbol{r}_k)$ is the internal vector potential from the particles of the system. It is a sum over all particles except particle number k and the second sum in L is thus a sum over pair interactions; therefore the factor one-half in front. The important thing in the Darwin formalism is that $\boldsymbol{A}_\mathrm{i}$ is divergence free (Coulomb gauge). The last sum is the usual one representing the interaction with the external vector potential $\boldsymbol{A}_\mathrm{e}$.

We will use spherical coordinates (r, θ, φ), so the velocity of a particle fixed on the rotating shell is

$$\boldsymbol{v}(\theta, \varphi, \dot{\varphi}) = \dot{\varphi}\boldsymbol{e}_z \times \boldsymbol{r} = R\sin\theta\dot{\varphi}\boldsymbol{e}_\varphi(\varphi). \tag{3}$$

For the kinetic energy we must integrate over the sphere $r = R$, and we find

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \boldsymbol{v}_k^2 = \frac{1}{2}\int_S \mathrm{d}m(\theta, \varphi)\boldsymbol{v}^2(\theta, \varphi, \dot{\varphi}) = \frac{1}{3}MR^2\dot{\varphi}^2, \tag{4}$$

in agreement with the fact that the moment of inertia of a spherical shell is $I_z = \frac{2}{3}MR^2$.

To find the vector potential of the current from the rotating shell, with charge $-Q$, is an elementary exercise [28]. Some useful formulae can be found in Essén [20]. At $r = R$ the result is

$$\boldsymbol{A}_\mathrm{i}(\theta, \varphi, \dot{\varphi}) = -\frac{\dot{\varphi}}{c}\frac{Q}{3}\sin\theta\boldsymbol{e}_\varphi(\varphi). \tag{5}$$

The self-interaction term in the Lagrangian is thus

$$L_\mathrm{i} = \frac{1}{2c}\int_S \mathrm{d}q(\theta, \varphi)\boldsymbol{v}(\theta, \varphi, \dot{\varphi}) \cdot \boldsymbol{A}_\mathrm{i}(\theta, \varphi, \dot{\varphi}) = \frac{RQ^2}{9c^2}\dot{\varphi}^2, \tag{6}$$

and is seen to be similar to the kinetic energy term.

For definiteness, we here compute the last term for the case of a homogeneous external field $\boldsymbol{B} = B_\mathrm{e}\boldsymbol{e}_z$. The vector potential is then

$$\boldsymbol{A}_\mathrm{e}(r, \theta, \varphi) = \tfrac{1}{2}B_\mathrm{e}(-y\boldsymbol{e}_x + x\boldsymbol{e}_y) = \tfrac{1}{2}B_\mathrm{e}r\sin\theta\boldsymbol{e}_\varphi(\varphi) \tag{7}$$

and one thus finds

$$L_{\mathrm{e}} = \frac{1}{c}\int_S \mathrm{d}q(\theta,\varphi)\boldsymbol{v}(\theta,\varphi,\dot{\varphi})\cdot\boldsymbol{A}_{\mathrm{e}}(R,\theta,\varphi) = -\frac{R^2 Q}{3c}B_{\mathrm{e}}\dot{\varphi}, \tag{8}$$

for the interaction Lagrangian of the rotating spherical shell with this field. This is the interaction needed to study diamagnetism. To investigate the London moment below we have to modify the external field.

Collecting terms we now get

$$L(\dot{\varphi}) = T + L_{\mathrm{i}} + L_{\mathrm{e}} = \frac{R^2}{3}\left[M\left(1+\frac{Q^2}{3RMc^2}\right)\dot{\varphi}^2 - \frac{Q}{c}B_{\mathrm{e}}\dot{\varphi}\right], \tag{9}$$

for our Lagrangian. If we use $Q = Ne$, $M = Nm$, and the classical electron radius $r_e = \frac{e^2}{mc^2}$, we can write

$$M\left(1+\frac{Q^2}{3RMc^2}\right) = Nm\left(1+\frac{Nr_e}{3R}\right) \equiv Nm(1+\epsilon N), \tag{10}$$

and rewrite the Lagrangian in the simple form,

$$L(\dot{\varphi}) = \frac{NmR^2}{3}\left[(1+\epsilon N)\dot{\varphi}^2 - \frac{e}{mc}B_{\mathrm{e}}\dot{\varphi}\right]. \tag{11}$$

