

21st Century Mechanics
Motion in accelerated reference frames

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Chapter 1

Motion in Accelerated Reference Frames

Until now we have studied motion in inertial reference frames. We have done this because such frames allows one to identify physical causes (forces) that produce the accelerations. These forces have identifiable sources such as a large mass producing gravity, matter producing contact forces, or charged particles producing electromagnetic fields. In many situations it may still be advantageous to use a non-inertial reference frame, i.e. one that is accelerated. If one knows how it accelerates one can find mathematically ‘fictitious’ forces that must be added to the real (physical) forces in order for the equation of motion in the accelerated frame to give correct results. The purpose of these fictitious forces is simply to give the particle the acceleration that it would have relative to the accelerated frame even if no real forces acted on it.

1.1 Kinematics in an Accelerated Reference Frame

The problem of describing the motion of a particle in an accelerated frame of reference is one of kinematics. Consider the position vector of a particle \mathcal{P} in a coordinate system, with origin at \mathcal{O} , fixed in an inertial reference frame \mathcal{O} . This position vector can be written in one of the following forms:

$$\mathbf{r} = \overline{\mathcal{OP}} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z. \quad (1.1)$$

We now introduce an accelerating reference frame \mathcal{A} and a coordinates system fixed in this frame. We denote the origin of this coordinate system \mathcal{A} . The position vector of the particle \mathcal{P} is then

$$\mathbf{r}^{\mathcal{A}} = \overline{\mathcal{AP}} = x^{\mathcal{A}} \mathbf{e}_x^{\mathcal{A}} + y^{\mathcal{A}} \mathbf{e}_y^{\mathcal{A}} + z^{\mathcal{A}} \mathbf{e}_z^{\mathcal{A}}. \quad (1.2)$$

Here the basis vectors $\mathbf{e}_x^{\mathcal{A}}, \mathbf{e}_y^{\mathcal{A}}, \mathbf{e}_z^{\mathcal{A}}$ represent directions fixed in the moving system \mathcal{A} . We will be interested in how the kinematic quantities position, velocity, and acceleration in the moving system \mathcal{A} relate to the corresponding quantities in the fixed system \mathcal{O} . For the position vector we have $\overline{\mathcal{OP}} = \overline{\mathcal{OA}} + \overline{\mathcal{AP}}$ so, if we put $\overline{\mathcal{OA}} = \mathbf{R}$, we get

$$\mathbf{r} = \mathbf{R} + \mathbf{r}^{\mathcal{A}}, \quad (1.3)$$

see figure 1.1. We now wish to take time derivatives of this relation in order to find relations between velocities and accelerations. It is then important to keep in mind that the motion and thus the time dependence of the vectors will depend on from which reference frame the measurements are made.

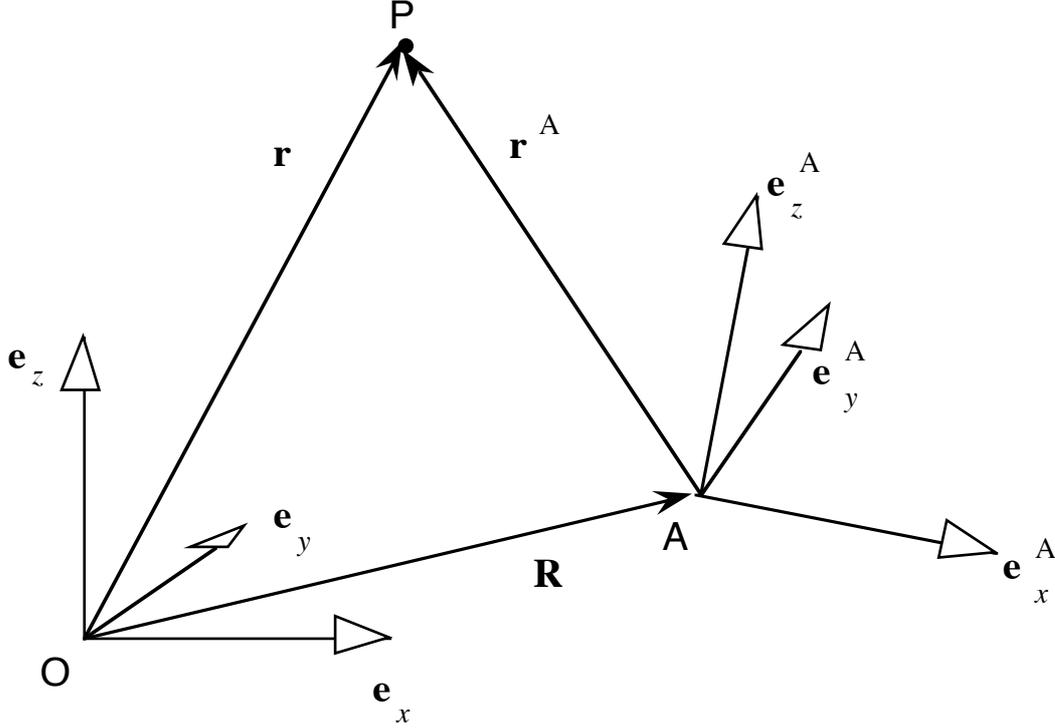


Figure 1.1: This figure illustrates the relation between the position vector, \mathbf{r} , of a particle at \mathcal{P} with respect to a fixed system O and the position vector, \mathbf{r}^A , of the same particle with respect to another, moving system A. This relation is also described by equation 1.3.

1.1.1 The Time Derivative of a Vector in a Rotating Reference Frame

In the fixed O-system, the point \mathcal{O} and the basis vectors \mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z , are at rest and time independent whereas the vectors \mathbf{r} and \mathbf{R} depend on time in way which depends on how the points \mathcal{P} and \mathcal{A} , respectively, move relative to the fixed frame. Also the basis vectors, \mathbf{e}_x^A , \mathbf{e}_y^A , \mathbf{e}_z^A , fixed in the A-system, will move and thus be time dependent if the moving frame rotates relative to the fixed frame. The time derivatives of the moving basis vectors are then given by

$$\dot{\mathbf{e}}_\lambda^A = \frac{d\mathbf{e}_\lambda^A}{dt} = \boldsymbol{\omega} \times \mathbf{e}_\lambda^A \quad \lambda = x, y, z, \quad (1.4)$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the moving frame as measured in the fixed frame¹. In this formula $\frac{d}{dt}$, or the over-dot, stands for a time derivative as measured from the fixed O-system. Time derivatives that refer to a time dependence observed from the A-system will be denoted by $\frac{{}^A d}{dt}$, or by a small circle instead of the dot. Using this notation we have

$$\overset{\circ}{\mathbf{e}}_\lambda^A = \frac{{}^A d\mathbf{e}_\lambda^A}{dt} = \mathbf{0} \quad \lambda = x, y, z, \quad (1.5)$$

since these vectors are constant (fixed) in the A-system.

Consider an arbitrary vector

$$\mathbf{b} = b_x \mathbf{e}_x + b_y \mathbf{e}_y + b_z \mathbf{e}_z = b_x^A \mathbf{e}_x^A + b_y^A \mathbf{e}_y^A + b_z^A \mathbf{e}_z^A. \quad (1.6)$$

What is the relationship between the time derivative of this vector as observed in the fixed system, $\dot{\mathbf{b}}$, and the derivative as measured in the moving system, $\overset{\circ}{\mathbf{b}}$? In order to

¹This result is derived in other chapters of this course.

find this relation one must note that the components of the vector are given by the scalar products

$$b_\lambda(t) = \mathbf{b} \cdot \mathbf{e}_\lambda, \quad b_\lambda^A(t) = \mathbf{b} \cdot \mathbf{e}_\lambda^A, \quad \lambda = x, y, z, \quad (1.7)$$

and thus are well defined real (scalar) functions of the time. For the time dependence of the components there is thus no need to distinguish between the two reference frames and we can use the overdot (or $\frac{d}{dt}$) to denote time derivative. The time derivative of the vector \mathbf{b} as measured in the fixed system is then

$$\dot{\mathbf{b}} = \dot{b}_x \mathbf{e}_x + \dot{b}_y \mathbf{e}_y + \dot{b}_z \mathbf{e}_z \quad (1.8)$$

and the corresponding quantity as measured in the A-system is

$$\overset{\circ}{\mathbf{b}} = \dot{b}_x^A \mathbf{e}_x^A + \dot{b}_y^A \mathbf{e}_y^A + \dot{b}_z^A \mathbf{e}_z^A. \quad (1.9)$$

If we instead take the time derivative of the vector in the fixed system but use the expansion in the moving basis we get (using equation 1.4)

$$\begin{aligned} \dot{\mathbf{b}} &= \frac{d}{dt} \left(b_x^A \mathbf{e}_x^A + b_y^A \mathbf{e}_y^A + b_z^A \mathbf{e}_z^A \right) = \\ &= \left(\dot{b}_x^A \mathbf{e}_x^A + \dot{b}_y^A \mathbf{e}_y^A + \dot{b}_z^A \mathbf{e}_z^A \right) + \left(b_x^A \dot{\mathbf{e}}_x^A + b_y^A \dot{\mathbf{e}}_y^A + b_z^A \dot{\mathbf{e}}_z^A \right) = \\ &= \overset{\circ}{\mathbf{b}} + \boldsymbol{\omega} \times \left(b_x^A \mathbf{e}_x^A + b_y^A \mathbf{e}_y^A + b_z^A \mathbf{e}_z^A \right). \end{aligned} \quad (1.10)$$

We thus have the important relation

$$\boxed{\dot{\mathbf{b}} = \overset{\circ}{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}} \quad (1.11)$$

connecting the time derivatives in the two systems.

