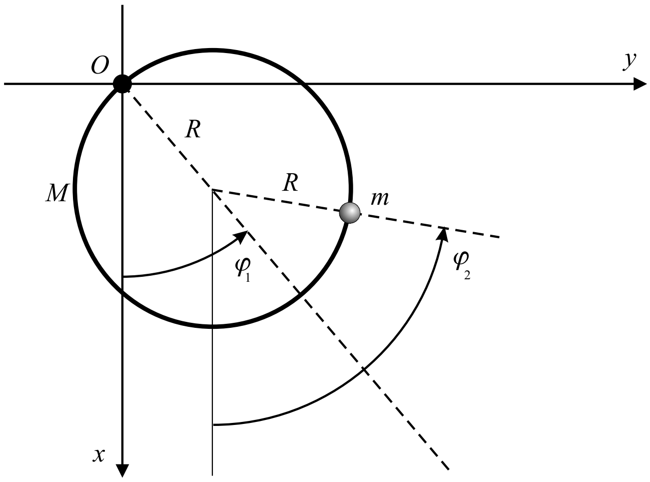
Solutions, Analytical Mechanics, 5C1121, Exam, 2005 03 11

Calculation problems

Problem 1: A metal ring has mass M and radius R. A point on its circumference is suspended from a fixed point O. At O there is a bearing which allows the ring to rotate in a vertical plane with negligible friction about an axis that is horizontal and perpendicular to the plane of the ring. A pearl can slide along the ring with negligible friction. The mass of the pearl is m. Calculate the angular frequencies for small oscillations near the equilibrium for this double pendulum.



Solution 1: The kinetic energy is

$$T = \frac{1}{2}\left(2MR^2\dot{\varphi}_1^2 + \frac{1}{2}m\boldsymbol{v}^2)\right)$$

where $\boldsymbol{r} = R\boldsymbol{e}_\rho(\varphi_1) + R\boldsymbol{e}_\rho(\varphi_2)$ so differentiation and squaring gives

$$\boldsymbol{v}^2 = R^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\cos(\varphi_1 - \varphi_2)\dot{\varphi}_1\dot{\varphi}_2).$$

We thus have that

$$T \approx \frac{1}{2}mR^2\left[\left(1 + \frac{2M}{m}\right)\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2\right],$$

keeping quadratic terms. The potential energy is

$$V = -MgR\cos\varphi_1 - mgR[\cos\varphi_1 + \cos\varphi_2] \approx V_0 + \frac{1}{2}mgR\left[\left(1 + \frac{M}{m}\right)\varphi_1^2 + \varphi_2^2\right].$$

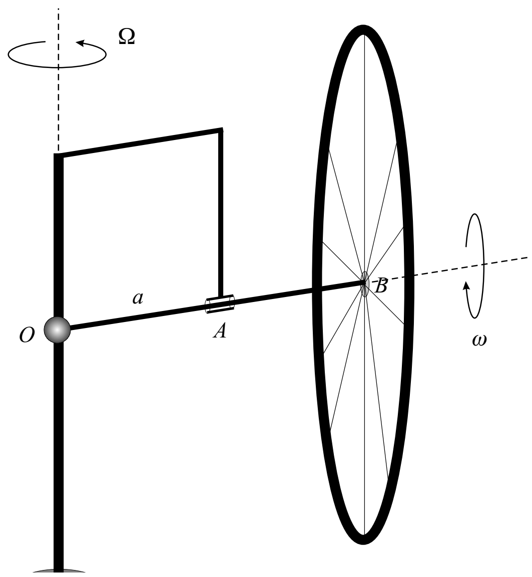
The secular equation then becomes

$$\left| -mR^2x\begin{pmatrix} 1+2M/m & 1 \\ 1 & 1 \end{pmatrix} + mgR\begin{pmatrix} 1+M/m & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

This gives

Answer: the angular frequencies are $\omega_1 = \sqrt{\frac{g}{2R}}$ and $\omega_2 = \sqrt{\frac{m+M}{M}\frac{g}{R}}$.