We see that the generalized coordinate φ is absent (i.e., cyclic) and that the generalized momentum is

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \frac{2NmR^2}{3}\left[(1+\epsilon N)\dot{\varphi} - \frac{e}{2mc}B_{\mathrm{e}}\right]. \tag{12}$$

The corresponding Hamiltonian is given by $H = \dot{\varphi}p_\varphi - L$ and

$$H(p_\varphi) = \frac{3}{4}\frac{N}{m(1+\epsilon N)}\left(\frac{p_\varphi}{NR} + \frac{eR}{3c}B_{\mathrm{e}}\right)^2 \tag{13}$$

is the result of the calculation.

4. Diamagnetism and Meissner effect

The Meissner effect [18] refers to the fact that a superconductor expels a magnetic field when cooled below the critical temperature. In this it is different thermodynamically from a so-called perfect conductor which merely has zero resistance, see Jackson [25], Pippard [29]. Here we will not discuss thermodynamics and phase transitions, but only the perfect diamagnetism of superconductors. That is, we will explain why an external field which is switched on does not enter the superconducting body.

Let us see what our model system predicts if we take the *initial conditions* to be $\dot{\varphi}(0) = 0$ when the external field is zero $B_{\mathrm{e}}(0) = 0$. The equation of motion is $\dot{p}_\varphi = \partial L/\partial\varphi = 0$, so the generalized momentum is conserved. The initial conditions give $p_\varphi = 0$ and then equation (12) gives

$$(1+\epsilon N)\dot{\varphi}(t) = \frac{e}{2mc}B_{\mathrm{e}}(t), \tag{14}$$

at all times. The angular velocity of the shell is completely determined by the external field at all times. Here this follows from our conservation law $p_\varphi =$ constant. Becker *et al* [2] explains this by stating that the electric field $\boldsymbol{E} = -(1/c)\partial\boldsymbol{A}_{\mathrm{e}}/\partial t$ causes acceleration of the shell.

The rotating shell will of course produce a magnetic field $B_{\rm i}$ of its own. Inside the shell ($r \leqslant R$) it is homogeneous and can be read off by comparing equations (5) and (7). This gives

$$B_{\rm i}(t) = -\frac{2}{3}\frac{Q}{R}\frac{\dot{\varphi}(t)}{c} = -N\frac{2}{3}\frac{e}{R}\frac{\dot{\varphi}(t)}{c}, \tag{15}$$

for the induced field inside the sphere (outside the shell one finds a pure dipole field [30]). Using (14) this can be expressed in terms of $B_{\rm e}$. The *total field inside the sphere* is then

$$B_{\rm dia} = B_{\rm e} + B_{\rm i} = B_{\rm e}\left(\frac{1}{1+\epsilon N}\right). \tag{16}$$

Here ϵ was defined in (10) and is

$$\epsilon = \frac{r_e}{3R}. \tag{17}$$

We see that *perfect diamagnetism* ($B_{\rm dia} \to 0$) corresponds to $N \to \infty$, so for finite N it cannot be achieved, but it gets better the larger the system.

One notes that our model for diamagnetism here is almost entirely like the old Langevin theory. The main difference is that we do not use Larmor's theorem and thus we do *not* assume that the external field is a *weak perturbation*, as is required for the use of Larmor's formula [23]. Instead everything is exact within the model. The smallness of ordinary diamagnetism, when the spheres are atoms, is due to the fact that $N \sim 10$ and $\epsilon \sim r_e/3a_0 \approx 1.78 \times 10^{-5}$, where a_0 is the Bohr radius. Clearly only a very small reduction of the external field is possible in this case.

