Problem 1.11 in Dynamics of Bodies (extended)

A homogeneous cylinder of radius r and mass m is given an angular velocity $\dot{\varphi}(0) = \omega$ and a translational velocity $\dot{x}_G(0) = v$ at $t = 0$ on a horizontal floor with coefficient of kinetic friction f. We take $\varphi(0) = x_G(0) = 0$.
1) Find a condition on ω that ensures that the cylinder returns to $x_G = 0$.
2) Find the largest distance that the cylinder reaches (at time $t = t_1$).
3) Where is the cylinder when it starts to roll at time $t = t_2$ assuming that $\omega = 3v/r$.
Solution: The motion has three phases: 1) the cylinder decelerating away $0 \leq t < t_1$, 2) the cylinder accelerated back towards the origin $t_1 \leq t < t_2$, 3) the cylinder rolling back for $t_2 \leq t$. These are illustrated in the figure below.

The equations of motion for the cylinder are,

$$m\ddot{x}_G = -F \quad (1)$$
$$J_G\ddot{\varphi} = -rF \quad (2)$$

as long as the sliding (kinetic) friction force $F = fN = fmg$ acts and this is the case as long as $\dot{x}_C > 0$, *i.e.* as long as the contact point has a forward velocity with respect to the floor. Since $J_G = mr^2/2$ we find

$$\ddot{x}_G = -fg \quad (3)$$
$$\ddot{\varphi} = -2fg/r \quad (4)$$

Integrating these using the given initial conditions gives,

$$\dot{x}_G(t) = v - fgt \quad (5)$$
$$\dot{\varphi}(t) = \omega - 2fgt/r \quad (6)$$

Putting $\dot{x}_G(t_1) = 0$ gives

$$t_1 = v/fg.$$

The angular velocity is then $\dot{\varphi}(t_1) = \omega - 2v/r$. In order for the cylinder to roll back this must be positive so we require that

$$\omega > 2v/r.$$

To find the distance that the cylinder has rolled at that time we must integrate $\dot{x}_G(t)$ once to get $x_G(t) = vt - fgt^2/2$. The largest distance of the cylinder is thus

$$x_{\max} = x_G(t_1) = v^2/2fg.$$

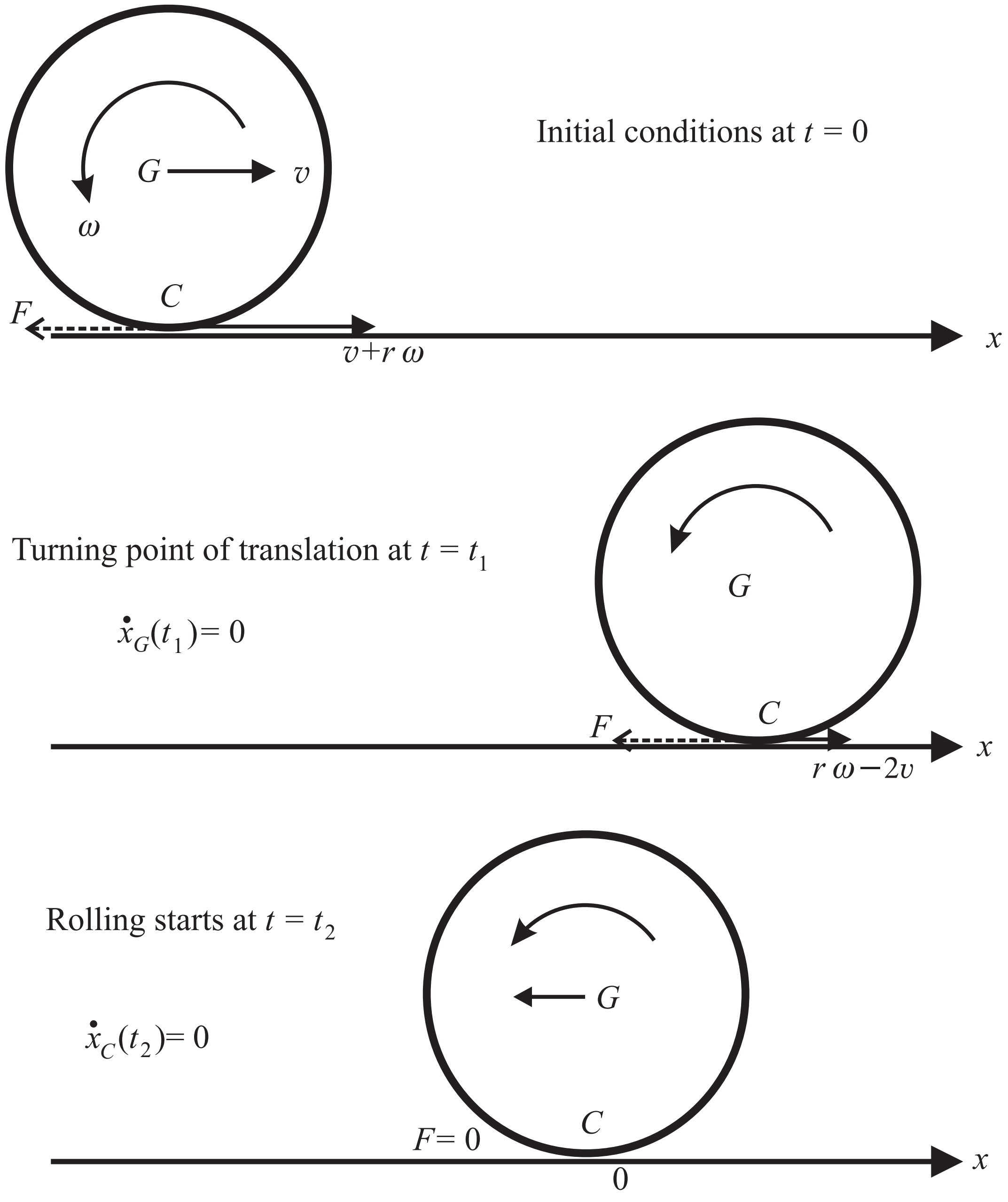
The velocity of the contact point with the floor is $\dot{x}_C = \dot{x}_G + r\dot{\varphi}$. Inserting the solutions (5) and (6) we find $\dot{x}_C(t) = v + r\omega - 3fgt$. The time $t = t_2$ at which rolling starts is given by the equation $\dot{x}_C(t_2) = 0$. The solution is

$$t_2 = (v + r\omega)/3fg.$$

Inserting $\omega = 3v/r$ we find that $t_2 = 4v/3fg$. The velocity of the cylinder is then $\dot{x}_G(t_2) = -v/3$, and the position of the cylinder is

$$x_{\text{roll}} = x_G(t_2) = 4v^2/9fg.$$

See the figure on the next page.


Initial conditions at $t = 0$
G
v
ω
C
F
$v+r\,\omega$
x
Turning point of translation at $t = t_1$
$\dot{x}_G(t_1)= 0$
G
C
F
$r\,\omega-2v$
x
Rolling starts at $t = t_2$
$\dot{x}_C(t_2)= 0$
G
C
$F= 0$
0
x