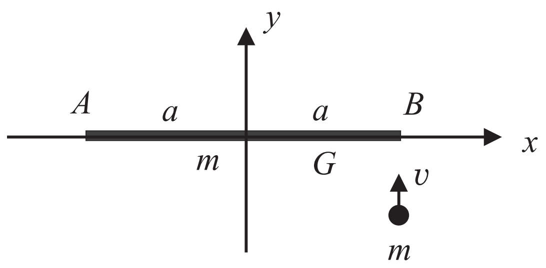
Rigid Body Dynamics, SG2150
Solutions to Exam, 2010 10 21

Calculational problems

Problem 1: A slender rod AB of mass m and length $2a$, rests on a smooth horizontal plane. A small ball of the same mass m impacts with the rod at its endpoint B and sticks there. Just before impact the ball has a velocity v in the horizontal plane perpendicular to the rod. Find the angular velocity of the body (rod plus ball) and the velocity of the point A, just after impact.



Solution 1: Momentum conservation $\boldsymbol{p}(t_f) = \boldsymbol{p}(t_i)$ for the system gives.

$$2m\boldsymbol{v}_G = mv\boldsymbol{e}_y$$

so that $\boldsymbol{v}_G = (v/2)\boldsymbol{e}_y$. Angular momentum conservation $\boldsymbol{L}_G(t_f) = \boldsymbol{L}_G(t_i)$ gives

$$J_G\,\omega\,\boldsymbol{e}_z = \boldsymbol{r}_{GB} \times mv\,\boldsymbol{e}_y$$

Here G is the center of mass of the system of ball plus rod, so $\boldsymbol{r}_{GB} = (a/2)\,\boldsymbol{e}_x$. This gives

$$J_G\,\omega = \frac{1}{2}mva.$$

For the moment of inertia of the system we find

$$J_G = \frac{1}{12}m(2a)^2 + m(a/2)^2 + m(a/2)^2 = \frac{5}{6}ma^2.$$

(**Answer i:**) Thus the angular velocity is

$$\boldsymbol{\omega} = \frac{3v}{5a}\,\boldsymbol{e}_z.$$

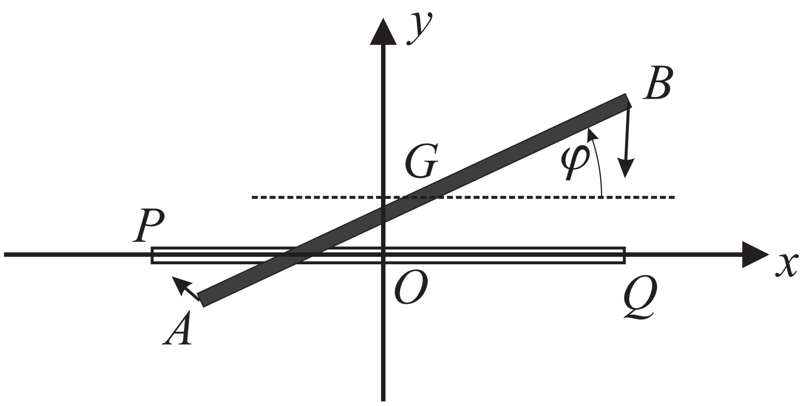
The velocity of the point A is then given by

$$v_A = v_G - (3a/2)\omega$$

along the y-axis. This gives: (**Answer ii:**) The velocity is

$$\boldsymbol{v}_A = -\frac{2}{5}v\boldsymbol{e}_y.$$

Problem 2: A slender rod AB of mass m and length $2a$, can move in a smooth horizontal plane. The endpoints of the rod are attached with elastic strings in such a way that at equilibrium the rod goes between the points P and Q a distance $2a$ apart and the forces on the rod are then zero. When the rod is displaced in the plane the force on the endpoints A and B pull them back towards P and Q respectively. The strength of these forces are proportional to the distances ($|\boldsymbol{F}_A| = k|AP|$, $|\boldsymbol{F}_B| = k|BQ|$). Find the kinetic energy, the potential energy and the equations of motion of the rod. Also find the angular frequencies for small amplitude motions. Use center of mass coordinates x, y and angle φ as generalized coordinates. Hint: the potential energy of the force $\boldsymbol{F}_A$ is $V_A = (k/2)(\boldsymbol{r}_A - \boldsymbol{r}_P)^2$.