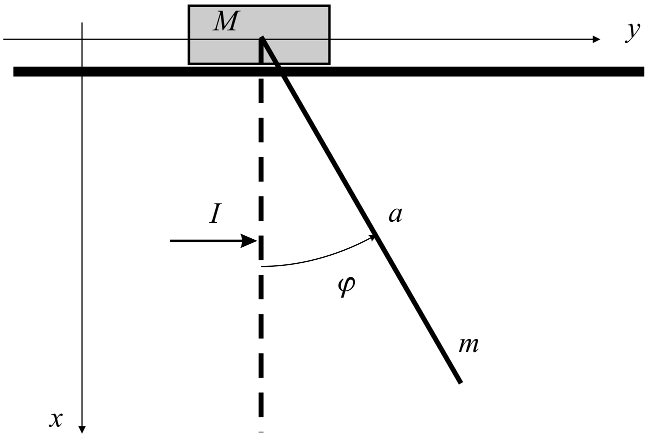
Problem 2: A bicycle wheel, of mass m and moment of inertia J with respect to its symmetry axis, is mounted on a light horizontal axis OB of length ℓ. The wheel may be assumed to be thin. The axis OB rotates about the vertical direction with angular velocity Ω and the wheel rotates about its symmetry axis OB with the angular velocity ω. At the distance a from a fix ball bearing at O the axis OB passes through a small well lubricated sleeve A which affects the axis OB with a force that is perpendicular to the axis. Calculate the size and direction of this force.



Solution 2: This problem is the same as Example 5.4 page 84 of Chapter 5 (Three dimensional motion of rigid bodies). Just replace d with a and $\dot{\varphi}$ with ω.

Answer: $F = \frac{mg\ell - J\Omega\omega}{a}$ upwards (if positive, otherwise downwards, of course).

Problem 3: A block of mass M can slide along a horizontal track. A pole, of mass m and length a, is suspended at one of its ends from the block. A bearing at that end allows the pole to rotate in the vertical plane of the track. When the system is at rest (in equilibrium) the pole is struck at its midpoint so that it receives an impulse I parallel to the track. Calculate the speed of the block and the angular velocity of the pole immediately after the strike.



Solution 3: The kinetic energy is

$$T = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{2}m\frac{a^2}{3}\dot{\varphi}^2 + \frac{1}{2}ma\dot{y}\dot{\varphi}\cos\varphi.$$

The work done by the force $F(t)$ of the strike, whose time integral is I, can be written

$$\delta W = \delta W_y + \delta W_\varphi = Q_y dy + Q_\varphi d\varphi = F dy + \frac{a}{2}F d\varphi$$

so that $Q_y = F$ and $Q_\varphi = (a/2)F$. The generalized impulses are thus $I_y = I$ and $I_\varphi = (a/2)I$.

Lagrange's equations for impact problems ($p_a \equiv \partial L/\partial \dot{q}_a = \partial T/\partial \dot{q}_a$)

$$\begin{aligned} p_y(t+\tau) - p_y(t) &= I_y \\ p_\varphi(t+\tau) - p_\varphi(t) &= I_\varphi \end{aligned}$$

therefore give (at $\varphi = 0, t = 0$)

$$\begin{aligned} (M+m)\dot{y} + \frac{1}{2}ma\dot{\varphi} &= I \\ m\frac{a^2}{3}\dot{\varphi} + \frac{1}{2}ma\dot{y} &= \frac{a}{2}I \end{aligned}$$

From these one can solve for the desired quantities.

Answer: $\dot{y} = \frac{I}{m+4M}$ and $\dot{\varphi} = \frac{6MI}{ma(m+4M)}$.

Idea problems:

Problem 4: One form of Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0.$$

Consider the case of a single particle and assume that $L = T - U$ where the work function U is given by, $U = -\frac{1}{2}m\Omega^2\rho^2 - m\Omega\rho^2\dot{\varphi}$, assuming cylinder coordinates (ρ, φ, z). Find the equations of motion and interpret them.

