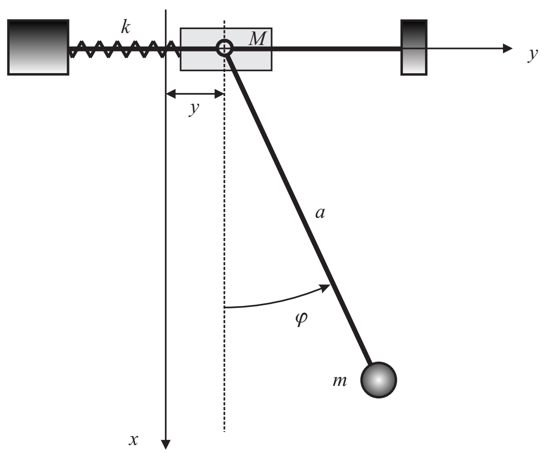
Rigid Body Dynamics, SG2150
Solutions to Exam, 2010 01 15

Calculational problems

Problem 1: A mathematical pendulum *i.e.* a particle of mass m in a light rod of length a, is attached to a block that slides along a horizontal track. The light rod can rotate about a horizontal axis perpendicular to the track in a vertical plane. The block has mass $M = 8m$ and is attached to one end of a horizontal spring of stiffness $k = 9mg/a$ which is fixed at the other end. Find the exact Lagrangian for the system. Also find its quadratic approximation and the periods for small oscillations about equilibrium.



Solution 1: Choosing the coordinates according to the figure so that $y = 0$ when the spring is unloaded we find the Lagrangian

$$L = T - V = \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}m\boldsymbol{v}^2 - \tfrac{1}{2}ky^2 - mga(1 - \cos\varphi).$$

Here the velocity $\boldsymbol{v}$ of the particle m is given by $\boldsymbol{v} = \dot{y}\,\boldsymbol{e}_y + a\dot{\varphi}\,\boldsymbol{e}_\varphi$. This gives us (**Answer 1:**) The exact Lagrangian is

$$L = T - V = \tfrac{1}{2}(m + M)\dot{y}^2 + \tfrac{1}{2}ma^2\dot{\varphi}^2 + m\dot{y}a\dot{\varphi}\cos\varphi - \tfrac{1}{2}ky^2 - mga(1 - \cos\varphi).$$

The quadratic approximation is obtained by putting $\cos\varphi \approx 1$ in the third term and $\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$ in the last term. The result can be written

$$L \approx \tfrac{1}{2}\left[(m + M)\dot{y}^2 + ma^2\dot{\varphi}^2 + 2ma\dot{y}\dot{\varphi}\right] - \tfrac{1}{2}\left[ky^2 + mga\varphi^2\right].$$

If we insert $M = 8m$ and $k = 9mg/a$ here we get, (**Answer 2a:**):

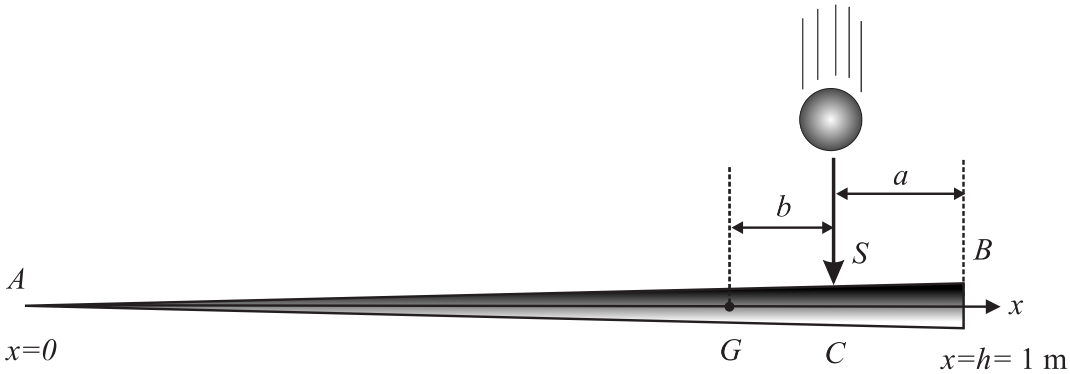
$$L \approx \tfrac{1}{2}m\left[9\dot{y}^2 + a^2\dot{\varphi}^2 + 2a\dot{y}\dot{\varphi}\right] - \tfrac{1}{2}\frac{mg}{a}\left[9y^2 + a^2\varphi^2\right].$$

This gives us the M- and K- matrices

$$\mathbf{M} = m\begin{pmatrix} 9 & a \\ a & a^2 \end{pmatrix}, \quad \mathbf{K} = \frac{mg}{a}\begin{pmatrix} 9 & 0 \\ 0 & a^2 \end{pmatrix}.$$

The secular equation, $\det(-\mathbf{M}x + \mathbf{K}) = 0$, becomes, $9\,(-x + g/a)^2 - x^2 = 0$. The roots, which are the eigen frequencies squared can be written, $x_{1,2} = \omega_{1,2}^2 = \frac{3}{3\pm1}\frac{g}{a}$. Thus the periods are (**Answer 2b:**) $T_{1,2} = 2\pi\sqrt{\frac{3\pm1}{3}\frac{a}{g}}$.

