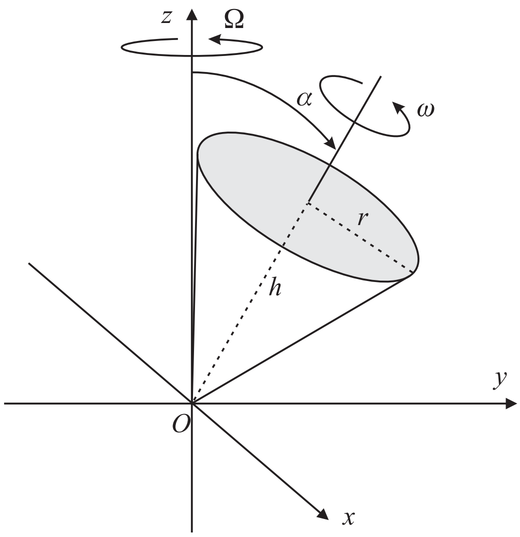
Advanced Engineering Dynamics, 5C1150
Solutions to Exam, 2006 10 24

Calculational problems

Problem 1: A top consists of a homogeneous, straight, circular cone of mass m, height h, and base radius r. The vertex of the cone is connected to a fixed point O by means of a ball and socket joint, so that the cone can rotate freely about O. The plane that passes through the axis of the cone and the vertical through O rotates with constant angular velocity Ω about the vertical through O. The angle α between the axis and the vertical remains fixed. Determine the angular velocity ω of the cone relative to this plane.



Solution 1: From $\dot{\boldsymbol{L}} = \boldsymbol{M}$ one obtains, using the Resal or B-system, that

$$\frac{\mathrm{d}^B}{\mathrm{d}t}\boldsymbol{L} + {}^O\boldsymbol{\omega}^B \times \boldsymbol{L} = \boldsymbol{M}$$

Here ${}^O\boldsymbol{\omega}^B = \Omega(\sin\alpha\, \boldsymbol{e}_2^B + \cos\alpha\, \boldsymbol{e}_3^B)$. The angular velocity of the cone is ${}^O\boldsymbol{\omega}^A = \Omega\sin\alpha\, \boldsymbol{e}_2^B + (\Omega\cos\alpha + \omega)\, \boldsymbol{e}_3^B$. The angular momentum of the cone with respect to O is then:

$$L_1\boldsymbol{e}_1^B + L_2\boldsymbol{e}_2^B + L_3\boldsymbol{e}_3^B = J_1\Omega\sin\alpha\, \boldsymbol{e}_2^B + J_3(\Omega\cos\alpha + \omega)\, \boldsymbol{e}_3^B$$

since the B-system is a principal axes system of the cone even if it is not body fixed. Here $J_3 = \frac{3}{10}mr^2$ and $J_1 = \frac{3}{5}m\left(h^2 + \frac{r^2}{4}\right)$. We also have that,

$$\boldsymbol{M} = \frac{3}{4}hmg\sin\alpha\, \boldsymbol{e}_1^B,$$

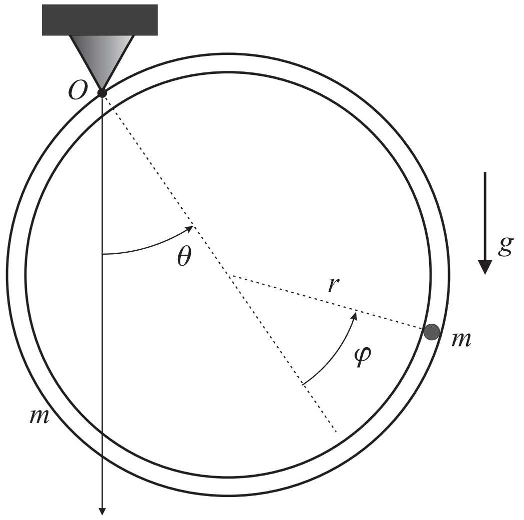
since the center of mass of the cone is at height $h/4$ from the base. Since all components of $\boldsymbol{L}$ in the B-system are constant we now get ${}^O\boldsymbol{\omega}^B \times \boldsymbol{L} = \boldsymbol{M}$ which gives

$$\frac{3}{10}mr^2\Omega(\Omega\cos\alpha + \omega)\sin\alpha - \frac{3}{5}m\left(h^2 + \frac{r^2}{4}\right)\Omega^2\cos\alpha\sin\alpha = \frac{3}{4}hmg\sin\alpha$$