1.1.2 Velocity in an Accelerated Reference Frame

Let us now go back to equation 1.3 and use it to find the relation between the velocity relative to the two systems O and A. Differentiation with respect to time in the fixed O-system gives (using equation 1.11)

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\mathbf{r}}^A = \dot{\mathbf{R}} + \overset{\circ}{\mathbf{r}}^A + \boldsymbol{\omega} \times \mathbf{r}^A. \quad (1.12)$$

The velocity relative to the moving A-system is by definition

$$\mathbf{v}^A \equiv \frac{{}^A d}{dt} \mathbf{r}^A = \overset{\circ}{\mathbf{r}}^A \quad (1.13)$$

so we get that the relation

$$\mathbf{v} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}^A + \mathbf{v}^A \quad (1.14)$$

between the two velocities.

This relation between the velocities is very important for understanding the relationship between the motions, as seen from the two reference frames, so we will pause to analyze it. If there is no relative rotation, so that $\boldsymbol{\omega} = \mathbf{0}$, then we simply get $\mathbf{v} = \dot{\mathbf{R}} + \mathbf{v}^A$. This tells us that the velocity, of the particle \mathcal{P} , in the fixed O-system is simply the (translational) velocity $\dot{\mathbf{R}}$, of the A-system relative to the O-system, plus the velocity \mathbf{v}^A relative to the A system. This is the law of addition of velocities which holds in classical mechanics²

²It does not hold when the speeds are comparable to that of light.

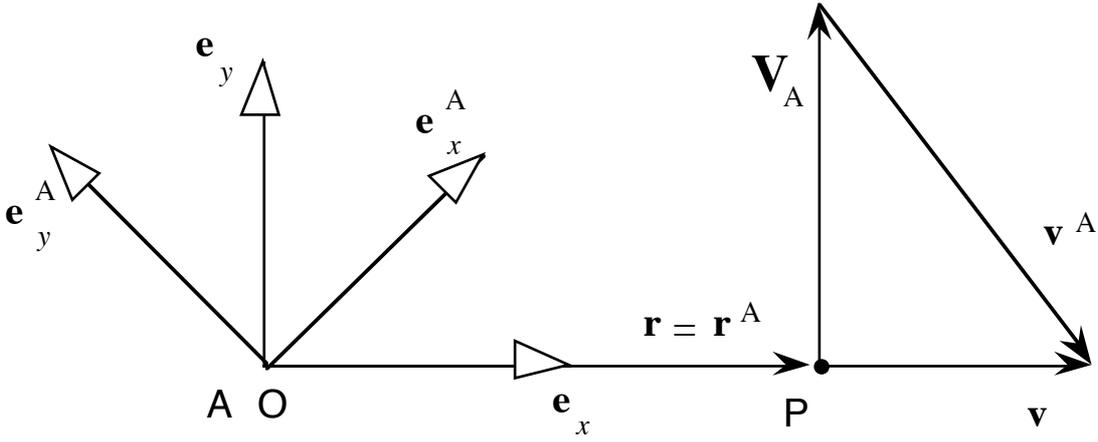


Figure 1.2: The moving A-system is assumed to rotate with angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z$, i.e. around the positive Z-axis (perpendicular to the plane of the figure). The velocity of a point fixed in the A-system is then $\mathbf{V}_A(\mathbf{r}^A) = \boldsymbol{\omega} \times \mathbf{r}^A$. If the particle \mathcal{P} moves along the positive X-axis in the fixed system (velocity \mathbf{v}) its velocity in the moving system will be given by \mathbf{v}^A as shown in the figure.

Assume that the particle \mathcal{P} is at rest in the moving frame A. This is the case, for example, if the moving frame is attached to a rigid body and \mathcal{P} is one of the particles of this body. In this case $\mathbf{v}^A = \mathbf{0}$ so that

$$\mathbf{v} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}^A \equiv \mathbf{V}_A(\mathbf{r}^A) \quad (1.15)$$

is the ‘system point velocity’, i.e. the velocity of a point fixed in the A-system with position vector \mathbf{r}^A . Using this notation we can write the result of equation 1.14 on the form

$$\mathbf{v} = \mathbf{V}_A(\mathbf{r}^A) + \mathbf{v}^A. \quad (1.16)$$

We again have a law of addition of velocities: the velocity of \mathcal{P} in the O-system is the vector sum of the velocity \mathbf{V}_A of the fixed point of the A-system that coincides with \mathcal{P} and the velocity \mathbf{v}^A of \mathcal{P} relative to the A-system. This is illustrated for a special case in figure 1.2 discussed in the example below.

Example 1.1 A particle \mathcal{P} moves along the positive X-axis with velocity $\dot{\mathbf{r}} = \mathbf{v} = \dot{x}(t) \mathbf{e}_x$ in a fixed coordinate system O. Introduce a rotating coordinate system A with the same (fixed) origin as the O-system and with Z-axis coinciding with that of the fixed system. Let its angular velocity vector be $\boldsymbol{\omega} = \omega \mathbf{e}_z$ relative to the fixed system. Calculate the velocity \mathbf{v}^A of \mathcal{P} relative to the moving system.

Solution: Since the fixed and moving system have the same fixed origin $\mathbf{R} = \dot{\mathbf{R}} = \mathbf{0}$, $\mathcal{O} = \mathcal{A}$, and $\mathbf{r} = \mathbf{r}^A$, see figure 1.2. The ‘system point velocity’ is then

$$\mathbf{V}_A(\mathbf{r}^A) = \boldsymbol{\omega} \times \mathbf{r}^A = \boldsymbol{\omega} \times \mathbf{r} = \quad (1.17)$$

$$= \omega \mathbf{e}_z \times x(t) \mathbf{e}_x = \omega x(t) \mathbf{e}_y. \quad (1.18)$$

Equation 1.16 then gives

$$\mathbf{v}^A = \mathbf{v} - \mathbf{V}_A = \dot{x}(t) \mathbf{e}_x - x(t) \omega \mathbf{e}_y. \quad (1.19)$$

This result is illustrated in figure 1.2. \square

1.1.3 Coriolis’ Theorem

We now proceed to find the relation between the accelerations \mathbf{a} and \mathbf{a}^A relative to the fixed O-system and the moving A-system respectively. We start by taking the time derivative, in the O-system, of equation 1.14. This gives

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}^A + \boldsymbol{\omega} \times \dot{\mathbf{r}}^A + \dot{\mathbf{v}}^A. \quad (1.20)$$

We now use equation 1.11 to replace the O-system time derivatives of \mathbf{r}^A and \mathbf{v}^A by expressions with time derivatives that refer to the A-system. This gives us

$$\mathbf{a} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}^A + \boldsymbol{\omega} \times \left(\dot{\mathbf{r}}^A + \boldsymbol{\omega} \times \mathbf{r}^A \right) + \left(\dot{\mathbf{v}}^A + \boldsymbol{\omega} \times \mathbf{v}^A \right). \quad (1.21)$$

Since $\dot{\mathbf{r}}^A = \mathbf{v}^A$ and $\dot{\mathbf{v}}^A = \mathbf{a}^A$ we can write this

$$\boxed{\mathbf{a} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}^A + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}^A \right) + 2\boldsymbol{\omega} \times \mathbf{v}^A + \mathbf{a}^A.} \quad (1.22)$$

This result, which sometimes is referred to as Coriolis' theorem, is the basis for the study of motion in accelerated reference frames.

Let us analyze it. What is the 'system point' (or 'transport') acceleration, i.e. the acceleration of a point fixed in the A-system? In order to find it, all we have to do is put $\mathbf{v}^A = \mathbf{a}^A = \mathbf{0}$ in the formula above. One finds

$$\mathbf{a} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}^A + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}^A \right) \equiv \mathbf{A}_A(\mathbf{r}^A). \quad (1.23)$$

Using this notation we can write the relation between the accelerations as follows:

$$\mathbf{a} = \mathbf{A}_A(\mathbf{r}^A) + 2\boldsymbol{\omega} \times \mathbf{v}^A + \mathbf{a}^A. \quad (1.24)$$

Evidently the addition law that we found for velocities (1.16) does not hold for accelerations. The acceleration in the fixed system is the vector sum of the 'system point' acceleration and the acceleration relative to the A-system *plus* an extra term $2\boldsymbol{\omega} \times \mathbf{v}^A$, the Coriolis acceleration.

