

# A Collection of Problems in Rigid Body and Analytical Mechanics

Hanno Essén  
KTH Mechanics  
100 44 Stockholm

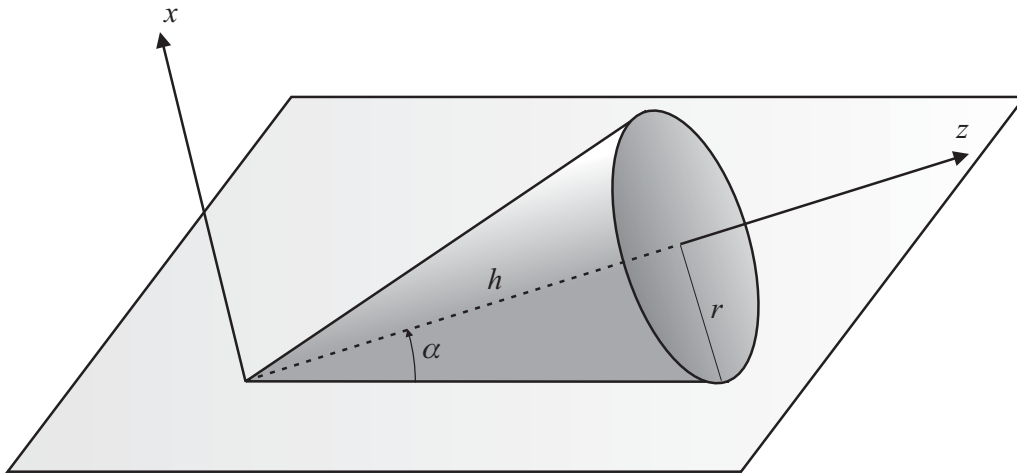
September 2006

## **Abstract**

This is an old collection of problems on rigid body and analytical mechanics. It has been modernized by means of  $\text{\LaTeX}$  and Corel-Draw. The texts have been edited and translated to English but the main work has been done on the illustrations. Minor errors have been corrected. Thank's to Gösta Wingårdh for checking the answers.

# 1 Problems on the three dimensional dynamics of rigid bodies

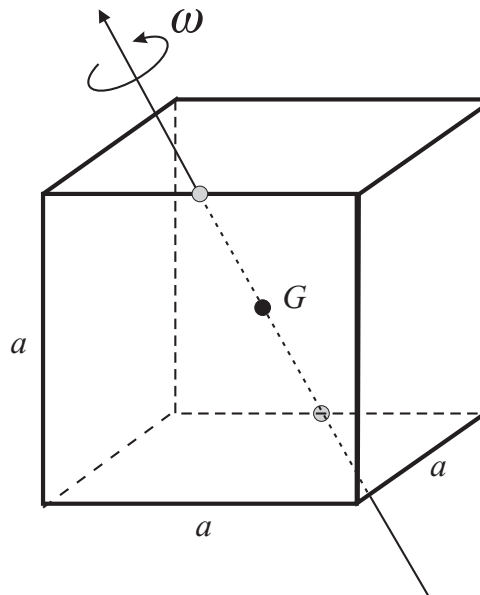
---



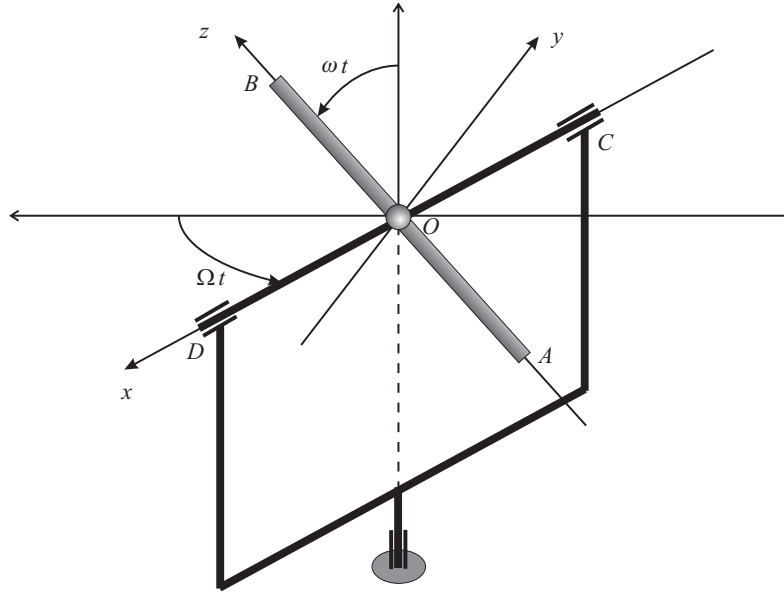
**Problem 1** A straight circular homogeneous cone of mass  $m$ , height  $h$ , and vertex angle  $2\alpha$ , rolls without slipping on a horizontal plane, with the vertex held at a fixed point. At a certain time the angular velocity is  $\omega$ . Calculate the kinetic energy.

Use the moments of inertia:  $J_z = \frac{3}{10}mr^2$  and  $J_x = \frac{3}{5}m\left(h^2 + \frac{r^2}{4}\right)$ .

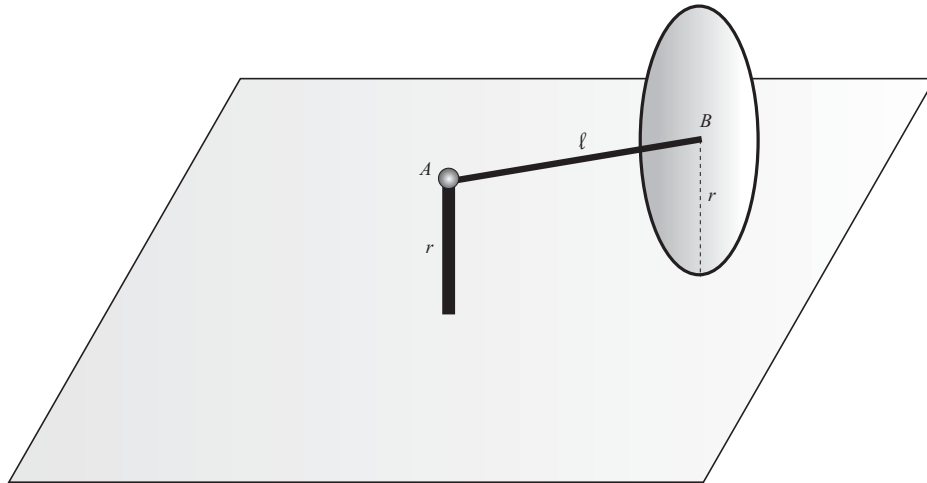
---



**Problem 2** A homogeneous cube of mass  $m$  and edge  $a$  rotates with angular velocity  $\omega$  about an axis through the center of mass  $G$  and the mid point of one of the edges. Find the kinetic energy.

