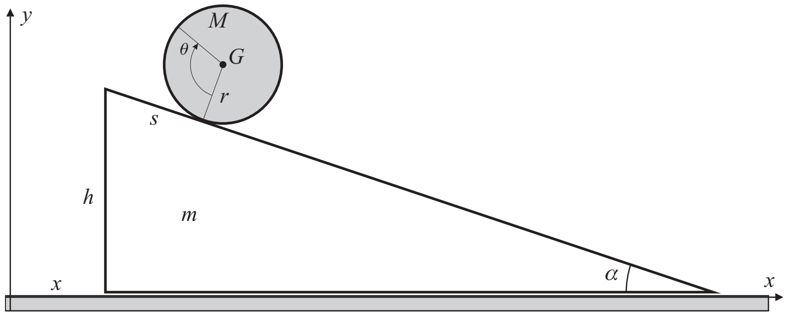
Rigid Body Dynamics, SG2150
Solutions to Exam, 2012 02 18

Calculational problems

Problem 1: A wedge of mass m can slide on a smooth horizontal plane. A cylinder of mass M and radius r can roll without slipping on the wedge . The cylinder is released from rest and starts to roll down the incline on the wedge, which makes an angle α with the horizontal. Find the acceleration of the wedge.



Solution 1: 1) Generalized coordinates: x, position of left side of the wedge, s, distance rolled by cylinder from left side of wedge. Rolling condition for cylinder: $s = r\theta$, where θ is the angle that the cylinder has turned. Kinetic energy of system:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\boldsymbol{v}_G^2 + \tfrac{1}{2}J_G\dot{\theta}^2.$$

Position of cylinder center of mass: $x_G = x + s\cos\alpha + r\sin\alpha$, $y_G = h - s\sin\alpha + r\cos\alpha$, where h is the height of the wedge. The cylinder center of mass velocity is thus given by:

$$\dot{x}_G = \dot{x} + \dot{s}\cos\alpha,\ \dot{y}_G = -\dot{s}\sin\alpha.$$

The moment of inertia of the cylinder is $J_G = \frac{1}{2}Mr^2$. The kinetic energy becomes

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x}^2 + 2\dot{x}\dot{s}\cos\alpha + \dot{s}^2) + \tfrac{1}{2}\tfrac{1}{2}Mr^2(\dot{s}/r)^2 =$$

$$\tfrac{1}{2}(m+M)\dot{x}^2 + M\cos\alpha\,\dot{x}\dot{s} + \frac{3}{4}M\dot{s}^2.$$

The potential energy is $V = Mgy_G$, or equivalently just $V = -Mg\sin\alpha\, s$. The Lagrange function is, $L = T - V$. First note that x is cyclic so that $\frac{\partial L}{\partial x} = 0$ and that therefore $\frac{\partial L}{\partial \dot{x}} =$ constant. This means that,

$$(m+M)\dot{x} + M\cos\alpha\,\dot{s} = \text{constant}.$$

The Lagrange equation of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$, becomes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\cos\alpha\,\dot{x} + \frac{3}{2}M\dot{s}\right) - Mg\sin\alpha = 0.$$

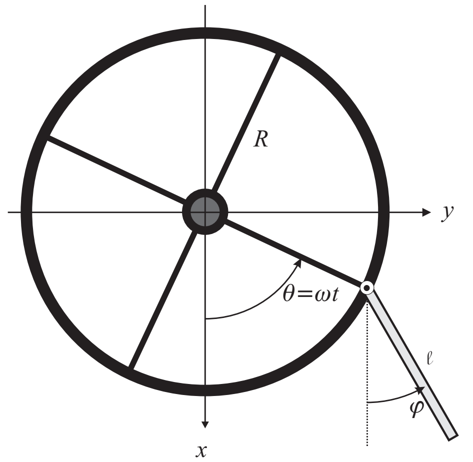
Combining the two equations we can eliminate $\ddot{s}$ and get,

$$\left(M\cos\alpha - \frac{3}{2}\frac{m+M}{\cos\alpha}\right)\ddot{x} = Mg\sin\alpha.$$

This gives the **Answer:**

$$\ddot{x} = \frac{M\sin\alpha\; g}{M\cos\alpha - \frac{3}{2}\frac{m+M}{\cos\alpha}} = -\frac{\sin(2\alpha)}{1 + 2\sin^2\alpha + \frac{3m}{M}}\; g.$$

Problem 2: A rod of length ℓ hangs at the edge of a vertical wheel of radius R (like a gondola on a ferris wheel). The wheel rotates with a constant given angular velocity ω about a horizontal axis through the midpoint. Find the Lagrangian and the equation of motion of the rod. Find the angular frequency of small amplitude motions of the rod in the two limiting cases $R\omega^2 \ll g$ and $R\omega^2 \gg g$. In the latter case the motion is of small amplitude relative the rotating system of the wheel.