Example 1.2 A particle \mathcal{P} moves along the positive X -axis with position vector given by $\mathbf{r} = x(t)\mathbf{e}_x$ in a fixed coordinate system O. Introduce a rotating coordinate system A with the same (fixed) origin as the O-system and with Z -axis coinciding with the that of the fixed system. Let its angular velocity vector be $\boldsymbol{\omega} = \omega(t)\mathbf{e}_z$ relative to the fixed system. Calculate the acceleration \mathbf{a}^A of \mathcal{P} relative to the moving system. Use the result for \mathbf{v}^A derived in example 1.1.

Solution: Since the fixed and moving system have the same origin $\mathbf{r} = \mathbf{r}^A$, see figure 1.2. In example 1.1 we found that $\mathbf{v}^A = \dot{x}\mathbf{e}_x - x\omega\mathbf{e}_y$. Using this the various terms of equation 1.22 become

$$\ddot{\mathbf{R}} = \mathbf{0}, \quad (1.25)$$

$$\dot{\boldsymbol{\omega}} \times \mathbf{r}^A = \dot{\omega}\mathbf{e}_z \times x\mathbf{e}_x = x\dot{\omega}\mathbf{e}_y, \quad (1.26)$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^A) = \omega\mathbf{e}_z \times (\omega\mathbf{e}_z \times x\mathbf{e}_x) = -x\omega^2\mathbf{e}_x, \quad (1.27)$$

$$2\boldsymbol{\omega} \times \mathbf{v}^A = 2\omega\mathbf{e}_z \times (\dot{x}\mathbf{e}_x - x\omega\mathbf{e}_y) = 2x\omega^2\mathbf{e}_x + 2\dot{x}\omega\mathbf{e}_y. \quad (1.28)$$

The acceleration in the fixed system is $\mathbf{a} = \ddot{x}\mathbf{e}_x$. Use of this and the above results in equation 1.22 gives, when all are added up,

$$\mathbf{a}^A = (\ddot{x} - x\omega^2)\mathbf{e}_x - (2\dot{x}\omega + x\dot{\omega})\mathbf{e}_y \quad (1.29)$$

for the acceleration relative to the rotating system A. \square

1.2 Fictitious Forces

When an accelerated reference frame is used Newton's second law, $m\mathbf{a} = \mathbf{F}$, must be modified. If we insert the expression for \mathbf{a} in terms of \mathbf{a}^A , equation 1.22, into $m\mathbf{a} = \mathbf{F}$ we find that an equation in the form

$$m\mathbf{a}^A = \mathbf{F} + \mathbf{F}_{\text{fict}} \quad (1.30)$$

is valid, in the A-system, provided that a *fictitious force*

$$\mathbf{F}_{\text{fict}} \equiv -m \left[\ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}^A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^A) + 2\boldsymbol{\omega} \times \mathbf{v}^A \right] \quad (1.31)$$

is added to the physical force. This fictitious force is, of course, only mass times acceleration moved to the other side of the equation. Note that the fictitious forces, like gravitational forces, thus are proportional to mass so that the motion of particles due to these forces is independent of their mass m . This is natural (for fictitious forces) since the observed motion is really due to a motion of the observer's reference frame. That gravity also behaves in this way is an important empirical fact, called the 'principle of equivalence', and one of the foundations of the relativistic theory of gravity.

Most applications of the theory of 'relative' motion that we have developed here deal either with purely *translational* acceleration of the A-system ($\boldsymbol{\omega} = \mathbf{0}$), in which case the fictitious force is simply

$$\mathbf{F}_{\text{fict}} = -m\ddot{\mathbf{R}}, \quad (1.32)$$

or with purely *rotational* acceleration. In the purely rotational case ($\ddot{\mathbf{R}} = \mathbf{0}$) there are three different contributions to the fictitious force

$$\text{the 'transverse' force:} \quad \mathbf{F}_{\text{tv}} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}^A, \quad (1.33)$$

$$\text{the centrifugal force:} \quad \mathbf{F}_{\text{cf}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^A), \quad (1.34)$$

$$\text{the Coriolis force:} \quad \mathbf{F}_{\text{cor}} = -m2\boldsymbol{\omega} \times \mathbf{v}^A. \quad (1.35)$$

Note that the transverse and centrifugal forces are force fields that depend on the position, \mathbf{r}^A , of the particle while the Coriolis force depends on velocity. We now first discuss the translational case in more detail.

1.2.1 Motion in a Translationally Accelerated Frame

A translationally accelerated reference frame is usually a frame defined by a moving vehicle of some kind, such as an accelerating car or elevator. Motion in such a frame can be investigated in the same way as motion in an inertial frame provided only that the fictitious force 1.32 is added to the real forces. The acceleration $\ddot{\mathbf{R}}$ is the acceleration of the vehicle relative to an inertial frame. This fictitious force is independent of position (and velocity). If the acceleration $\ddot{\mathbf{R}}$ is also time independent, i.e. if $\ddot{\mathbf{R}} = \mathbf{A} = \text{const.}$, then the fictitious force provides a constant (homogeneous) force field just like the field of gravity near the surface of the earth. A very efficient way of understanding this situation is to view the fictitious force as providing an extra gravity. If the acceleration due to gravity is \mathbf{g} in the inertial frame, then the accelerated frame can be considered as having an 'effective' acceleration due to gravity \mathbf{g}_{eff} given by

$$\mathbf{g}_{\text{eff}} \equiv \mathbf{g} - \mathbf{A}. \quad (1.36)$$

This is illustrated in figure 1.3.

The force field $m\mathbf{g}_{\text{eff}}$ will clearly be a conservative force field so that it may be practical to use conservation of energy in the accelerated system. The potential energy of a particle of mass m and position vector \mathbf{r} (we will not use the superscript A on quantities referring to the A-system unless we wish to compare them with those of an inertial system) is given by

$$\Phi(\mathbf{r}) = -m\mathbf{g}_{\text{eff}} \cdot \mathbf{r}. \quad (1.37)$$

The force is then obtained as $\mathbf{F} = -\nabla\Phi = m\mathbf{g}_{\text{eff}}$.

Example 1.3 A box stands on a horizontal floor in the back of a van. The coefficient of static friction between box and floor is μ and the coefficient of sliding friction is $f (< \mu)$.

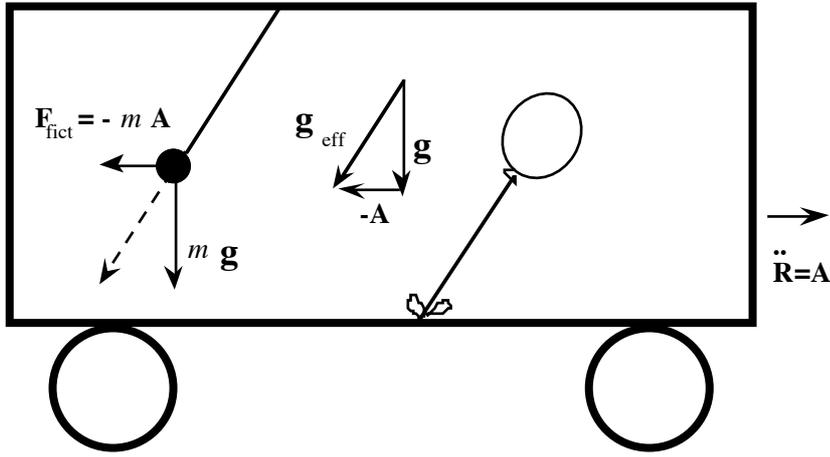


Figure 1.3: Inside a vehicle that accelerates to the right, with constant acceleration $\mathbf{A} = \ddot{\mathbf{R}}$, one will experience an effective gravity $\mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{A}$ as indicated in this figure. A weight that hangs from the ceiling in a string will tend backwards to make the string parallel to the new effective plumb-line. In a similar way a balloon tied to the floor will tend forwards since the direction of the buoyancy force on the balloon must be opposite to that of \mathbf{g}_{eff} .

- a) What maximum acceleration A can the van have if the box is to remain at rest?
b) Assume that the van breaks with an acceleration of a magnitude that just exceeds A so that the box slides forward. Find the speed of the box when it hits the wall of the drivers cabin assuming that the initial distance to this wall is d .