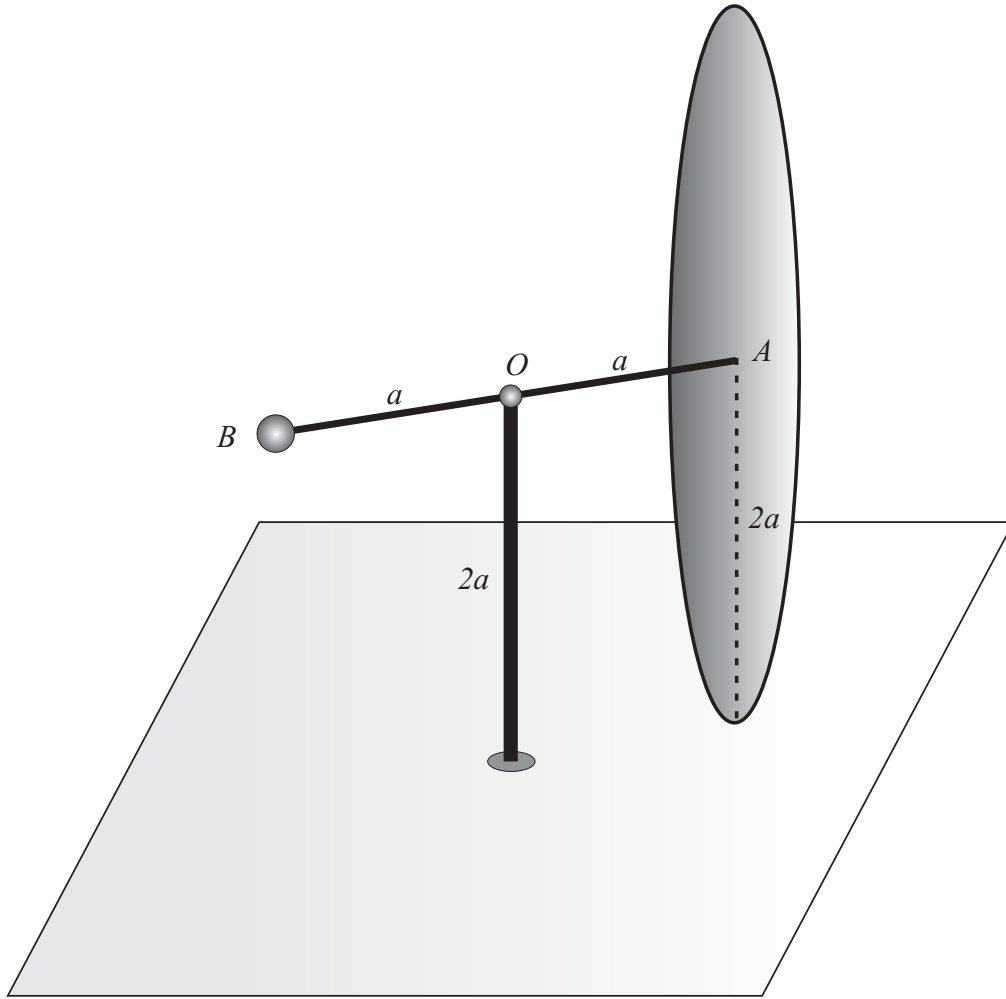


**Problem 3** A thin homogeneous rod  $AB$ , of mass  $m$  and length  $\ell$ , rotates with constant angular velocity  $\omega$  about a horizontal axis  $CD$ , that passes through the mid point  $O$  of the rod and is perpendicular to the rod. At the same time the axis  $CD$  rotates about the vertical direction through  $O$  with constant angular velocity  $\Omega$ . Determine the moment of force, as a function of time, that acts on the rod with respect to  $O$ , that gives the described motion. The answer should be given in terms of the components with respect to the body fixed system of axes  $xyz$ .



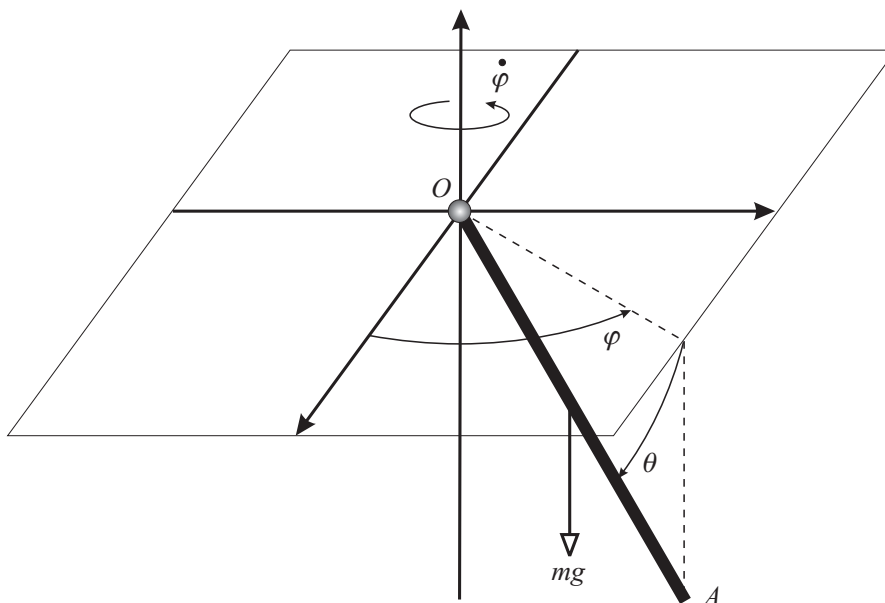
**Problem 4** The end point  $B$  of a straight homogeneous rod  $AB$  of length  $\ell$  is attached to the center of a homogeneous circular disc of radius  $r$  so that the rod is perpendicular to the disc. The other end point  $A$  of the rod is connected to a fixed point by a smooth joint, so that it can rotate freely in all directions. The disc rolls on a rough horizontal floor that lies at a distance  $r$  below  $A$ . The rod  $AB$  is thus horizontal. The mass of the disc is  $m$  and the rod is light compared to the disc.

Determine the force on the floor from the disc if  $AB$  rotates with constant angular velocity  $\omega_0$  about the vertical through  $A$ .



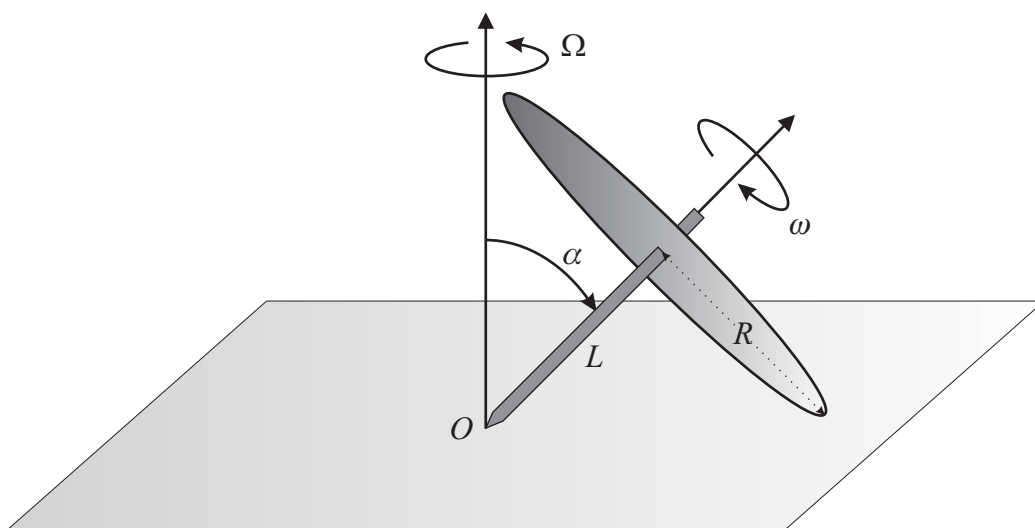
**Problem 5** A rod  $AB$  of negligible mass and of length  $2a$  can rotate freely about the fixed midpoint  $O$  and remains horizontal at all times. A particle of mass  $m$  is attached to the end point  $B$  and at the other end point  $A$  a wheel is attached. The wheel can be regarded as a homogeneous circular disc of radius  $2a$  and mass  $m$ . It can rotate with negligible friction about the axis  $AB$ , which is normal to the wheel and passes through its midpoint. The wheel is in contact with a flat rough horizontal floor. Initially the system is set in motion so that the wheel is given the angular velocity  $4\omega_0$  in the direction  $\overrightarrow{BA}$ , while the axis  $AB$  is given the angular velocity  $\omega_0$  vertically downwards. To start with the wheel slips but eventually friction forces make the wheel roll without slipping. It is assumed that the wheel is in contact with the floor at all times and the friction acts in the direction tangent to the wheel.