Solution 4: Form $L = T - U$ and find

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + \frac{1}{2}m\Omega^2\rho^2 + m\Omega\rho^2\dot{\varphi}.$$

The equations of motion become

$$\begin{aligned} m\ddot{\rho} - m\rho\dot{\varphi}^2 &= m\rho\Omega^2 + 2m\Omega\rho\dot{\varphi} \\ m\rho\ddot{\varphi} + 2m\dot{\rho}\dot{\varphi} &= -2m\Omega\dot{\rho} \\ m\ddot{z} &= 0 \end{aligned}$$

On the right hand sides we here have mass times acceleration expressed in cylinder coordinates. On the left hand sides we therefore have generalized (canonical) forces.

The interpretation of these is that the equations are the equations for a free particle moving relative to a system the rotates about the z-axis with angular velocity ω. The forces on the particle are therefore the (fictitious) centrifugal and Coriolis forces. Note that the Lagrange function can be rewritten on the following form

$$L = \frac{1}{2}m[\dot{\rho}^2 + \rho^2(\Omega + \dot{\varphi})^2 + \dot{z}^2].$$

From this form it is obvious that the particle has an angular velocity Ω even when it is at rest, which means that it is observed from a rotating reference frame.

Problem 5: Assume that the forces on a system are large during a very short time, that is, one is dealing with an impact. It is characterized by the fact that the position (configuration) changes negligibly during the short time, but not the velocities. How can Lagrange's equations of the system be adapted so that they answer the question: how does the impact change the velocity state of the system?

Solution 5: See Theory of Lagrange's equations, Section 9.2.

turn over

Problem 6: A homogenous cube of mass m and edge length a is divided into eight identical cubes. A new body is formed by gluing together seven of these again, in the original cuts. Find the inertia tensor of the new body with respect to a system with origin in the midpoint (center of mass) of the original intact cube and with the coordinates axes parallel to the sides of the cube.

Solution 6: For the original cube we have the inertia tensor

$$\mathbf{I}^{\text{cube}} = \frac{1}{6}ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Due to additivity we also have

$$\mathbf{I}^{\text{cube}} = \mathbf{I}^{\text{body}} + \mathbf{I}^{\text{octant}}$$

where $\mathbf{I}^{\text{body}}$ is the desired inertia tensor of the new body, and $\mathbf{I}^{\text{octant}}$ is the inertia tensor of the missing piece (one of the octants). We also have that

$$\mathbf{I}^{\text{octant}} = \mathbf{I}^{\text{small cube}} + \text{Steiner, or parallel axis, contribution.}$$

For a small (one eighths) cube we have

$$\mathbf{I}^{\text{small cube}} = \frac{1}{6}\frac{m}{8}\left(\frac{a}{2}\right)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

while the Steiner, or parallel axes, contribution is the same as for a particle of mass $m/8$ and position $\boldsymbol{r} = (a/4)\boldsymbol{e}_x + (a/4)\boldsymbol{e}_y + (a/4)\boldsymbol{e}_z$, and this gives:

$$\text{Steiner, or parallel axis, contribution} = \frac{m}{8}a^2 \begin{pmatrix} 1/8 & -1/16 & -1/16 \\ -1/16 & 1/8 & -1/16 \\ -1/16 & -1/16 & 1/8 \end{pmatrix}.$$

We therefore now finds that

$$\mathbf{I}^{\text{body}} = \mathbf{I}^{\text{cube}} - (\mathbf{I}^{\text{small cube}} + \text{Steiner, or parallel axis, contribution})$$

This gives **Answer:**

$$\mathbf{I}^{\text{body}} = \frac{1}{6}ma^2 \begin{pmatrix} 7/8 & 3/64 & 3/64 \\ 3/64 & 7/8 & 3/64 \\ 3/64 & 3/64 & 7/8 \end{pmatrix}.$$

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