Solution: a) In the accelerated reference frame of the van the box will remain at rest if the friction force can balance the fictitious force. The friction force is given by $F_\mu = \mu N = \mu mg$ so we get

$$\mu mg = mA. \quad (1.38)$$

The maximum allowed acceleration is thus $A = \mu g$.

b) Once the box starts to move it is acted on by the fictitious force and the sliding friction force. The force of sliding friction has magnitude $F_f = fmg$ and is directed opposite to the velocity of the box. The fictitious force has magnitude $F_{\text{fict}} = mA = m\mu g$ and is parallel to the velocity of the box. The change in kinetic energy of the box is thus

$$T_2 - T_1 = \frac{1}{2}m(v^2 - 0^2) = \int_0^d (m\mu g - mfg) dx = mg(\mu - f)d. \quad (1.39)$$

The speed attained by the box is thus found to be

$$v = \sqrt{2g(\mu - f)d} \quad (1.40)$$

(relative to the moving system of the van). It is seen to be greater the larger the difference between static and sliding friction is. \square

1.2.2 The Rotational Fictitious Forces

In the majority of applications one can assume that a rotating reference frame rotates with an angular velocity of fixed direction. We will then mostly use the convention that this direction coincides with the Z -axis (of both the fixed and the moving system) so that

$$\boldsymbol{\omega}(t) = \omega(t) \mathbf{e}_z. \quad (1.41)$$

In order to understand the three rotational fictitious forces, 1.33-1.35, better we now calculate them explicitly for this case.

For the first two forces the simplest expressions are obtained if cylindrical coordinates,

$$\mathbf{r}^A = \mathbf{r} = \rho \mathbf{e}_\rho + z \mathbf{e}_z, \quad (1.42)$$

are used (in the moving A-system). The ‘transverse’ force becomes

$$\mathbf{F}_{\text{tv}} = -m \dot{\boldsymbol{\omega}} \times \mathbf{r} = -m \dot{\omega} \rho \mathbf{e}_\varphi. \quad (1.43)$$

It is thus non-zero only if $\dot{\omega} \neq 0$. The centrifugal force is found to be given by

$$\mathbf{F}_{\text{cf}} = -m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m \omega^2 \rho \mathbf{e}_\rho. \quad (1.44)$$

It always points radially outward from the rotation axis. For the Coriolis force we return to Cartesian coordinates and put $\mathbf{v} = \dot{x} \mathbf{e}_x + \dot{y} \mathbf{e}_y + \dot{z} \mathbf{e}_z$. This force, which vanishes if the particle is at rest (in the moving system), is then

$$\mathbf{F}_{\text{cor}} = -m 2 \boldsymbol{\omega} \times \mathbf{v} = 2m\omega(\dot{y} \mathbf{e}_x - \dot{x} \mathbf{e}_y). \quad (1.45)$$

All these forces are seen to be in the plane perpendicular to the fixed rotation axis.

Example 1.4 A free particle (unaffected by any physical forces) is observed from a reference frame that rotates with a constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z$ with respect to an inertial frame ($\omega = \text{const.}$). Find the equation of motion for the particle in the rotating frame and solve it.

Solution: The equation of motion is $m\mathbf{a} = \mathbf{F} + \mathbf{F}_{\text{fict}}$ and in this case $\mathbf{F} = \mathbf{0}$ while the fictitious forces that contribute are the centrifugal and the Coriolis force, so we get

$$m\mathbf{a} = -m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m 2 \boldsymbol{\omega} \times \mathbf{v}. \quad (1.46)$$

If we insert $\boldsymbol{\omega} = \omega \mathbf{e}_z$, $\mathbf{r} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z$, and $\mathbf{v} = \dot{x} \mathbf{e}_x + \dot{y} \mathbf{e}_y + \dot{z} \mathbf{e}_z$ in this equation some calculation gives a vector equation with the following components:

$$m\ddot{x} = m\omega^2 x + 2m\omega\dot{y}, \quad (1.47)$$

$$m\ddot{y} = m\omega^2 y - 2m\omega\dot{x}, \quad (1.48)$$

$$m\ddot{z} = 0. \quad (1.49)$$

The z-equation is easily solved and gives the general solution $z(t) = z(0) + \dot{z}(0)t$. The two first equations are coupled. One way of solving them is to introduce the complex quantity

$$\zeta(t) \equiv x(t) + iy(t). \quad (1.50)$$

Then the x-equation plus $i (= \sqrt{-1})$ times the y-equation gives

$$\ddot{\zeta} + 2i\omega\dot{\zeta} - \omega^2\zeta = 0. \quad (1.51)$$

The ‘ansatz’ $\exp(\lambda t)$ then gives the characteristic equation

$$\lambda^2 + 2i\omega\lambda - \omega^2 = (\lambda + i\omega)^2 = 0. \quad (1.52)$$

Thus $\lambda = -i\omega$ is a double root. The ‘ansatz’ $\exp(\lambda t)$ therefore gives only one linearly independent solution in this case. The new ‘ansatz’ $t \exp \lambda t$, inserted into the differential equation gives a characteristic equation of the form

$$(\lambda + i\omega)^2 t + (\lambda + i\omega) = 0. \quad (1.53)$$

so this ‘ansatz’ gives a new independent solution for the same value $\lambda = -i\omega$. The general solution is thus

$$\zeta(t) = A \exp(-i\omega t) + B t \exp(-i\omega t). \quad (1.54)$$

Here A and B are complex constants that can be determined in terms of the initial conditions. One obtains

$$\zeta(0) \equiv x(0) + iy(0) = A \quad (1.55)$$

$$\dot{\zeta}(0) \equiv \dot{x}(0) + i\dot{y}(0) = -i\omega A + B \quad (1.56)$$

so that the solution is $A = x(0) + iy(0)$ and $B = [\dot{x}(0) - \omega y(0)] + i[\dot{y}(0) + \omega x(0)]$. The explicit solutions for $x(t)$ and $y(t)$ can then be found by separating the real and imaginary parts in $\zeta(t)$. One gets

$$x(t) = [x(0) + t(\dot{x}(0) - \omega y(0))] \cos \omega t + [y(0) + t(\dot{y}(0) + \omega x(0))] \sin \omega t \quad (1.57)$$

$$y(t) = [y(0) + t(\dot{y}(0) + \omega x(0))] \cos \omega t - [x(0) + t(\dot{x}(0) - \omega y(0))] \sin \omega t \quad (1.58)$$

$$z(t) = z(0) + \dot{z}(0)t. \quad (1.59)$$

The general shape of this trajectory is that of a spiral. Note that the initial conditions refer to the rotating system. If the particle is at rest in the fixed system its initial conditions in the moving system might be

$$x(0) = R, \quad \dot{x}(0) = 0, \quad (1.60)$$

$$y(0) = 0, \quad \dot{y}(0) = -R\omega, \quad (1.61)$$

$$z(0) = 0, \quad \dot{z}(0) = 0. \quad (1.62)$$

When these initial conditions are inserted in the general solution above the result for the x-y-motion is seen to be motion in a circle with radius R and angular velocity $-\omega$. The minus sign here is due to the fact that if a fixed particle is viewed from a frame rotating with $\omega \mathbf{e}_z$ it will seem to rotate in the opposite direction. \square

The above example shows that the simple linear motion of a free particle becomes quite complicated when viewed from a rotating system. Under such circumstances the use of a non-inertial (accelerated) frame is, of course, pointless. There are, needless to say, also many situations in which the use of this theory of motion relative to a moving frame either is necessary or, at least, advantageous. If, for example, the motion in the moving system is known one can use this theory to calculate the forces. This is indicated in the following example.

Example 1.5 A small box stands on a rotating horizontal rough plane. The box stands at a distance R from the rotation axis which is vertical. The angular velocity ω is constant. The coefficient of friction between the box and the plane is μ . How large may R be if the box is to remain at rest?

Solution: In the rotating system this is a statics problem. The box will remain at rest if the total force, including fictitious forces, on it is zero. In this case the only forces acting in the horizontal plane are friction and the centrifugal force. We get

$$\mathbf{0} = \mathbf{F}_f + \mathbf{F}_{cf} \quad (1.63)$$

so use of equation 1.34 with $\rho = R$ gives

$$F_f = mR\omega^2. \quad (1.64)$$

The friction force F_f must fulfill

$$F_f < \mu N = \mu mg \quad (1.65)$$

so one obtains

$$mR\omega^2 < \mu mg. \quad (1.66)$$

which gives the answer $R < \mu g/\omega^2$. \square

1.2.3 Work and Rotational Fictitious Forces

When the angular velocity vector of the rotating system is constant, so that $\boldsymbol{\omega} = \omega \mathbf{e}_z$ with $\omega = \text{const.}$, the only fictitious forces are the Coriolis and the centrifugal forces. The work done by the Coriolis force is always zero since the displacement of the particle $d\mathbf{r} = \mathbf{v} dt$ is always perpendicular to the fictitious force $\mathbf{F}_{\text{cor}} = 2m\mathbf{v} \times \boldsymbol{\omega}$ so that $dW = \mathbf{F}_{\text{cor}} \cdot d\mathbf{r} = 0$. The centrifugal force, on the other hand, does work, but is conservative as we'll now see.