Calculate the angular velocities of the wheel and of the axis when slipping is over and rolling has started. Also calculate the force of the wheel on the ground in this case.

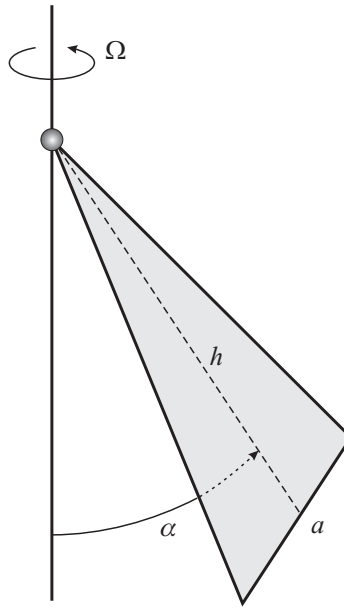


**Problem 6** A straight homogeneous rod  $OA$  (length  $2\ell$  and mass  $m$ ) can rotate freely about the fixed end point  $O$ . Initially the rod is horizontal ( $\theta = 0$ ) and rotates about the vertical direction through  $O$  with angular velocity  $\dot{\varphi} = \omega$ . Under the influence of gravity it starts to rotate about a horizontal axis through  $O$ . Calculate  $\dot{\varphi}$  as a function of  $\theta$  in the ensuing motion and calculate the turning points in the  $\theta$ -motion.

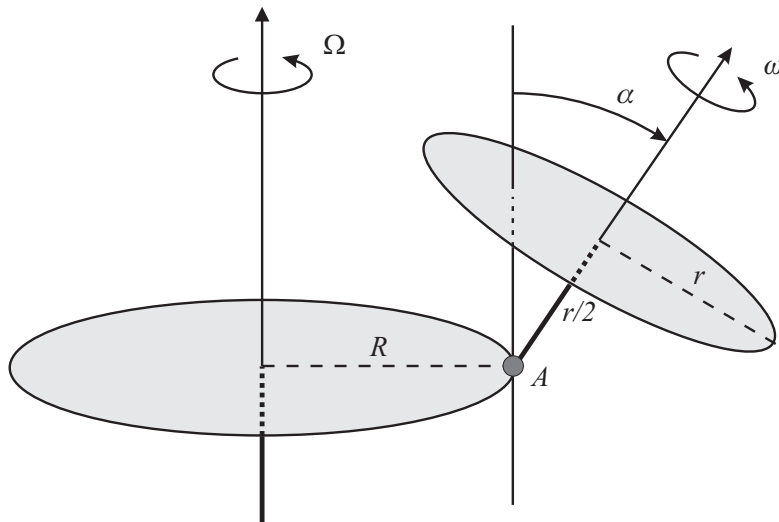
---



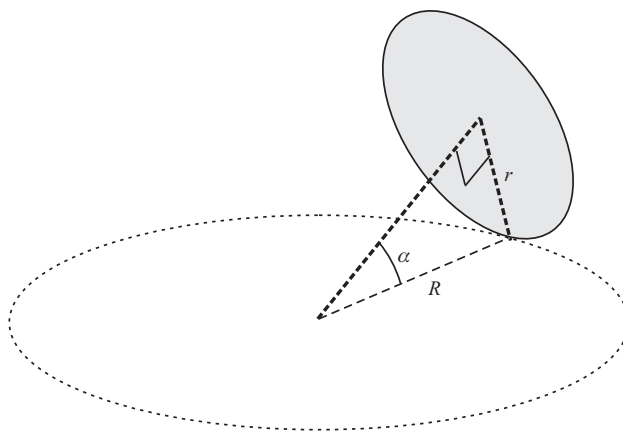
**Problem 7** A top consists of a disc of radius  $R$  and a straight light pole that constitutes the axis of the top. This axis passes through the fixed point  $O$ . The distance between  $O$  and the disc is  $L$ . The angle  $\alpha$  between the axis and the vertical remains fixed. The plane that passes through the axis of the top and the vertical through  $O$  rotates with constant angular velocity  $\Omega$  about the vertical through  $O$ . Determine the angular velocity  $\omega$  of the top relative to this plane.



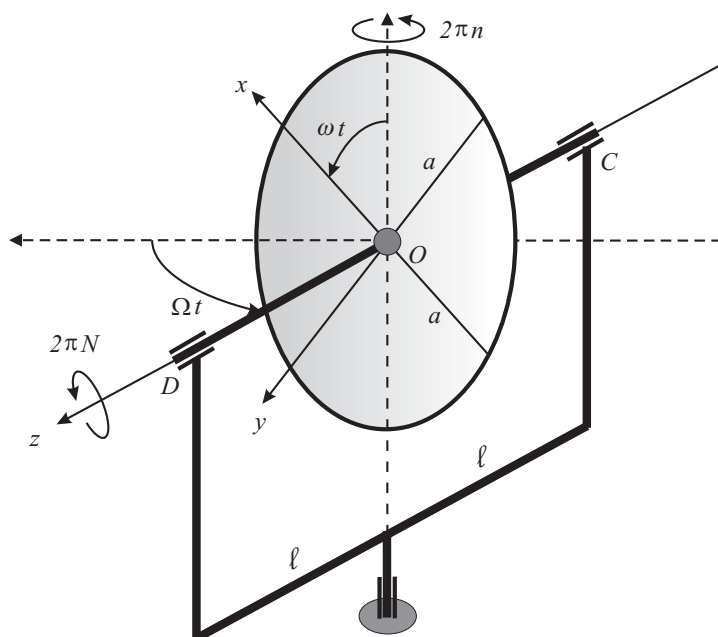
**Problem 8** A homogeneous isosceles triangular plate with base  $a$  and altitude  $h$  rotates with constant angular velocity  $\Omega$  about a vertical axis through the vertex of the triangle, where it is attached to a fixed point by means of a smooth ball and socket joint. The base of the triangle remains horizontal throughout the motion. Calculate the angle  $\alpha$  that the altitude of the triangle makes with the vertical.