What about the macroscopic superconducting spheres? For $R = 1$ cm one finds that $\epsilon \approx 10^{-13}$. Does the quantity $\epsilon N = N r_e/3R$ grow sufficiently to produce nearly perfect diamagnetism? One might assume that $N \propto R^3$ but this is not correct. The conduction electrons and thus also the superconducting condensate consist of electrons from a thin layer at the Fermi surface in momentum space. Since this is a two-dimensional object the number of relevant electrons must obey $N \propto R^2$ (Essén [24]). Incidentally, this gives the physical result that the surface charge density $\sigma_- = -Ne/4\pi R^2$, of our model, can remain constant as R increases. The simplest possible minimum estimate assumes that each surface atom contributes one Fermi surface electron and that only these participate in the condensate. This gives $N \approx R^2/a_0^2$. We then find that $\epsilon N \approx (r_e/3R)\left(R^2/a_0^2\right) = 3.3 \times 10^5\,\text{m}^{-1} R$. For $R = 1$ cm this gives $\epsilon N \approx 3300$, so macroscopic spheres should in fact be highly diamagnetic.

5. Rotation and the London moment

We now come to the main task of this work. What is the field of a rotating superconductor? Since our model managed to predict strong diamagnetism it might also give decent results in this case. The external field is no longer assumed to be a homogenous field. Instead we now start rotating the heavy sphere with the positive surface charge density $\sigma_+ = Ne/4\pi R^2$. When this sphere rotates with angular velocity Ω it will produce the field,

$$B_{\rm e}(t) = N\frac{2}{3}\frac{e}{R}\frac{\Omega(t)}{c}, \tag{18}$$

for $r \leqslant R$, in analogy with equation (15). (Outside the sphere it is a dipole field and goes to zero at infinity, just as the field $B_{\rm i}$ above.)

Assuming initial conditions $\dot{\varphi}(0) = 0$ when $\Omega(0) = 0$, we again get equation (14) for the induced angular velocity $\dot{\varphi}(t)$ of the freely rotating negatively charged shell. Equation (14) now relates $\dot{\varphi}(t)$ and $\Omega(t)$ at all times. To find the internal (London) field in this case all we have to do is to use equation (16) and replace B_{e} on the right-hand side with expression (18). This produces the result

$$B_{\mathrm{Lond}} = B_{\mathrm{e}} + B_{\mathrm{i}} = \frac{2mc}{e}\Omega\left(\frac{\epsilon N}{1+\epsilon N}\right), \tag{19}$$

after some simple algebra. When $N \to \infty$ this *approaches the London moment* $\left(B_{\mathrm{Lond}} \to \frac{2mc}{e}\Omega\right)$ of equation (1). Just as was the case above with the perfect diamagnetism we find that the London moment is exact only in the limit of infinitely many particles. If we trace the origin of the terms we see that the extra 1 in the denominator of (19) is due to the contribution to inertia from electron mass, while ϵN comes from the inductive inertia that reflects the energy cost of building up a magnetic field. In electric circuit theory, one is used to considering only the inductive inertia. Inertia due to electron mass is usually negligible in such experiments. In high-precision measurements, however, the electron inertia may play a role and thus the correction term to the London moment suggested by equation (19) may have to be taken seriously.

As mentioned above there is disagreement on the correct theoretical approach to corrections to the London moment [11–16]. The most accurate experiment was done using a superconducting ring [17], not a sphere. Our calculations can be adapted to a ring, but for a ring of negligible thickness the term L_{i} of equation (6) becomes infinite, which presumably means that our correction would go to zero. It also means that a finite thickness is needed for accurate calculation, but then the algebra is considerably more complicated.

6. Discussion and conclusions

The beauty of our embarrassingly simple model is that it does not just give the London moment, as many other oversimplified studies. Instead, it gives the London moment only as a limit for $N \to \infty$, *and* it shows how this limit is intimately connected with the limit of perfect diamagnetism. This is no mean achievement for such a small investment and must be regarded as physics pedagogics at its best.

While most textbooks seem to ignore the London moment there is still a fair amount of active research in this and related areas [31–33]. It has been pointed out that the universality of the London moment, and its sign in particular, mean that the superconducting charge carriers are always electrons, not holes [34]. If nothing else, this paper would therefore, at least, like to make the theoretical and experimental fact of the London moment better known. It is just as remarkable as zero resistivity and perfect diamagnetism, not to mention the Josephson effect.

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