The work of the centrifugal force for a particle that moves from \mathbf{r}_a to \mathbf{r}_b is

$$W_{ab} = \int_a^b \mathbf{F}_{cf} \cdot d\mathbf{r} = \int_a^b m\omega^2 \rho \mathbf{e}_\rho \cdot d\mathbf{r}. \quad (1.67)$$

If we now express the displacement in terms of cylindrical coordinates,

$$d\mathbf{r} = d(\rho \mathbf{e}_\rho + z \mathbf{e}_z) = d\rho \mathbf{e}_\rho + \rho d\mathbf{e}_\rho + dz \mathbf{e}_z, \quad (1.68)$$

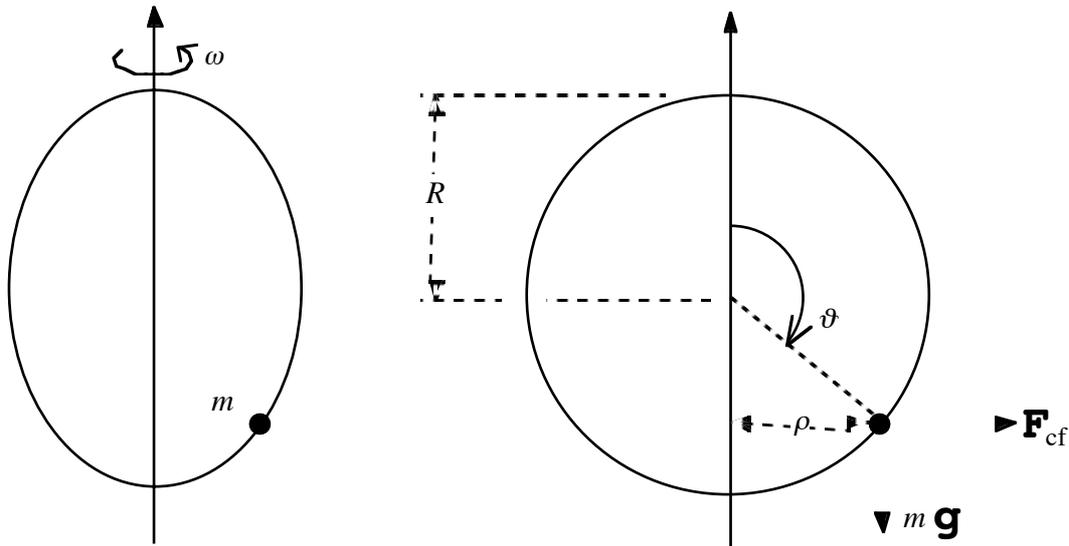


Figure 1.4: This figure shows the situation in example 1.6. To the left the rotating circular wire is shown. On the right the same wire is viewed from a rotating reference frame in which it is at rest, but in which fictitious forces act.

and use the fact that both $d\mathbf{e}_\rho = \frac{d\mathbf{e}_\rho}{dt} dt$, being the small change in a unit vector, as well as \mathbf{e}_z are perpendicular to \mathbf{e}_ρ we find

$$W_{ab} = \int_a^b m\omega^2 \rho \mathbf{e}_\rho \cdot (d\rho \mathbf{e}_\rho + \rho d\mathbf{e}_\rho + dz \mathbf{e}_z) = m\omega^2 \int_{\rho_a}^{\rho_b} \rho d\rho. \quad (1.69)$$

The work of the centrifugal force thus only depends on (the ρ -values of) the end-points of the particle trajectory and this force is thus a conservative (fictitious) force. The potential energy of a particle in this force field is given by

$$\Phi_{\text{cf}}(\rho) = - \int_0^\rho \mathbf{F}_{\text{cf}} \cdot d\mathbf{r} = -\frac{1}{2} m\omega^2 \rho^2, \quad (1.70)$$

where the lower integration limit is conventionally set to zero since this gives the simplest expression. The solution of many a problem involving the centrifugal force is facilitated by the use of this formula.

Example 1.6 A stiff smooth wire in the shape of a circle of radius R is mounted on a fixed vertical axis in the plane of the circle through its mid-point. It rotates around this axis with constant angular velocity ω , see figure 1.4. A small pearl of mass m can slide along the wire with negligible friction. Find the equilibrium positions of the pearl on the circle as function of ω .

Solution: Consider the situation in a reference frame that rotates so that the circular wire is at rest while there are fictitious forces acting on the pearl. The forces acting are then the normal force from the wire, gravity, the Coriolis force, and the centrifugal force. The normal and the Coriolis forces do no work so only the conservative forces, gravity and centrifugal force do work on the pearl. The pearl will be in an equilibrium position on the circle if the potential energy of the forces acting on it has a minimum. We introduce the angle ϑ between the radius to the pearl and the vertical axis as the relevant coordinate, see figure 1.4. The potential energy of gravity is then

$$\Phi_{\text{grav}}(\vartheta) = mgR \cos \vartheta. \quad (1.71)$$

The distance between the pearl and the rotation axis is $\rho = R \sin \vartheta$ so the potential energy of the centrifugal force is, according to equation 1.70,

$$\Phi_{\text{cf}}(\vartheta) = -\frac{1}{2} m\omega^2 (R \sin \vartheta)^2. \quad (1.72)$$

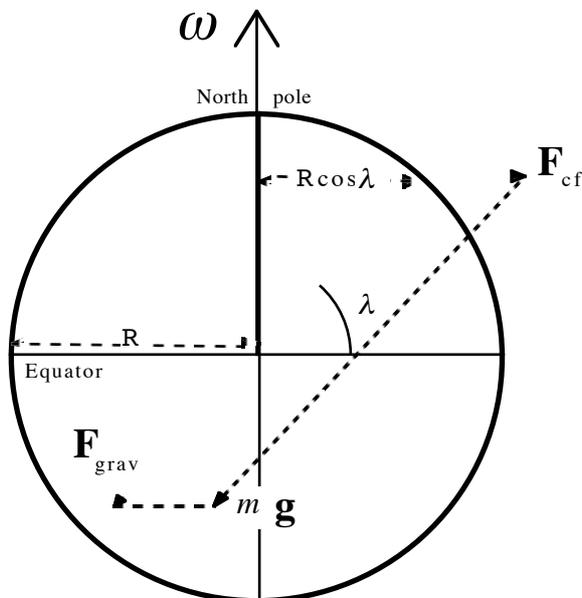


Figure 1.5: This figure shows how the local acceleration due to ‘gravity’ \mathbf{g} is the result of gravitation and centrifugal force. The magnitude of the centrifugal force is $F_{\text{cf}} = mR \cos \lambda \omega^2$ where the angle λ is the so called ‘geocentric’ latitude.

The total potential energy of the forces acting on the pearl is thus

$$\Phi(\vartheta) = mR(g \cos \vartheta - \frac{1}{2}\omega^2 R \sin^2 \vartheta). \quad (1.73)$$

Equilibrium requires that

$$\frac{d\Phi(\vartheta)}{d\vartheta} = -mR \sin \vartheta (g + \omega^2 R \cos \vartheta) = 0 \quad (1.74)$$

The solutions to this equation are given by the solutions of

$$\sin \vartheta = 0, \quad (1.75)$$

which are $\vartheta = 0, \pi$, and the solution of

$$\cos \vartheta = -\frac{g}{\omega^2 R}, \quad (1.76)$$

which exists only when $\omega^2 R \geq g$, in which case it is $\vartheta = \arccos(-\frac{g}{\omega^2 R})$. It is left as an exercise to the reader to study the stability properties of these equilibrium positions. \square

1.3 Motion Relative to the Rotating earth

Until now we have treated problems concerning motion on our planet as if it defined an inertial frame. Usually this gives useful results but when higher accuracy is needed one must take into account the fact that our planet rotates. The angular velocity of this rotation is

$$\omega = \frac{360^\circ}{23 \text{ h } 56 \text{ m } 4 \text{ s}} = \frac{2\pi \text{ rad}}{86164 \text{ s}} = 7.292 \cdot 10^{-5} \frac{\text{rad}}{\text{s}}, \quad (1.77)$$

so it can be regarded as small for most purposes. Note that the rotational period of the earth is *not* 24 hours but four minutes less. 24 hours is the period of rotation with respect to the direction defined by the Sun but the Sun is not in a fixed direction since the earth goes around it once a year. The direction of the earth’s angular velocity vector is from the centre to the north pole (see figure 1.5).