**Problem 9** A top consists of of a homogeneous, circular disc of radius  $r$ , and a light axis of length  $r/2$  perpendicular to the disc through its center. The axis is connected by a joint to a point  $A$  on the periphery of a horizontal circular merry-go-round of radius  $R$ , so that the axis of the top can move with negligible friction in a vertical plane containing the axis of the merry-go-round. The merry-go-round has the constant angular velocity  $\Omega$  about its axis, and the top has the constant angular velocity  $\omega$ , relative to the merry-go-round, about its axis. Find the angle  $\alpha$  between the axis of the top and the vertical.



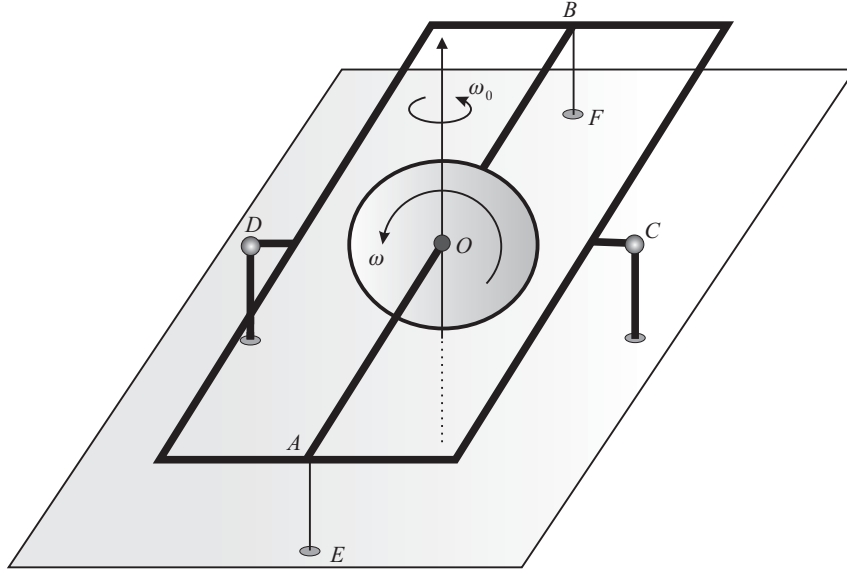
**Problem 10** A coin rolls with negligible rolling resistance on a horizontal table along a circle of radius  $R$ . The coin can be considered as a thin homogeneous disc of radius  $r$  and mass  $m$ . The plane containing the coin makes the angle  $\alpha = \arcsin(r/R)$  with the vertical plane through the line of intersection of the plane of the coin and the table surface. How much time does it take for the coin to roll once round the circle on the table?



**Problem 11** A homogeneous circular disc of mass  $M$  and radius  $a$  rotates about its horizontal axis  $CD$  with constant angular velocity  $\omega = 2\pi N$ . The axis  $CD$  itself turns (see figure) round a vertical axis passing through the mid point of the disc with constant angular velocity  $\Omega = 2\pi n$ . The disc is mounted in the middle of the axis  $CD$  which is of length  $\ell$  and of negligible weight.

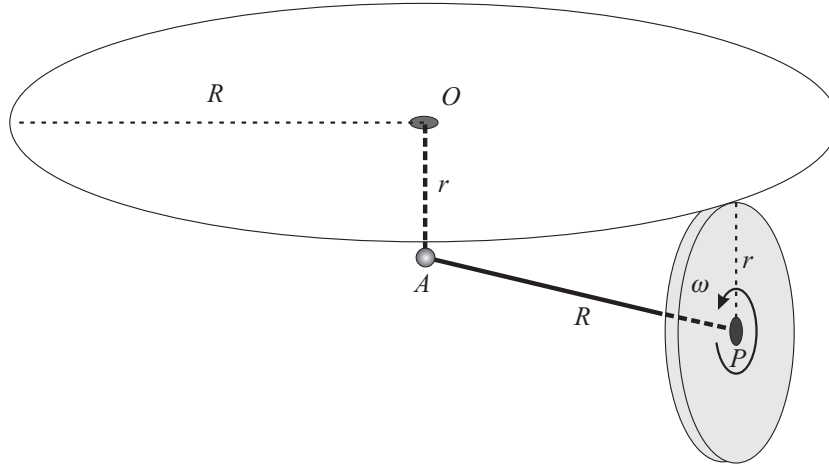
- Find the reaction forces in the bearings at  $C$  and at  $D$ .
- Find  $n$  such the reaction force in one of the bearings is zero.

Numerical values:  $\ell = a = 0.5$  m,  $N = 1800$  revolutions per minute, acceleration due to gravity  $g \approx \pi^2$  m/s<sup>2</sup>.

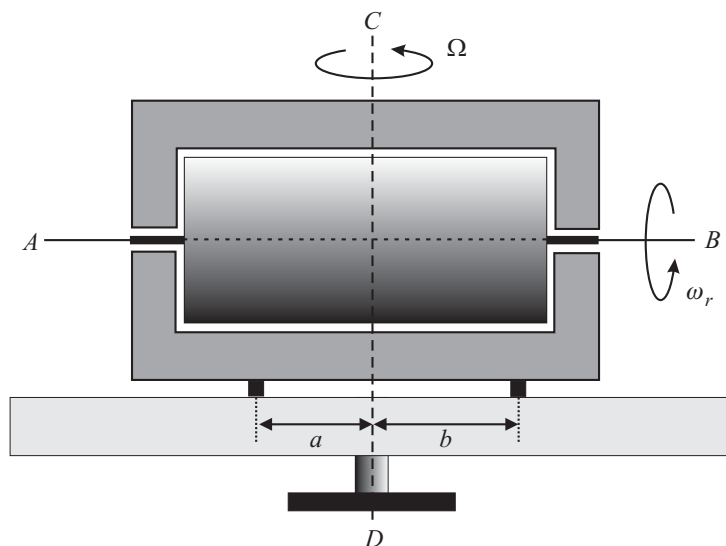


**Problem 12** The axis  $AB$  of a symmetric top is mounted in a rectangular frame that can rotate about the axis  $CD$  which is  $\perp AB$  and passes through the center of mass  $O$  of the top. The distances are  $OA = OB = L$ . The contraption is mounted on a turn table. The rotation of the frame about  $CD$  is hindered by two threads  $AE$  and  $BF$  that are  $\perp AB$  and  $CD$ . Initially these are straight but not under tension. The moment of inertia of the top is  $J$  and its angular velocity relative to the frame is  $\omega$ , counter clockwise as seen from  $A$ . If the entire contraption is turned about an axis  $\perp AB$  and  $CD$  with angular velocity  $\omega_0$ , counter clockwise as seen from above, a tension will arise in one of the threads. Which one and how large is it?

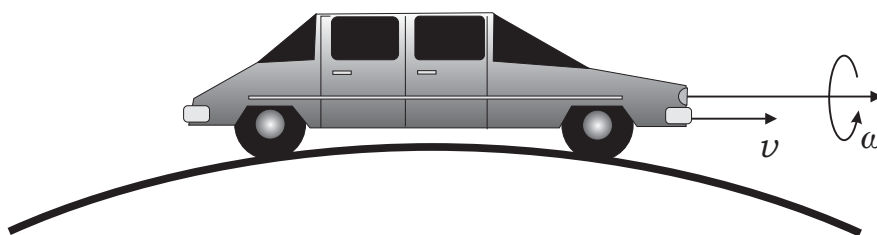
---



**Problem 13** A vertical pole of length  $r$  is attached in a horizontal ceiling. In the lower end of the pole  $A$  is attached a light rod  $AP$  of length  $R$  by means of a smooth universal joint. A circular homogeneous disc is mounted with its mid point in  $P$ , perpendicularly to  $AP$ . The disc (mass  $m$ , radius  $r$ ) rolls in the ceiling without slipping. The contact point describes a circle of radius  $R$ . The acceleration of gravity is  $g$  and the angular velocity of the disc relative to the  $OAP$  plane is  $\omega$ . Find the force that the disc exerts on the ceiling and the smallest  $\omega$  needed to sustain contact.