Solution 2: This is a problem with a time dependent constraint. The kinetic energy of the rod is,

$$T = \tfrac{1}{2} m \boldsymbol{v}_G^2 + \tfrac{1}{2}\left(\frac{m\ell^2}{12}\right)\dot{\varphi}^2$$

The position of the center of mass of the rod is $\boldsymbol{r}_G = R\boldsymbol{e}_r(\theta) + \frac{\ell}{2}\boldsymbol{e}_r(\varphi)$. The center of mass velocity of the rod, $\dot{\boldsymbol{r}}_G$, is thus given by,

$$\boldsymbol{v}_G = R\omega\, \boldsymbol{e}_\varphi(\theta) + \frac{\ell}{2}\dot{\varphi}\, \boldsymbol{e}_\varphi(\varphi).$$

The potential energy of the rod is,

$$V = -mgx_G = mg\left(-R\cos(\omega t) - \frac{\ell}{2}\cos\varphi\right).$$

Skipping a constant term in the kinetic energy, and the term in the potential involving $\cos(\omega t)$, since it will not affect the equation of motion for the rod, one finds,

$$L = \frac{m\ell^2}{6}\dot{\varphi}^2 + mR\omega\frac{\ell}{2}\dot{\varphi}\cos(\varphi - \omega t) + mg\frac{\ell}{2}\cos\varphi,$$

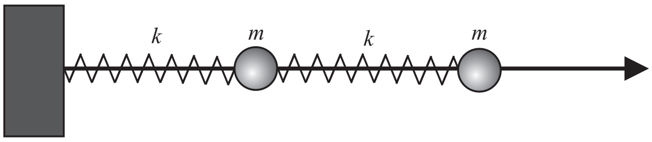
for the Lagrangian.

The usual calculations $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$ will now give,

$$\ddot{\varphi} = -\frac{3}{2}\left(\frac{g}{\ell}\sin\varphi + \frac{R}{\ell}\omega^2\sin(\varphi - \omega t)\right),$$

which is the equation of motion for the rod. For $g \gg R\omega^2$ neglect the second term on the right hand side and find the usual equation of motion for a physical pendulum consisting of a rod. The angular frequency is **Answer:** $\omega_g = \sqrt{\frac{3}{2}\frac{g}{\ell}}$. For the other case we introduce the variable $u = \varphi - \omega t$, and note that then $\ddot{\varphi} = \ddot{u}$. With $g \ll R\omega^2$ the equation of motion for the u-coordinate is $\ddot{u} = -\frac{3}{2}\frac{R\omega^2}{\ell}\sin u$. For small amplitude one then gets **Answer:** $\omega_\omega = \sqrt{\frac{3}{2}\frac{R\omega^2}{\ell}}$.

Problem 3: Calculate the angular eigen frequencies for the two degree of freedom coupled oscillator problem in the figure below. The spring on the left is attached to a fixed wall at its left end. The two identical particles of mass m can slide with negligible friction along the horizontal track. The two springs of stiffness k are also identical.



Solution 3: The kinetic energy is given by,

$$T = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

and the potential energy is,

$$V = \tfrac{1}{2}k[x_1^2 + (x_2 - x_1)^2] = \tfrac{1}{2}k(2x_1^2 + x_2^2 - 2x_1x_2)$$

where x_1, x_2 denote the deviations from the equilibrium positions of the left and right particles respectively.

From this we can read off the M- and K-matrices:

$$\mathbf{M} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K} = k\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Putting $\omega^2 = x$ we now find the secular equation,

$$\det(-\mathbf{M}y + \mathbf{K}) = \begin{vmatrix} -my + 2k & -k \\ -k & -my + k \end{vmatrix} = 0,$$

which gives the roots,

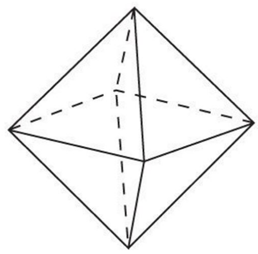
$$y_{1,2} = \tfrac{1}{2}(3 \pm \sqrt{5})\frac{k}{m} = \tfrac{1}{2}\tfrac{1}{2}(\sqrt{5} \pm 1)^2\frac{k}{m}$$

We thus find the two angular eigen frequencies (**Answer:**)

$$\omega_{1,2} = \tfrac{1}{2}(\sqrt{5} \pm 1)\sqrt{\frac{k}{m}}.$$

Idea problems:

Problem 4: Twelve identical slender homogeneous rods, each of mass m and length a are welded together at the endpoints so that they constitute the edges of a regular octahedron. Find the moment of inertia of this body with respect to an axis through the midpoint.



Solution 4:
We use that the moment of inertia for a rod of mass M and length ℓ is $J_m = \frac{M\ell^2}{12}$ for a perpendicular axis though the midpoint, and $J_e = \frac{M\ell^2}{3}$ for a perpendicular axis though an endpoint.

Consider the octahedron seen from straight above on corner. There are then four rods in a horizontal plane at a perpendicular distance $a/2$ from the vertical axis through the center. These four rods contribute the moment of inertia $J_h = 4\left[\frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2\right] = \frac{4}{3}ma^2$ according to the parallel axis theorem. The remaining eight rods extend a distance $a/\sqrt{2}$ from the vertical axis and this contributes $J_v = 8\left[\frac{m}{3}\left(\frac{a}{\sqrt{2}}\right)^2\right] = \frac{4}{3}ma^2$ to the total.

Adding these now gives the **Answer:**

$$J = \frac{8}{3}ma^2.$$

Since the octahedron is symmetric all moments of inertia for axes through the midpoint are the same.

Problem 5: Find the components of the vector equation $\dot{\boldsymbol{L}} = \boldsymbol{M}$ for a rigid body using the body fixed principal axes system of basis vectors, *i.e.* Euler's dynamic equations.

Solution 5:
See Section 4.2, pages 64-65, in Essén's *Dynamics of Bodies.*

Problem 6: Use the Lagrange formalism to show that any system with a Lagrangian that does not depend explicitly on time is characterized by a conserved quantity (normally the energy).
Solution 6:
This is shown in Section 19.1, pages 25-26, in Essén's *The Theory of Lagrange's Method.*

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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