1.3.1 The Effect of the Centrifugal Force

When we study motion near the surface of the earth and consider the earth as fixed, we must thus take into account not only the force of gravity, contact forces etc. but also the fictitious forces due to the rotation. These are the centrifugal force and the Coriolis force. The centrifugal force will usually *not* require any special treatment; its effect is already included in, what we call the acceleration due to gravity, \mathbf{g} . The local magnitude, g , and direction (the plumb-line), $-\mathbf{e}_z$, of \mathbf{g} ($= -g\mathbf{e}_z$) will be determined by both gravity and the centrifugal force. Because of this \mathbf{g} would not be constant in magnitude and would not be directed towards the centre even if the earth was a perfect sphere, see figure 1.5. This effect is fairly small, for the earth, since the maximum value of the centrifugal acceleration, which is attained at the equator, $\lambda = 0$, is

$$R\omega^2 = (6.4 \cdot 10^6 \text{ m}) \cdot (7.3 \cdot 10^{-5} \frac{1}{\text{s}})^2 = 3.4 \cdot 10^{-2} \frac{\text{m}}{\text{s}^2}. \quad (1.78)$$

This should be compared with $g = 9.8 \text{ m/s}^2$. The effect of the centrifugal force is thus roughly three orders of magnitude smaller than that of gravity. In figure 1.5 the size of the effect of the rotation is exaggerated for clarity.

In figure 1.5 the flattening effect of the rotation on the shape of the earth has been neglected. This flattening is such that the direction of \mathbf{g} is everywhere perpendicular to the surface of the earth. It is difficult to calculate this flattening analytically since the formula for the gravitational field from a non-spherical body is quite complicated. In spite of this already Newton found a decent value for this flattening by means of hydrostatics and some simplifying assumptions.

1.3.2 The Effects of the Coriolis Force

Once it is understood that the effect of the centrifugal force is included in the local acceleration due to gravity, \mathbf{g} , there remains to study the effects of the Coriolis force. If we assume that $m\mathbf{g}$ and the Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, are the only forces on a particle of mass m its equation of motion can be written

$$\dot{\mathbf{v}} = 2\mathbf{v} \times \boldsymbol{\omega} + \mathbf{g}. \quad (1.79)$$

When $\omega = 0$ we know that the solution of this equation is

$$\mathbf{v}_0(t) = \mathbf{v}(0) + \mathbf{g}t. \quad (1.80)$$

Since ω is small it is reasonable to assume that the correct solution is close to \mathbf{v}_0 . We thus make the ‘ansatz’ that \mathbf{v} is \mathbf{v}_0 plus a small correction $\delta\mathbf{v}$,

$$\mathbf{v}(t) = \mathbf{v}_0(t) + \delta\mathbf{v}(t). \quad (1.81)$$

When we insert this into equation 1.79 we find that $\delta\mathbf{v}$ obeys $\delta\dot{\mathbf{v}} = 2\mathbf{v}_0 \times \boldsymbol{\omega} + 2\delta\mathbf{v} \times \boldsymbol{\omega}$. The assumption that $\delta\mathbf{v}$ is small means that the last term here is the product of two small terms, so we assume that we may neglect it. The equation for $\delta\mathbf{v}$ is then, using equation 1.80 for \mathbf{v}_0 ,

$$\delta\dot{\mathbf{v}} = [2\mathbf{v}(0) \times \boldsymbol{\omega}] + (2\mathbf{g} \times \boldsymbol{\omega})t. \quad (1.82)$$

We now integrate this with respect to time, assuming that $\delta\mathbf{v}(0) = \mathbf{0}$, to get

$$\delta\mathbf{v}(t) = [2\mathbf{v}(0) \times \boldsymbol{\omega}]t + (\mathbf{g} \times \boldsymbol{\omega})t^2. \quad (1.83)$$

Combining this with formula 1.80 gives the solution

$$\mathbf{v}(t) = \mathbf{v}(0) + \mathbf{g}t + [2\mathbf{v}(0) \times \boldsymbol{\omega}]t + (\mathbf{g} \times \boldsymbol{\omega})t^2 \quad (1.84)$$

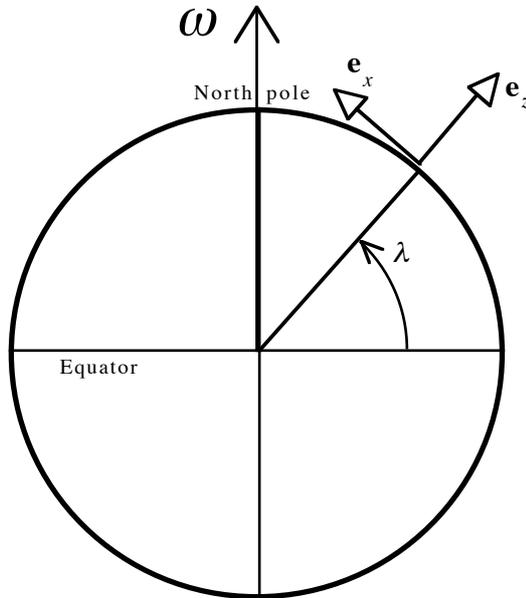


Figure 1.6: This figure shows the angular velocity vector $\boldsymbol{\omega}$ of the planet earth. It also defines the latitude, the angle λ , of a location on the northern hemisphere and the basis vectors \mathbf{e}_x and \mathbf{e}_z of the local coordinate system used in example 1.7. The angular velocity vector is given by $\boldsymbol{\omega} = \omega(\cos \lambda \mathbf{e}_x + \sin \lambda \mathbf{e}_z)$ in terms of this basis. The formula is also valid on the southern hemisphere if the angle λ is taken to be negative there.

to our original problem, equation 1.79. A second integration with respect to time gives us the trajectory of the particle

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)t + \frac{1}{2}\mathbf{g}t^2 + [\mathbf{v}(0) \times \boldsymbol{\omega}]t^2 + \frac{1}{3}(\mathbf{g} \times \boldsymbol{\omega})t^3. \quad (1.85)$$

The first three terms represent the well known parabolic trajectory of the constant gravitational field. The two last terms are corrections due to the Coriolis force. This solution is, of course, only valid for times t small enough so that the correction terms remain small.

Example 1.7 A particle is dropped from a vertical tower of height h at latitude λ . Calculate its deflection from the vertical path. Neglect air resistance.

Solution: We use the coordinate system indicated in figure 1.6 with \mathbf{e}_x to the north, \mathbf{e}_y to the west, and \mathbf{e}_z vertical so that

$$\mathbf{r}(0) = h \mathbf{e}_z, \quad (1.86)$$

$$\mathbf{v}(0) = \mathbf{0}, \quad (1.87)$$

$$\mathbf{g} = -g \mathbf{e}_z, \quad (1.88)$$

$$\boldsymbol{\omega} = \omega(\cos \lambda \mathbf{e}_x + \sin \lambda \mathbf{e}_z). \quad (1.89)$$

When these expressions are inserted into equation 1.85 some calculation gives

$$\mathbf{r}(t) = \left(h - \frac{1}{2}gt^2\right) \mathbf{e}_z - \left(\frac{1}{3}g\omega \cos \lambda t^3\right) \mathbf{e}_y. \quad (1.90)$$

The particle hits the ground at the time $t = T$ which is the root of the equation $z(t) = h - \frac{1}{2}gt^2 = 0$, i.e. when $t = \sqrt{2h/g}$. The y -coordinate at this time gives the sideways deflection of the particle which thus is

$$y(T) = -\left(\frac{1}{3}g\omega \cos \lambda T^3\right) = -\frac{2}{3}h\sqrt{\frac{2h}{g}}\omega \cos \lambda. \quad (1.91)$$

The minus sign here means that the deflection is to the east. For a tower of $h = 10$ m at the equator ($\lambda = 0$) this gives roughly a 1 mm deflection. \square

In contrast to the above example, where the effect of the Coriolis force is quite small, there are many large scale phenomena on the earth for which the Coriolis force is responsible. These are, in particular, the circulation of the atmosphere and the oceans. For example, the counterclockwise whirling of cyclones on the northern hemisphere (clockwise on the southern hemisphere) comes about when the Coriolis force deflects the air rushing in to fill a low-pressure region perpendicular to the velocity.

1.3.3 Foucault's Pendulum

Consider a simple pendulum suspended at the north pole. If such a pendulum is released from rest, with its string making an angle with the vertical, it will swing back and forth in a 'fixed' plane. Fixed relative to what one may ask? Is it fixed with respect to the earth or to the universe as a whole (the fixed stars)? If one can minimize the effect of air resistance the pendulum will swing for a long time and, at the north pole, one would observe how the plane of the pendulum would rotate and complete one full circuit in one sidereal day (23 h 56 m 4 s). The plane is thus fixed with respect to the fixed stars and not the earth. This behavior of *Foucault's pendulum* is historically important, as the most direct and conclusive evidence that the earth rotates.