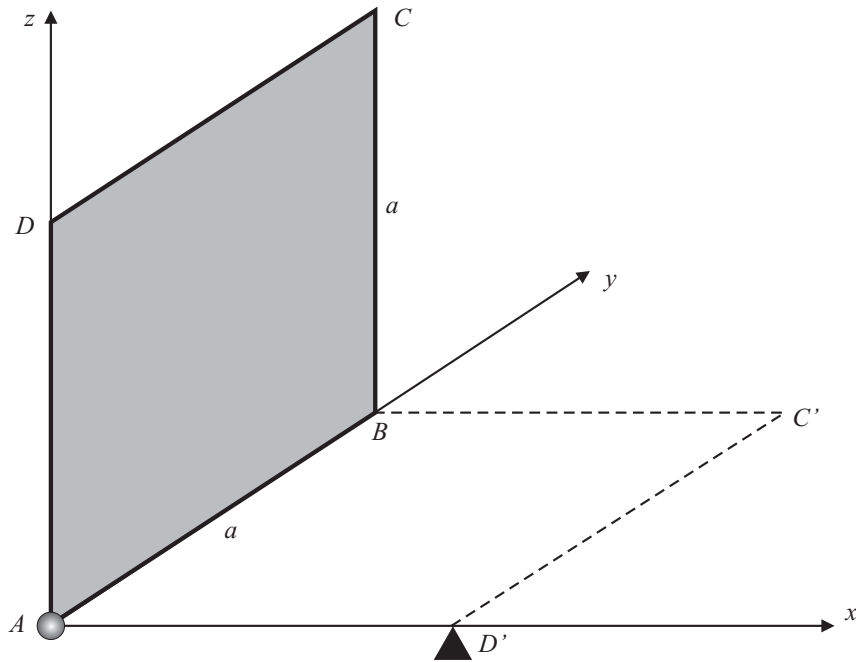


**Problem 14** An electric motor of weight  $mg$  rests on two narrow strips which are attached to a horizontal table that turns with constant angular velocity  $\Omega$ , counter clockwise as seen from above, about a vertical axis  $CD$ . This axis intersects the rotation axis  $AB$  of the motor. The strips are perpendicular to  $AB$  and at distances  $a$  and  $b$  from  $CD$ , see Figure. The rotor turns relative to the stator with constant angular velocity  $\omega_r$  counter clockwise as seen from  $B$ . The stator is symmetric with respect to the vertical plane through  $AB$  and with respect to the perpendicular plane through  $CD$ . The center of mass of the rotor is in the intersection of  $AB$  and  $CD$  and the ellipsoid of inertia is rotationally symmetric around  $AB$ . The moment of inertia of the rotor with respect to  $AB$  is  $J$ . Find the maximum allowed  $\Omega$  for which the motor remains at rest on the strips?

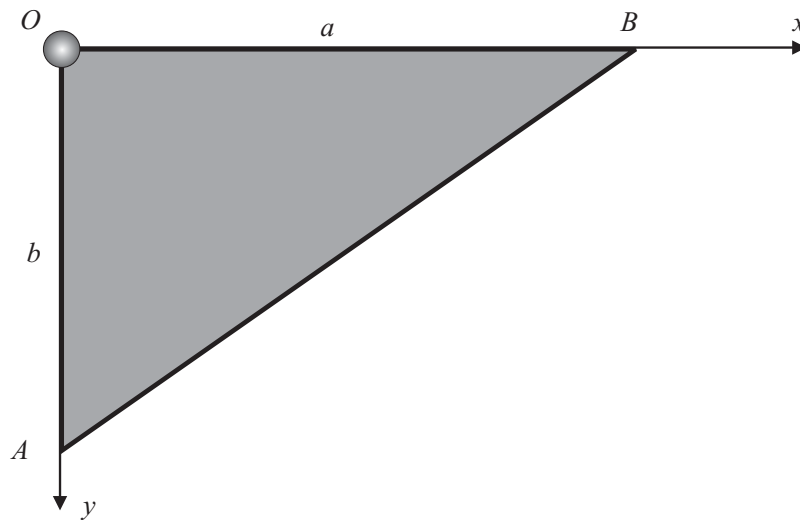


**Problem 15** A fast car with a gas turbine engine drives along a straight road and passes a hill top with speed  $v$ . The hill top has radius of curvature  $\rho$ . The axis of the gas turbine is oriented along the length of the car. Its rotor has moment of inertia  $J$  with respect to the turbine axis, and angular velocity  $\omega$  relative to the car, so that the angular velocity vector  $\boldsymbol{\omega}$  points in the forward direction. The driver notices that the car has a tendency to turn to sideways.

- In which direction and how much torque is required to keep the car from turning?
- Find the magnitude of the torque in SI-units if  $v = 30 \text{ m/s}$ ,  $\rho = 100 \text{ m}$ ,  $J = 1 \text{ kg m}^2$  and  $\omega = 2000 \text{ radians per second}$ !