Solving for ω one obtains: **Answer:**

$$\omega = \left(\frac{4h^2 - r^2}{2r^2}\right)\Omega\cos\alpha + \frac{5}{2}\frac{hg}{r^2\Omega}.$$

Problem 2: A small tube of mass m is bent in the form of a circle of radius r and is pivoted about a fixed point O on its circumference. A particle, also of mass m, can slide with negligible friction inside the tube. a) Use the Lagrangian method to obtain the equations of motion, assuming plane motion. b) Now assume small amplitude motion and obtain the natural frequencies and the corresponding amplitude ratios.



Solution 2: The kinetic energy is $T = \frac{1}{2}J_O\dot{\theta}^2 + \frac{1}{2}m\boldsymbol{v}^2$, with $J_O = 2mr^2$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ where $\boldsymbol{r} = r\boldsymbol{e}_\rho(\theta) + r\boldsymbol{e}_\rho(\theta+\varphi)$. We get $\boldsymbol{v} = r\dot{\theta}\boldsymbol{e}_\varphi(\theta) + r(\dot{\theta}+\dot{\varphi})\boldsymbol{e}_\varphi(\theta+\varphi)$ so $\boldsymbol{v}^2 = r^2[\dot{\theta}^2 + (\dot{\theta}+\dot{\varphi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\varphi})\cos(\varphi)]$. The kinetic energy is thus $T = \frac{1}{2}mr^2[\dot{\theta}^2(4+2\cos\varphi)+\dot{\varphi}^2+2\dot{\theta}\dot{\varphi}(1+\cos\varphi)]$. The potential energy is $V = -2mgr\cos\theta - mgr\cos(\theta+\varphi)$. Lagrange's equations of motion for the θ - motion and the φ - motion respectively are:

$$\begin{aligned} mr^2[2\ddot{\theta}(2+\cos\varphi) + \ddot{\varphi}(1+\cos\varphi) - \dot{\varphi}(2\dot{\theta}+\dot{\varphi})\sin\varphi] - mgr[2\sin\theta + \sin(\theta+\varphi)] &= 0 \\ mr^2[\ddot{\varphi} + \ddot{\theta}(1+\cos\varphi) + \dot{\theta}^2\sin\varphi] + mgr\sin(\theta+\varphi) &= 0 \end{aligned}$$

For small amplitude motion we have

$$L = T - V = \frac{1}{2}mr^2(6\dot{\theta}^2 + 4\dot{\theta}\dot{\varphi} + \dot{\varphi}^2) + \frac{1}{2}mgr[3\theta^2 + 2\theta\varphi + \varphi^2]$$

The secular equation

$$\det(-\mathbf{M}x + \mathbf{K}) = \begin{vmatrix} -6rx+3g & -2rx+g \\ -2rx+g & -rx+g \end{vmatrix} = 0$$

becomes $(6rx - 3g)(rx - g) - (2rx - g)^2 = 0$. The roots give

$$x_1 = \omega_1^2 = \frac{g}{2r}, \quad x_2 = \omega_2^2 = \frac{2g}{r}.$$

Then solving

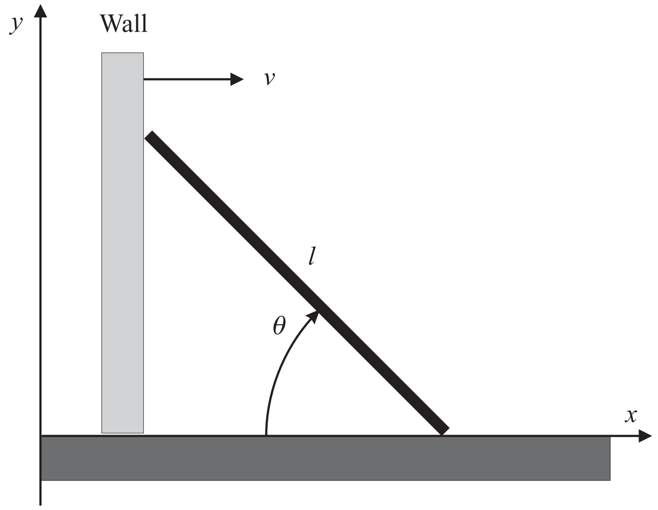
$$(-\mathbf{M}x_a + \mathbf{K})\boldsymbol{a}_a = \mathbf{0}$$

gives

$$a_{1\varphi}/a_{1\theta} = 0, \quad a_{2\varphi}/a_{2\theta} = -3$$

and this concludes this problem.