When such a pendulum is not at the north pole one must calculate the angular velocity of rotation of the plane by taking the Coriolis force into account in the equations of motion. Let the bob of the pendulum have mass m , and denote the force of the string by \mathbf{S} . The equation of motion for the pendulum is

$$m\ddot{\mathbf{r}} = \mathbf{S} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}. \quad (1.92)$$

We choose a coordinate system with the origin at the point of suspension and the direction of the axes as in figure 1.6 (see example 1.7). We then have $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = \ell$, where ℓ is the length of the string so that

$$\mathbf{S} = -S\mathbf{r}/|\mathbf{r}| = -S\frac{(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)}{\sqrt{x^2 + y^2 + z^2}} = -\frac{S}{\ell}(x\mathbf{e}_x + y\mathbf{e}_y - \sqrt{\ell^2 - x^2 - y^2}\mathbf{e}_z). \quad (1.93)$$

The Coriolis force is

$$-2m\boldsymbol{\omega} \times \dot{\mathbf{r}} = -2m\omega(\cos\lambda\mathbf{e}_x + \sin\lambda\mathbf{e}_z) \times (\dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z) = \quad (1.94)$$

$$= -2m\omega[\cos\lambda(\dot{y}\mathbf{e}_z - \dot{z}\mathbf{e}_y) + \sin\lambda(\dot{x}\mathbf{e}_y - \dot{y}\mathbf{e}_x)]. \quad (1.95)$$

When this is inserted into equation 1.92 one obtains a vector equation with the following components:

$$m\ddot{x} = -\frac{S}{\ell}x + 2m\omega\sin\lambda\dot{y} \quad (1.96)$$

$$m\ddot{y} = -\frac{S}{\ell}y - 2m\omega\sin\lambda\dot{x} + 2m\omega\cos\lambda\dot{z} \quad (1.97)$$

$$m\ddot{z} = \frac{S}{\ell}\sqrt{\ell^2 - x^2 - y^2} - mg - 2m\omega\cos\lambda\dot{y} \quad (1.98)$$

So far no approximations have been made. In order to solve this system we now assume that the string is long and the amplitude of the oscillations small. We thus assume that x, y, \dot{x} , and \dot{y} are small so that products of these terms can be neglected. Since

$$\dot{z} = -\frac{d}{dt}\sqrt{\ell^2 - x^2 - y^2} = \frac{x\dot{x} + y\dot{y}}{\sqrt{\ell^2 - x^2 - y^2}} \quad (1.99)$$

we can thus neglect \dot{z} and also \ddot{z} . The third (z) equation then gives $S = mg + 2m\omega\cos\lambda\dot{y}$ for the tension in the string. When this is inserted in the x- and y-equations the term

proportional to \dot{y} becomes multiplied with x and with y respectively, and so can be neglected. This gives us the approximate equations

$$\ddot{x} = -\frac{g}{\ell}x + 2\omega \sin \lambda \dot{y}, \quad (1.100)$$

$$\ddot{y} = -\frac{g}{\ell}y - 2\omega \sin \lambda \dot{x}. \quad (1.101)$$

These coupled equations can be solved with the complex technique illustrated in example 1.4 above. First introduce the complex function

$$\zeta(t) \equiv x(t) + iy(t). \quad (1.102)$$

The two equations are then equivalent to the single equation

$$\ddot{\zeta} + 2i\omega \sin \lambda \dot{\zeta} + \omega_0^2 \zeta = 0, \quad (1.103)$$

where we have put

$$\omega_0 \equiv \sqrt{\frac{g}{\ell}} \quad (1.104)$$

for the angular frequency of the (simple) pendulum.

Note that we have not made any special assumptions about the initial conditions. Because of this the motion will not be confined to a plane even in the absence of a Coriolis force ($\omega = 0$). The problem is then that of the ‘spherical pendulum’ and the general (small amplitude) solution trajectory turns out to be an ellipse. The simple pendulum problem is obtained as a special case when the minor axis of the ellipse goes to zero.

In order to solve equation 1.103 we, as usual, make the ‘ansatz’ $A \exp \nu t$ and get the characteristic equation

$$\nu^2 + 2i\omega \sin \lambda \nu + \omega_0^2 = 0 \quad (1.105)$$

with the roots

$$\nu_{\pm} = -i \left(\omega \sin \lambda \pm \sqrt{\omega_0^2 + \omega^2 \sin^2 \lambda} \right). \quad (1.106)$$

If we introduce the quantity

$$\omega_1 \equiv \omega_0 \sqrt{1 + \left(\frac{\omega}{\omega_0} \sin \lambda \right)^2}, \quad (1.107)$$

which normally will be quite close to ω_0 since the frequency of the pendulum (ω_0) is much larger than the frequency of the earth’s rotation (ω), we can write the two roots in the form

$$\nu_{\pm} = \pm i\omega_1 - i\omega \sin \lambda. \quad (1.108)$$

The general solution of equation 1.103 is thus

$$\zeta(t) = \exp(-i\omega \sin \lambda t) [A \exp(i\omega_1 t) + B \exp(-i\omega_1 t)] \quad (1.109)$$

where A and B are complex constants that must be determined in terms of the initial conditions. When $\omega = 0$ this solution corresponds to an ellipse in the xy -plane which is traversed with angular velocity $\omega_1 \approx \omega_0$. When the Coriolis force is taken into account the ellipse will rotate (that is, its major axis will rotate) with angular velocity $-\omega \sin \lambda$ (see figure 1.7). At the north pole this gives the angular velocity $-\omega$ as it should.

One should note that the terms that we have neglected, when we ‘linearized’ the spherical pendulum, also cause precession (slow rotation) of the ellipse. In general this precession will be greater than that caused by the Coriolis force but fortunately it is proportional to the (conserved) angular momentum, $L_z = m(xy\dot{y} - y\dot{x}x)$, of the bob with respect to the Z -axis. It is thus zero when the initial conditions give plane motion; the rotation of the plane of the ‘simple’ pendulum is thus, really, due only to the Coriolis force, i.e. the rotation of the earth.

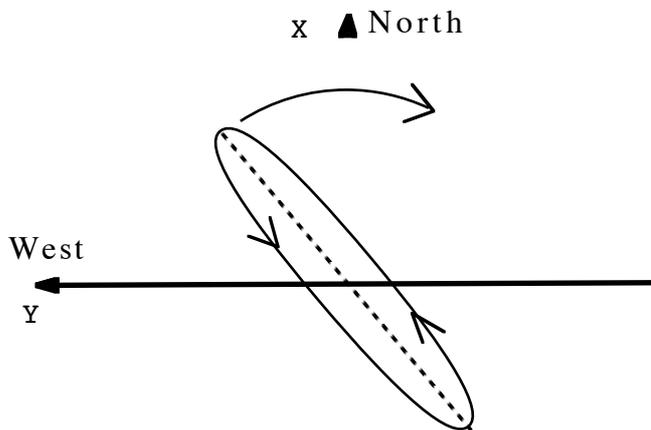


Figure 1.7: This figure shows the trajectory of the bob of Foucault's pendulum on the northern hemisphere. The initial conditions determine the shape of the ellipse as well as the direction in which it is traversed. The Coriolis force makes the ellipse rotate in the direction of the curved arrow, with angular velocity of magnitude $|\omega \sin \lambda|$, when the pendulum is on the northern hemisphere. On the southern hemisphere it rotates in the opposite direction.

1.4 Problems

Problem 1.1 The reference frame A rotates relative to a fixed frame with an angular velocity vector $\boldsymbol{\omega}$ which is given by

$$\boldsymbol{\omega} = \omega_x^A \mathbf{e}_x^A + \omega_y^A \mathbf{e}_y^A + \omega_z^A \mathbf{e}_z^A = \frac{\omega_0}{\sqrt{2}} [\sin(\omega_0 t) \mathbf{e}_x^A + \cos(\omega_0 t) \mathbf{e}_y^A + \mathbf{e}_z^A]$$

when it is expanded in a basis fixed in the moving frame A. Find the first and second time derivatives, $\dot{\boldsymbol{\omega}}$ and $\ddot{\boldsymbol{\omega}}$ respectively, of this vector relative to the fixed system but expanded in the moving frame basis vectors $(\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A)$.

Problem 1.2 The acceleration \mathbf{a} of a particle relative to a fixed frame has the components $(2, 3, 1)\text{m/s}^2$ in some basis. This particle is observed from an accelerated reference frame A, with angular velocity components $\frac{1}{4}(3, 2, 1)\text{s}^{-1}$ and angular acceleration components $2(1, 2, 3)\text{s}^{-2}$ (relative to the fixed frame). Relative to a coordinate system fixed in this moving frame the position vector, \mathbf{r}^A , of the particle has components $-2(1, 2, 3)\text{m}$. The velocity vector, \mathbf{v}^A , in this frame, has the components $(1, 2, 3)\text{m/s}$, and the acceleration, \mathbf{a}^A , components are $(3, 1, 2)\text{m/s}^2$. All components are given with respect to the same basis. Find the acceleration of the origin \mathcal{A} of the moving frame.