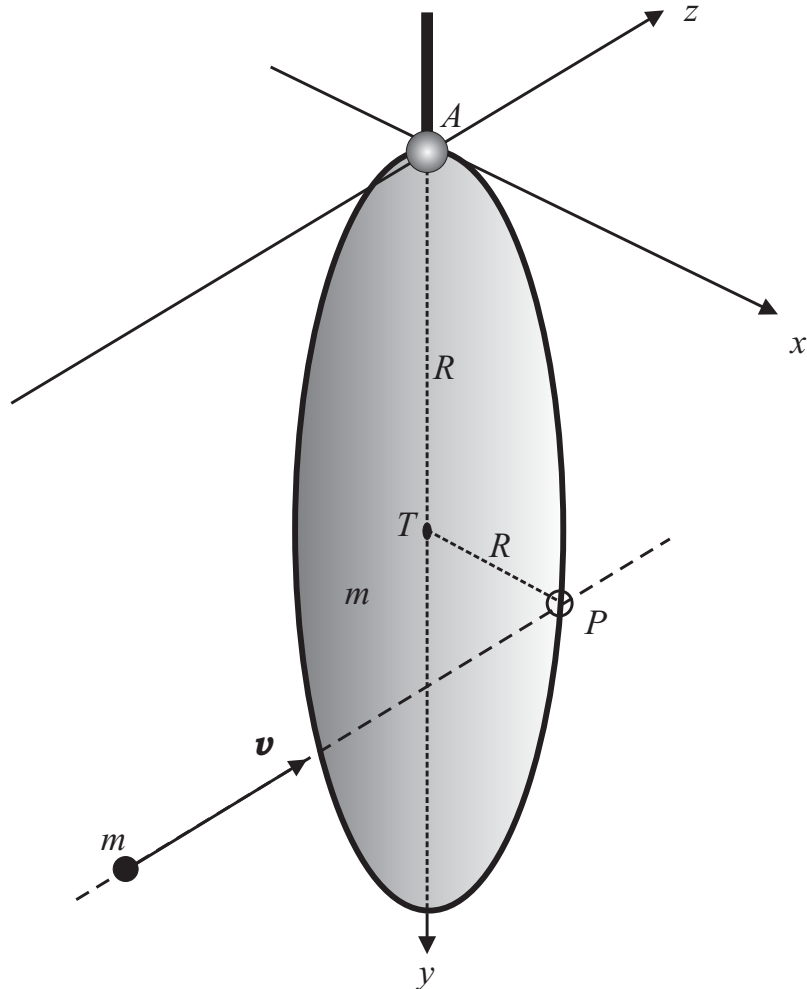


**Problem 16** A quadratic homogeneous hatch  $ABCD$  with edges of length  $a$  can rotate freely about the horizontal edge  $AB$ . The hatch is initially standing vertically but is slightly perturbed and falls. When  $AD$  becomes horizontal the corner  $D$  hits a fixed point  $D'$  and gets stuck. It is assumed that the corner  $B$  simultaneously comes loose (without impact forces), and that the hatch starts to rotate about  $AD'$  with an angular velocity  $\omega_x$ . Find this angular velocity  $\omega_x$ .



**Problem 17** A homogenous thin plate in the form of a right triangle  $AOB$  of mass  $m$  can rotate freely about a fixed ball and socket joint at the right angle  $O$ . The perpendicular edges have lengths  $OB = a$  and  $OA = b$ . Initially the point  $A$  is kept still, and the plate rotates with angular velocity  $\omega$  about  $OA$ . The corner  $A$  is suddenly released, and the corner  $B$  is fixed instead.

- Find the new angular velocity  $\omega'$  about  $OB$  immediately after  $B$  becomes fixed!
- How large is the impulse  $S_z$  that acts at  $B$ ?

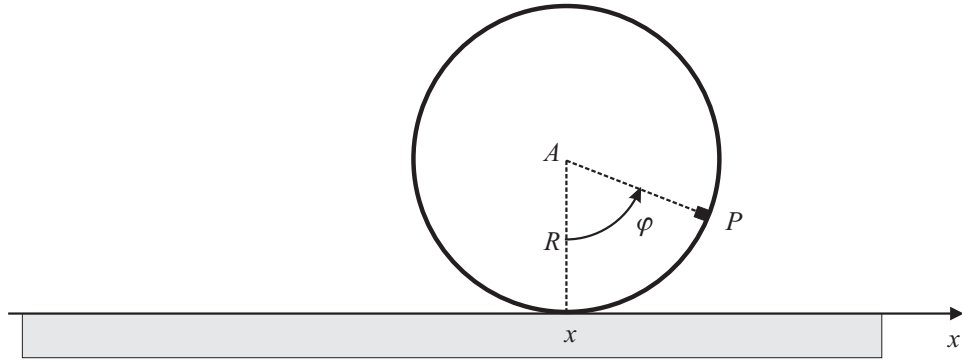


**Problem 18** A thin homogenous circular disc of radius  $R$  and mass  $m$  hangs at rest from the fixed point  $A$  on its periphery. It can rotate freely about a smooth universal joint at  $A$ . A particle of mass  $m$ , and speed  $v$  perpendicular to the disc, hits the disc in a point  $P$  on the periphery and gets stuck there. The point  $P$  is the rightmost point on the disc and is at the same height as the center  $T$  of the disc.  $ATP$  is thus a right angle.

- Determine the magnitude and the direction of the angular velocity immediately after the impact.
- Find the reaction impulse at  $A$ .

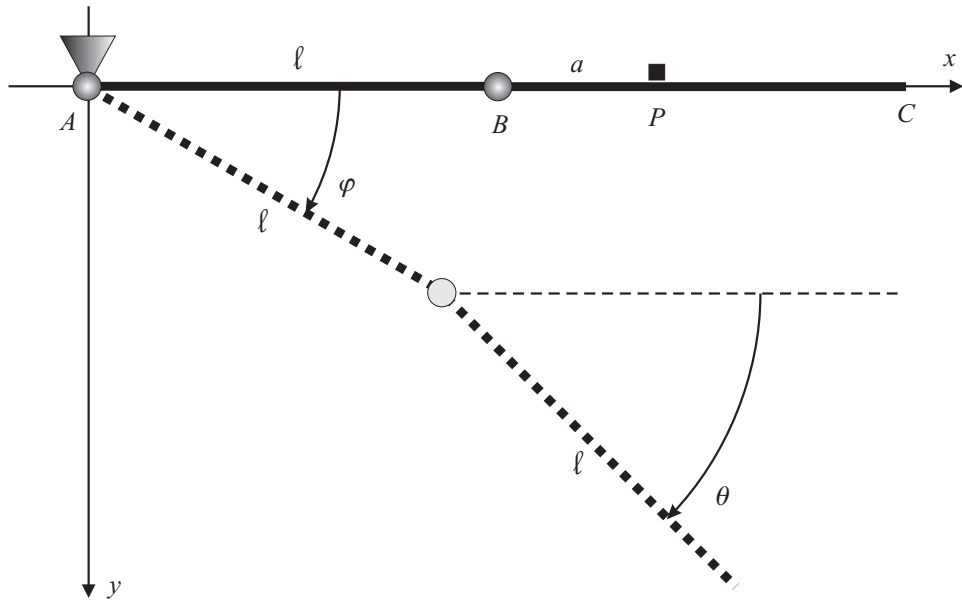
## 2 Problems on the Lagrange's method

---

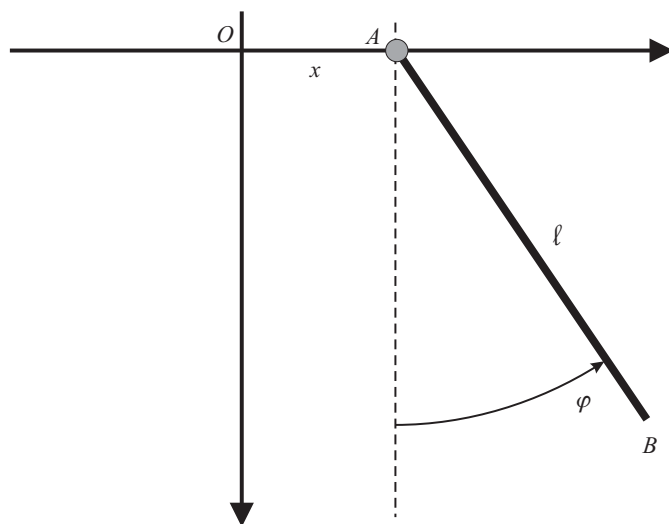


**Problem 19** A straight, circular, cylindrical shell of mass  $M$  and radius  $R$  can roll without slipping on a horizontal surface. Inside the shell slides a particle  $P$  of mass  $m$ , with negligible friction. Denote by  $\varphi$  the angle between the vertical and  $AP$ , the line from the axis of the cylinder to the particle. The system is released from rest with the particle at  $\varphi = \pi/2$ . Calculate the position  $x$  of the axis of the cylinder, assuming  $x = 0$  initially, as a function of  $\varphi$ .