Problem 2: A bat has the shape of a long narrow (straight circular homogeneous) cone. A person holding the the bat at the pointed end A of the cone hits a ball that transfers an impact S to the bat in a direction perpendicular to the axis of the cone. Assume that the bat has length one meter. Find the distance a from the thick end B where the ball should be hit if there is to be no reaction impulse in the hand of the batter.



Solution 2: We have,

$$x_G = \frac{3}{4}h, \tag{1}$$

and,

$$J_A = J_z = \frac{3}{5}mh^2, \tag{2}$$

according to problem 4.

Steiner's theorem (the parallel axis theorem) now gives,

$$J_A = J_G + mx_G^2 \;\;\Rightarrow\;\; J_G = \frac{3}{80}mh^2, \tag{3}$$

for the moment of inertia with origin at G.

The impact equations now give:

$$\Delta \boldsymbol{p} = mv_G\,\boldsymbol{e}_y = S\,\boldsymbol{e}_y,$$

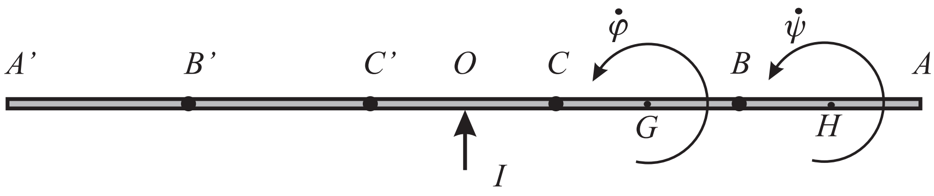
$$\Delta \boldsymbol{L}_G = J_G\omega\,\boldsymbol{e}_z = bS\,\boldsymbol{e}_z,$$

where $b = x_C - x_G$. Thus $\omega = bS/J_G$. The condition that there is no reaction impulse at A requires that there is no velocity v_A at A just after the impact. The connection formula for velocities in a rigid body gives for this case that,

$$v_A = 0 \;\;\Leftrightarrow\;\; v_G - \omega x_G = (S/m) - (bS/J_G)(3h/4) = 0.$$

We find that $b = h/20$. Thus the distance from the thick end to the impact point should be $|BC| = a = h - (x_G + b) = h/5$. This gives us the
Answer: $a = (1/5)\,\text{m} = 20\,\text{cm}$, since $h = 1$ m.

Problem 3: A linkage consists of five identical thin rods each of mass m and length a. They are hinged to each other at the endpoints so that the linkage as a whole can deform in a plane. Initially the linkage is placed on a horizontal plane, at rest in a straight configuration, when it is stuck at the midpoint O in a direction perpendicular to the linkage. The impulse delivered is I. Assume symmetry about the axis through the midpoint perpendicular to the linkage. Find the angular velocities of the rods immediately after impact.



Solution 3: The kinetic energy is,

$$T = \frac{m}{2} v_O^2 + 2\left(\frac{m}{2} v_G^2 + \frac{m}{2} v_H^2 + \frac{J}{2}\dot{\varphi}^2 + \frac{J}{2}\dot{\psi}^2\right).$$

If we choose the x axis in the direction of the impact we have $v_O = \dot{x}$. Because of the geometry all translational velocities are in the x-direction immediately after the impact. Symmetry gives that $v_C = v_G = \dot{x}$. The connection formula then gives $v_G = v_C + (a/2)\dot{\varphi} = \dot{x} + (a/2)\dot{\varphi}$. Further $v_B = \dot{x} + a\dot{\varphi}$ and consequently $v_H = v_B + (a/2)\dot{\psi} = \dot{x} + a\dot{\varphi} + (a/2)\dot{\psi}$. The moments of inertia are $J = ma^2/12$. Calculations then give,

$$T = m\left(\frac{5}{2}\dot{x}^2 + \frac{4}{3}a^2\dot{\varphi}^2 + \frac{1}{3}a^2\dot{\psi}^2 + 3\dot{x}a\dot{\varphi} + \dot{x}a\dot{\psi} + a^2\dot{\varphi}\dot{\psi}\right),$$