Problem 1.3 A railway tank car rolls straight down an incline that makes an angle α with the horizontal, see figure 1.8. What angle β does the surface of the liquid inside make with the horizontal? Neglect air resistance, friction, the moments of inertia of the wheels, and initial wave motion.

Problem 1.4 A train of mass m travels along a straight track on horizontal ground. The latitude of the track is λ and it goes from south-west to north-east, see figure 1.9. The speed of the train is v and it moves towards north-east. Calculate the total side-ways force on the rails from the train and indicate on which rail the force mainly acts.

Problem 1.5 A car starts to accelerate with the door on the right hand side wide open. The angle between the door and the direction of acceleration is 90° . This is also the

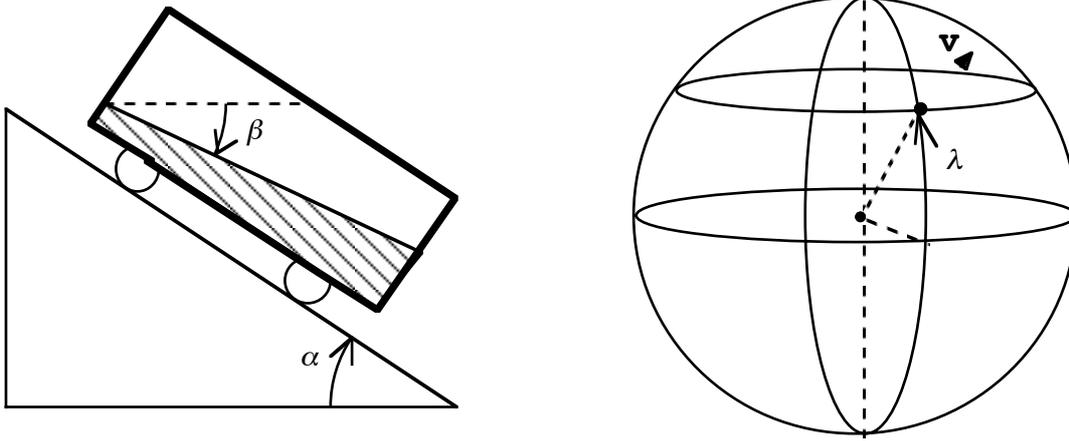


Figure 1.8: The figure on the left refers to problem 1.3. The railway tank car rolls down an incline of angle α . What angle β does the surface of the liquid inside make with the horizontal?

Figure 1.9: The figure on the right refers to problem 1.4. The velocity \mathbf{v} , to the north-east, of the train at latitude λ is indicated.

angle that the door must rotate in order to close, see figure 1.10. The door can rotate, with negligible friction, around a vertical axis. The horizontal distance from the axis to the centre of mass of the door is d and the radius of gyration of the door with respect to its axis is d_g . If the door is to lock properly when it closes it must have a minimum angular velocity ω . What is the minimum (constant) acceleration that the car must have in order for the door to lock when it closes?

Problem 1.6 A motor cycle has a foot brake that acts on the back wheel, and a hand brake that acts on the front wheel. The coefficient of friction between ground and wheels is f . Investigate the maximum allowed braking acceleration, if no skidding is to take place, when

- there is optimal braking on both wheels,
- only the front wheel brakes,
- only the back wheel brakes.

The horizontal distance between the center of the front wheel and the center of mass is a , that between the center of mass and the back wheel b , and the vertical distance between the ground and the center of mass is h , see figure 1.11.

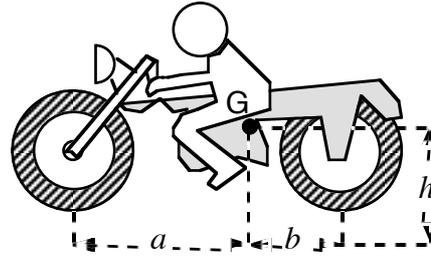
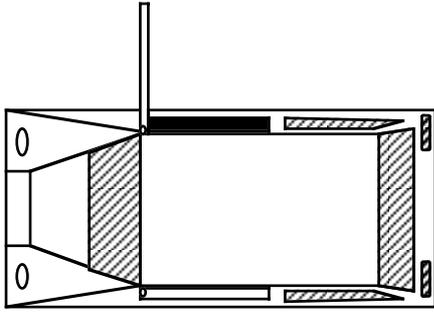


Figure 1.10: The figure on the left refers to problem 1.5. When the car starts to accelerate the angle between the door and the direction of acceleration is 90° . This is also the angle that the door must rotate in order to close.

Figure 1.11: The figure on the right refers to problem 1.6. \mathcal{G} is the centre of mass of the motor cycle (with driver).

1.5 Hints and Answers

Answer 1.1 Use formula 1.11. One finds

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \dot{\boldsymbol{\omega}}$$

and thus that

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}} = \frac{\omega_0^2}{\sqrt{2}} [\cos(\omega_0 t) \mathbf{e}_x^A - \sin(\omega_0 t) \mathbf{e}_y^A].$$

For the second time derivative we get

$$\ddot{\boldsymbol{\omega}} = (\dot{\dot{\boldsymbol{\omega}}}) + \boldsymbol{\omega} \times \dot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}.$$

Calculations then give

$$\ddot{\boldsymbol{\omega}} = -\frac{\omega_0^3}{2} [(\sqrt{2} - 1) \sin(\omega_0 t) \mathbf{e}_x^A + (\sqrt{2} - 1) \cos(\omega_0 t) \mathbf{e}_y^A + \mathbf{e}_z^A]$$

for the second time derivative of the angular velocity vector.

Answer 1.2 Note that the answer is given by the vector $\ddot{\mathbf{R}}$ of formula 1.22. Elementary calculations give the components

$$(-1, 5, -7)\text{m/s}^2$$

for this vector.

Answer 1.3 First find the effective acceleration of gravity \mathbf{g}_{eff} in the car. It turns out to be perpendicular to the incline. The liquid surface will be perpendicular to \mathbf{g}_{eff} and it will thus make an angle α with the horizontal. The answer is thus that $\beta = \alpha$.

Answer 1.4 The sideways force is given by the Coriolis force. Introduce a suitable local coordinate system and express $\boldsymbol{\omega}$ and \mathbf{v} , the velocity of the train, in this system. Then calculate the Coriolis force.

With \mathbf{e}_z vertical and \mathbf{e}_x pointing to the north we have that the angular velocity vector, at the position of the train, is

$$\boldsymbol{\omega} = \omega(\cos \lambda \mathbf{e}_x + \sin \lambda \mathbf{e}_z).$$

The velocity, \mathbf{v} , of the train, is then

$$\mathbf{v} = \frac{v}{\sqrt{2}}(\mathbf{e}_x - \mathbf{e}_y).$$

The Coriolis force is thus

$$\begin{aligned}\mathbf{F}_{\text{cor}} &= -2m\omega \frac{v}{\sqrt{2}}(\cos \lambda \mathbf{e}_x + \sin \lambda \mathbf{e}_z) \times (\mathbf{e}_x - \mathbf{e}_y) = \\ &= \sqrt{2}m\omega v[-\sin \lambda(\mathbf{e}_x + \mathbf{e}_y) + \cos \lambda \mathbf{e}_z].\end{aligned}$$

The horizontal component of this force, which acts on the rails, is $-\sqrt{2}m\omega v \sin \lambda(\mathbf{e}_x + \mathbf{e}_y)$. This is seen to be perpendicular to the velocity of the train and therefore to the direction of the rails. For a person looking in the direction of the trains velocity it is directed to the right. The right hand rail must compensate this force with a suitable horizontal normal force.

Answer 1.5 Use the accelerated reference frame of the car. The fictitious forces on the door correspond to a system of parallel forces of the same type as gravity. They thus have a resultant acting at the centre of mass. Consequently the door can be treated as a physical pendulum and conservation of energy can be used to find its final angular velocity. In this way one finds that

$$a = \frac{d_g^2 \omega^2}{2d}$$

is the smallest acceleration that will lock the door.

Answer 1.6 In the accelerated frame of the motor cycle the relevant forces are the normal forces, N_1, N_2 , from the ground on the wheels, the friction forces F_i ($i = 1, 2$) from the ground on the wheel(s) that are braked, and the resultant of the fictitious (translational) forces acting forwards at the centre of mass. In the accelerated frame the problem is one of statics. One finds the answers:

a) $F_i = fN_i$, $i = 1, 2$, gives maximum retardation

$$\ddot{x} = fg.$$

b) $F_2 = 0$ and $F_1 = fN_1$ gives maximum retardation

$$\ddot{x} = \frac{fg}{a + b - fh}.$$

b) $F_1 = 0$ and $F_2 = fN_2$ gives maximum retardation

$$\ddot{x} = \frac{fg}{a + b + fh}.$$

For typical values of the parameters the last case will be the worst case.