---

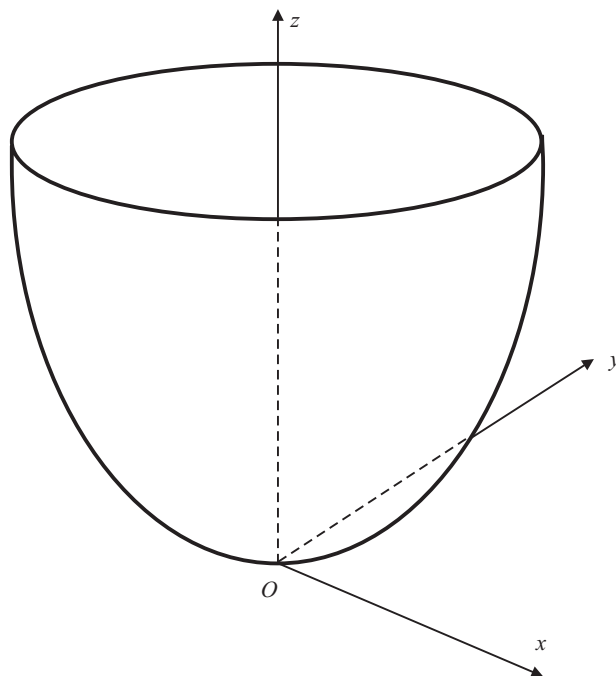


**Problem 20** Two identical homogeneous rods  $AB$  and  $BC$  of length  $\ell$  are connected by a smooth joint at  $B$ . The end point  $A$  can rotate about a fixed smooth joint. Initially the rods are held at rest in a straight horizontal line through  $A$ . A small light particle is placed in a point  $P$  on the upper side of  $BC$ . The system is then released, from rest, and starts to fall. At what distance  $a$  from  $B$  can the particle be placed, if it is not to lose contact with the rod immediately?

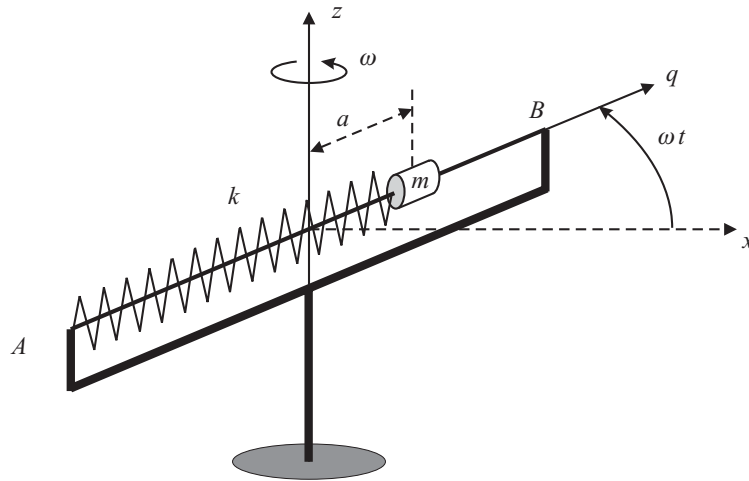


**Problem 21** The end point  $A$  of a rod  $AB$  can move with negligible friction along a straight horizontal track (the  $x$ -axis). The rod is thin and homogeneous. Its length is  $\ell$  and its mass  $m$ . It moves in a vertical plane where it can rotate freely about  $A$ . The point  $A$  is acted on by a periodic force  $F_x = \frac{mg}{3} \cos(\omega t)$ , where  $\omega^2 = g/\ell$ , directed along the  $x$ -axis.

- Find the equations of motion for this pendulum!
- Determine its motion assuming that the angle  $\varphi$  and the angular velocity  $\dot{\varphi}$  always remain small. Use the initial values (at  $t = 0$ ):  $x(0) = \dot{x}(0) = \varphi(0) = \dot{\varphi}(0) = 0$ , where  $x$  is  $A$ 's position.



**Problem 22** A bowl has the shape of an elliptic paraboloid given by the equation  $7(x^2 + y^2) - 2xy = 24cz$ . It is fixed with vertical  $z$ -axis upwards, so that a particle can stay at rest at the origin  $O$ . The inside surface of the bowl is smooth. Find the angular frequencies  $\omega_1$  and  $\omega_2$  of the two eigen-modes of the particle as it oscillates near  $O$ .



**Problem 23** A weight of mass  $m$  can slide along a straight smooth horizontal track  $AB$ . The weight and the end point  $A$  are connected by spiral spring, of negligible mass, with stiffness  $k$ . As shown in the Figure the equilibrium position of the weight is a distance  $a$  beyond the midpoint of the track as seen from  $A$ .

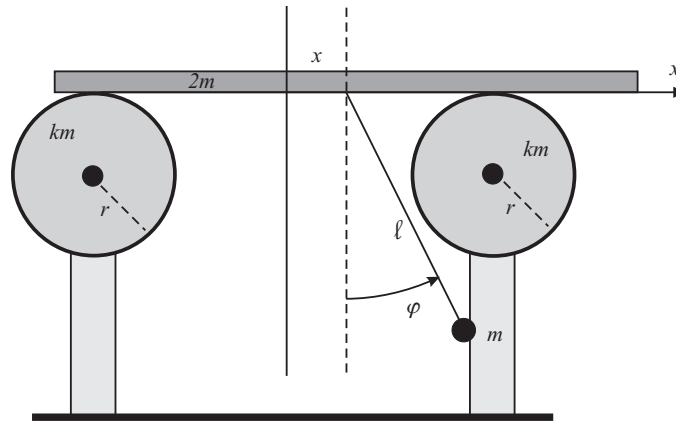
The track is mounted in such a way that it can rotate about a vertical axis through the mid point of the track.

a) Calculate the angular frequency  $\Omega$  of oscillations of the weight when the track is fixed.

Now assume that the track is rotated with constant angular velocity  $\omega$ . Find:

b) the new equilibrium position of the weight.

c) the new frequency  $\nu$  for small oscillations expressed in terms of  $\Omega$ .



**Problem 24** A horizontal rectangular plate of mass  $2m$  rests on two identical circular cylinders, which can rotate freely about their fixed parallel axes. The plate can roll on the cylinders without slipping and without rolling resistance. Two sides of the plate remain parallel to the cylinder axes at all times. The cylinders are homogeneous, and have radius  $r$  and mass  $km$ . A simple pendulum of mass  $m$  and length  $\ell$  is suspended from the mid point of the plate. Find Lagrange's equations of motion for the motion of the system and integrate them for small oscillations of the pendulum. Initially the system is at rest, the mid point of the plate at the mid point between the cylinders,  $x(0) = 0$ , and the pendulum makes the angle  $\varphi(0) = \alpha$  with the vertical.