Problem 3: A rigid rod of mass m and length l is motionless with $\theta = 45°$ when, at $t = 0$, the vertical wall begins to move in the positive x-direction at constant speed v. Denote the coordinates of the midpoint (center of mass) of the rod x, y. Assuming smooth surfaces, and that contact with the surfaces are maintained, solve for the initial values of $\dot{x}, \dot{y}$, and $\dot{\theta}$. Find the impulse exerted on the rod by the wall at $t = 0$.



Solution 3: The lower end remains on the horizontal surface so there are two degrees-of-freedom and we can use generalized coordinates x, θ. For the y-coordinate of the center of mass we then get $y = \frac{l}{2}\sin\theta$. This gives $\dot{y} = \frac{l}{2}\dot{\theta}\cos\theta$ and the kinetic energy is therefore

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J_G\dot{\theta}^2 = \frac{1}{2}m[\frac{l^2}{12}(1 + 3\cos^2\theta)\dot{\theta}^2 + \dot{x}^2].$$

At the initial position $\cos\theta = \sin\theta = 1/\sqrt{2}$.

The position of the point of contact with the wall is

$$\boldsymbol{r}_P(x, \theta) = x_P\boldsymbol{e}_x + y_P\boldsymbol{e}_y = (x - \frac{l}{2}\cos\theta)\boldsymbol{e}_x + l\sin\theta\,\boldsymbol{e}_y.$$

The work done by the impact force is then $\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}_P = F_x\boldsymbol{e}_x \cdot \left(\frac{\partial \boldsymbol{r}_P}{\partial x}\mathrm{d}x + \frac{\partial \boldsymbol{r}_P}{\partial \theta}\mathrm{d}\theta\right)$. This gives $\mathrm{d}W = F_x\mathrm{d}x + F_x\frac{l}{2}\sin\theta\,\mathrm{d}\theta$. The generalized forces are thus $Q_x = F_x$ and $Q_\theta = F_x\frac{l}{2}\sin\theta$. If we call the impulse I_x we now get for the generalized impulses $Q_x = I_x$ and $I_\theta = I_x\frac{l}{2}\sin\theta$. The Lagrange impact equations now become:

$$p_x - 0 = m\dot{x} - 0 = I_x$$

$$p_\theta - 0 = m\frac{5l^2}{24}\dot{\theta} - 0 = I_x\frac{l}{2\sqrt{2}}$$

We also have that

$$v = \dot{x}_P = \dot{x} + \frac{l}{2\sqrt{2}}\dot{\theta}$$

Eliminating I_x from the first pair of equations gives $\dot{\theta} = \frac{12\dot{x}}{5\sqrt{2}l}$. Minor calculations then finally give:

Answer: $\dot{x} = \frac{5}{8}v$, $\dot{y} = \frac{3}{8}v$, $\dot{\theta} = \frac{3}{2\sqrt{2}}\frac{v}{l}$, and $I_x = m\frac{5}{8}v$.

Idea problems:

Problem 4: Show that the moments of inertia of a homogeneous, straight, circular cone of mass m, height h, and base radius r are $J_z = \frac{3}{10}mr^2$ and $J_x = \frac{3}{5}m\left(h^2 + \frac{r^2}{4}\right)$. Here the z-axis is the symmetry axis of the cone and the x-axis is a perpendicular axis through the vertex.

Problem 5: Euler's dynamic equations are the components of the equation of motion $\dot{\boldsymbol{L}} = \boldsymbol{M}$ along the body fixed principal axes system of a rigid body. Derive these and show how they simplify for a symmetric top. Assume $\boldsymbol{M} = \boldsymbol{0}$ and solve them for the symmetric top case.

Problem 6: An n-degree-of-freedom system is described by the Lagrangian,

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n).$$

How are the generalized momenta defined? Under what conditions are these constants of the motion? Prove this!

Each problem gives maximum 3 points. A minimum of three points on each of the calculational and idea parts is required.

Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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