for the kinetic energy immediately after impact. The generalized impulses are $I_x = I, I_\varphi = 0, I_\psi = 0$. The generalized momenta are all zero before impact. The values after impact are thus,

$$\frac{\partial T}{\partial \dot{x}} = p_x = 5m\dot{x} + 3ma\dot{\varphi} + ma\dot{\psi} \quad = \quad I, \tag{4}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = p_\varphi = 3ma\dot{x} + (5/2)ma^2\dot{\varphi} + ma^2\dot{\psi} \quad = \quad 0, \tag{5}$$

$$\frac{\partial T}{\partial \dot{\psi}} = p_\psi = ma\dot{x} + ma^2\dot{\varphi} + (2/3)ma^2\dot{\psi} \quad = \quad 0. \tag{6}$$

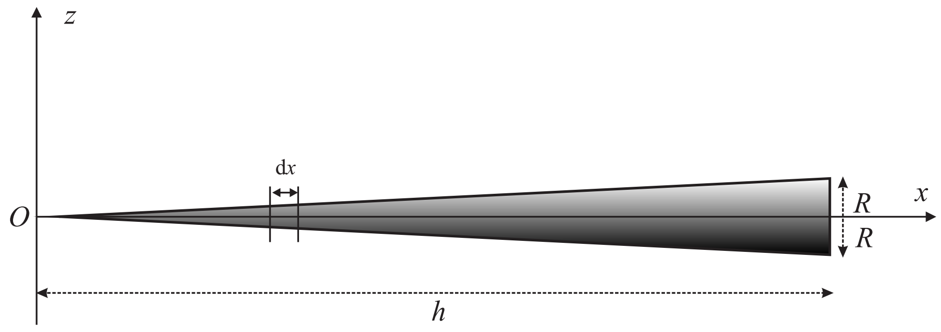
Solving this system gives the **Answer:**

$$\dot{\varphi} = -\frac{6I}{5ma},\ \dot{\psi} = \frac{3I}{5ma}.$$

One also finds that $\dot{x} = 4I/(5m)$.

Idea problems:

Problem 4: Find the position of the center of mass, and the moment of inertia J_z with respect to a perpendicular axis, for a long narrow (slim) cone with respect to an origin O at the vertex of the cone. Assume that the cone is straight, circular and homogeneous of height h and base radius R ($R \ll h$).



Solution 4:
Assume that the cone has density ρ. Put an x-axis along the cone with origin at O. The radius of the circular cross section at x is then $r = xR/h$ and the volume of the cylindrical mass element of height $\mathrm{d}x$ at x is then $\mathrm{d}V = \pi r^2 \mathrm{d}x = \pi(R^2/h^2)x^2\mathrm{d}x$. The mass of this volume element is thus $\mathrm{d}m = \rho \mathrm{d}V = \rho\pi(R^2/h^2)x^2\mathrm{d}x$. Put $\kappa = \rho\pi(R^2/h^2)$ so that $\mathrm{d}m = \kappa x^2 \mathrm{d}x$. This means that the mass of the cone (bat) is,

$$m = \int_0^h \mathrm{d}m = \int_0^h \kappa x^2 \mathrm{d}x = \kappa h^3/3. \tag{7}$$

So $\kappa = 3m/h^3$. It is now easy to find the center of mass G of the cone. We have,
(**Answer 1:**)

$$x_G = \frac{1}{m}\int_0^h x\,\mathrm{d}m = \frac{\kappa h^4/4}{\kappa h^3/3} = \frac{3}{4}h. \tag{8}$$

The moment of inertia with respect to a perpendicular axis through the origin is then,
(**Answer 2:**)

$$J_z = \int_0^h x^2 \mathrm{d}m = \kappa\frac{h^5}{5} = \frac{3}{5}mh^2. \tag{9}$$

Note that we assume here that the bat is narrow so that it can be approximated with a one dimensional mass distribution, $\lambda(x) = \kappa x^2$, along the x-axis.

Problem 5: A rigid body rotates about a fixed point O. Derive a formula for the components of the angular momentum vector $\boldsymbol{L}_O$ given the components of the angular velocity vector $\boldsymbol{\omega}$ and the masses m_i and position vectors $\boldsymbol{r}_i$ of the particles of the body.

Solution 5:
The solution can be found in the text to this course.

Problem 6: Derive Euler's dynamic equations for a rigid body that rotates about a fixed point O. Consider the components of the vector equation $\dot{\boldsymbol{L}} = \boldsymbol{M}$ in a body fixed principal axes system.

Solution 6:
The solution can be found in the text to this course.

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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