### 3 Answers and hints

#### Answer 1

Note that the angular velocity vector is parallel to the line of contact between the cone and the horizontal plane. The kinetic energy is,

$$T = \frac{3m\omega^2}{40} \frac{6h^2 + r^2}{h^2 + r^2} r^2.$$

#### Answer 2

Note that the kinetic energy is the same for all directions of the rotation axis through the center of mass; the cube is a spherical top. The kinetic energy is,

$$T = \frac{1}{12} ma^2 \omega^2.$$

#### Answer 3

The components of the moment of force are:

$$\begin{aligned} M_x &= -\frac{m\ell^2}{12} \Omega^2 \sin \omega t \cos \omega t, \\ M_y &= \frac{m\ell^2}{6} \omega \Omega \cos \omega t, \\ M_z &= 0. \end{aligned}$$

#### Answer 4

The force becomes,

$$F = mg + \frac{mr\omega_0^2}{2}.$$

#### Answer 5

The angular velocity of the wheel becomes  $\omega_0$ .

The axis  $BA$  ends up with the angular velocity  $2\omega_0$  vertically downwards.

The force on the ground becomes  $F = 4ma\omega_0^2$ .

#### Answer 6

Use the fact that the vertical component of the angular momentum is conserved. Using this one obtains:

$$\dot{\varphi}(\theta) = \frac{\omega}{\cos^2 \theta}.$$

Use of the conservation of energy and the fact that the turning points in the  $\theta$ -motion occur when  $\dot{\theta} = 0$  one obtains that the turning points are given by  $\theta_{\min} = 0$  and

$$\theta_{\max} = \arcsin \left( \frac{\ell\omega^2}{3g} \left[ \sqrt{1 + \left( \frac{3g}{\ell\omega^2} \right)^2} - 1 \right] \right).$$

#### Answer 7

This system is called the precessing heavy symmetric top. The result should be,

$$\omega = \frac{2Lg}{R^2\Omega} + \left[ 2 \left( \frac{L}{R} \right)^2 - \frac{1}{2} \right] \Omega \cos \alpha.$$

Note that if  $\omega \gg 1$  one must necessarily have  $\Omega \ll 1$  and the second term in the expression can then be neglected compared to the first. For certain values of the parameters  $R$  and  $L$  the second term is identically zero. What does this mean?

**Answer 8**

You need the moments of inertia of a triangle. One find that the angle can be either  $\alpha = 0$  or,

$$\alpha = \arccos\left(\frac{4g}{3h\Omega^2}\right).$$

**Answer 9**

The angle is,

$$\alpha = \arctan\left(\frac{R\Omega^2}{r\omega\Omega - g}\right).$$

**Answer 10**

Note that the coin rolls in such a way that a point, rigidly connected to the coin, and situated on its axis, is fixed. This is the point in the middle of the circle on which it rolls. Be careful with the signs in the rolling constraints.

The period becomes,

$$T = \pi \sqrt{\frac{r}{g} \frac{4 \cot^2 \alpha \cos \alpha \sin \alpha + 2 \cot \alpha - \cos \alpha \sin \alpha}{\sin \alpha}}.$$

Alternatively this can be written,

$$T = \frac{\pi}{\sqrt{g}} \sqrt{6R \cot \alpha - 5r \cos \alpha},$$

provided  $R$  is not expressed in terms of  $r$  and  $\alpha$ .

**Answer 11**

a) The forces on the bearings become,

$$N_{C,D} = \frac{M}{2} \left( g \pm \frac{2\pi^2 n N a^2}{\ell} \right).$$

b) The force is zero on one of the bearings when,

$$n = \frac{g\ell}{2\pi^2 N a^2}.$$

Using the numerical values the force is zero when  $n = 2$  revolutions/minute.

**Answer 12**

The thread  $AE$  acquires the tension,

$$S_{AE} = \frac{J\omega\omega_0}{L}.$$

**Answer 13**

The force on the disc is,

$$N = \frac{mr^3\omega^2}{2R^2} - mg,$$

and the angular velocity must fulfill,

$$\omega^2 \geq \frac{2gR^2}{r^3}.$$

**Answer 14**

The largest angular velocity that is allowed is,

$$\Omega_{\max} = \frac{mga}{J\omega}.$$

**Answer 15**

a) The car tends to turn to the left and the torque is

$$M = \frac{Jv\omega}{\rho}.$$

b) With the given data the torque becomes  $M = 600 \text{ Nm}$ .

**Answer 16**

The angular velocity is,

$$\omega_x = -\frac{3}{4}\sqrt{\frac{3g}{a}}.$$

**Answer 17**

a) The angular velocity is,

$$\omega' = -\frac{a}{2b}\omega.$$

b) The impulse is

$$S_z = \frac{ma\omega}{8}.$$

**Answer 18**

The angular velocity vector immediately after the impact is,

$$\boldsymbol{\omega} = \frac{v}{29R}(4\mathbf{e}_x - 20\mathbf{e}_y).$$

The reaction impulse at  $A$  is given by,

$$\mathbf{S} = -\frac{mv}{29}\mathbf{e}_z.$$

**Answer 19**

One finds the relation,

$$x = \frac{mR}{2M+m}(1 - \sin \varphi),$$

for the position of the cylinder as a function of the position angle of the particle.

**Answer 20**

The particle maintains contact with the rod if the acceleration downwards is less than  $g$  at its position. Note that you only need the accelerations at the initial moment, *i.e.* when  $\varphi = \theta = \dot{\varphi} = \dot{\theta} = 0$ . Using this one finds that

$$a \geq 2\ell/3$$

must hold if contact survives.

**Answer 21**

a) Using Lagrange's method the equations of motion become

$$\begin{aligned}\ddot{x} + \frac{1}{2}\ell\ddot{\varphi}\cos\varphi - \frac{1}{2}\ell\dot{\varphi}^2\sin\varphi &= \frac{1}{3}g\cos\omega t, \\ \ddot{x}\cos\varphi + \frac{2}{3}\ell\ddot{\varphi} &= -g\sin\varphi.\end{aligned}$$

b) Linearization and solution gives the result:

$$\begin{aligned}\varphi &= \frac{2}{5}\left[\cos\left(\sqrt{6g/\ell}t\right) - \cos\left(\sqrt{g/\ell}t\right)\right], \\ x &= \frac{\ell}{3} - \frac{\ell}{5}\left[\cos\left(\sqrt{6g/\ell}t\right) + \frac{2}{3}\cos\left(\sqrt{g/\ell}t\right)\right].\end{aligned}$$

**Answer 22**

Note that the contribution to the kinetic energy from  $\dot{z}$  can be neglected near the equilibrium. Using the theory for couple oscillations one obtains:

$$\omega_1^2 = \frac{g}{2c}, \quad \omega_2^2 = \frac{2g}{3c}.$$

**Answer 23**

a) The angular frequency when the track is not rotating is,

$$\Omega = \sqrt{\frac{k}{m}}.$$

b) The new equilibrium position when the track rotates is,

$$q_0 = a\frac{\Omega^2}{\Omega^2 - \omega^2}.$$

c) For the new angular frequency  $\Omega'$  one obtains  $\Omega'^2 = \Omega^2 - \omega^2$ . The frequency is then,  $\nu = \Omega'/2\pi$ . The frequency for small oscillations near equilibrium is thus,

$$\nu = \frac{1}{2\pi}\sqrt{\Omega^2 - \omega^2}.$$

**Answer 24**

The angle of the pendulum is given by

$$\varphi = \alpha \cos \omega t$$

and the position of the mid point of the plate is,

$$x = \frac{\alpha\ell}{3+k}(1 - \cos \omega t)$$

where,

$$\omega^2 = \frac{3+k}{2+k} \frac{g}{\ell}.$$