Solution 2: The kinetic energy is,

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}J_G\dot{\varphi}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \frac{a^2}{3}\dot{\varphi}^2\right),$$

and the potential energy is,

$$V = V_A + V_B = \frac{1}{2}k(\boldsymbol{r}_G - a\,\boldsymbol{e}_\rho - \boldsymbol{r}_P)^2 + \frac{1}{2}k(\boldsymbol{r}_G + a\,\boldsymbol{e}_\rho - \boldsymbol{r}_Q)^2.$$

Here $\boldsymbol{r}_G = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y$, $\boldsymbol{e}_\rho = \cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y$, $\boldsymbol{r}_P = -a\,\boldsymbol{e}_x$, $\boldsymbol{r}_Q = a\,\boldsymbol{e}_x$. We thus get,

$$V = \frac{1}{2}k\left\{[(x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y) - a(\boldsymbol{e}_\rho - \boldsymbol{e}_x)]^2 + [(x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y) + a(\boldsymbol{e}_\rho - \boldsymbol{e}_x)]^2\right\}.$$

Simplification gives, using $(\boldsymbol{e}_\rho - \boldsymbol{e}_x)^2 = (\cos\varphi - 1)^2 + \sin^2\varphi = 2(1 - \cos\varphi)$,

$$V(x, y, \varphi) = \frac{1}{2}k\left[2(x^2 + y^2) + 2a^2(\boldsymbol{e}_\rho - \boldsymbol{e}_x)^2\right] = \frac{1}{2}(2k)\left[x^2 + y^2 + 2a^2(1 - \cos\varphi)\right].$$

The problem separates into three independent problems, two simple harmonic oscillators for the translational motion and a pendulum problem for the angle variable. The Lagrangian is:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{m}{3}a^2\dot{\varphi}^2 - \frac{1}{2}(2k)(x^2 + y^2) + \frac{1}{2}(4k)a^2\cos\varphi,$$

when the constant term in the potential energy has been removed. The equations of motion are,

$$m\ddot{x} = -2kx \quad (1)$$

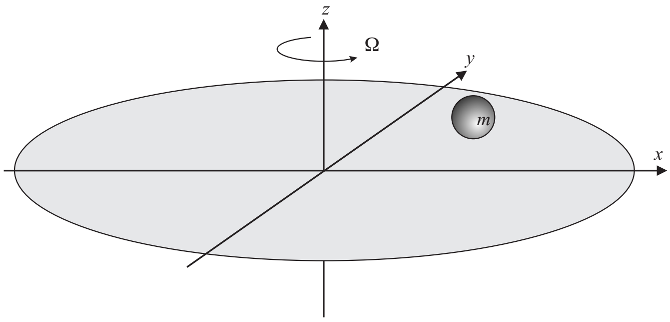
$$m\ddot{y} = -2ky \quad (2)$$

$$m\ddot{\varphi} = -6k\sin\varphi \quad (3)$$

For small amplitude motion $\sin\varphi$ is approximated by φ. The angular frequencies for the motion are then.

$$\omega_x = \sqrt{\frac{2k}{m}},\ \omega_y = \sqrt{\frac{2k}{m}},\ \omega_\varphi = \sqrt{\frac{6k}{m}}.$$

Problem 3: A rough horizontal table rotates with constant angular velocity Ω about a fixed vertical axis. Find the equations of motion for the center of mass of a ball, of mass m and radius R, that rolls on the table. Solve them. Hints: Use a fixed Cartesian coordinate system. Find the conditions for rolling without slipping by expressing the velocity of a point of the table and equating it to the velocity of the lowest point on the ball. Use the vector equations of motion, $m\ddot{\boldsymbol{r}}_G = \boldsymbol{F}$, $\dot{\boldsymbol{L}}_G = -R\boldsymbol{e}_z \times \boldsymbol{F}$. Use $J_G = 2mR^2/5$.



Solution 3: The rolling condition is that the velocity of the point of the table in contact with the ball,

$$\boldsymbol{v}_{Ct} = \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega\,\boldsymbol{e}_z \times (x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y) = -\Omega y\,\boldsymbol{e}_x + \Omega x\,\boldsymbol{e}_y,$$

is the same as the velocity of the lowest point of the ball (in contact with the table,

$$\boldsymbol{v}_{Cb} = \boldsymbol{v}_G + \boldsymbol{\omega} \times (-R\,\boldsymbol{e}_z) = (\dot{x}\,\boldsymbol{e}_x + \dot{y}\,\boldsymbol{e}_y) - R(\omega_x\,\boldsymbol{e}_x + \omega_y\,\boldsymbol{e}_y) \times \boldsymbol{e}_z = (\dot{x} - R\omega_y)\boldsymbol{e}_x + (\dot{y} + R\omega_x)\boldsymbol{e}_y.$$

This gives us the two equations,

$$\dot{x} - R\omega_y = -\Omega y \tag{4}$$
$$\dot{y} + R\omega_x = \Omega x \tag{5}$$

Noting that $\boldsymbol{L}_G = J_G\boldsymbol{\omega}$ and combining the two vector equations of motion to eliminate $\boldsymbol{F}$ we find,

$$J_G\dot{\boldsymbol{\omega}} = -R\,\boldsymbol{e}_z \times m\ddot{\boldsymbol{r}}_G$$

The two horizontal components of this equation gives, using $\boldsymbol{r}_G = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + R\,\boldsymbol{e}_z$,

$$J_G\dot{\omega}_x = Rm\ddot{y} \tag{6}$$
$$J_G\dot{\omega}_y = -Rm\ddot{x}. \tag{7}$$

Now take the time derivatives of equations (4) and (5) and use the result to eliminate the angular accelerations in equations (6) and (7). One finds the two equations of motion for the center of mass,

$$\left(m + \frac{J_G}{R^2}\right)\ddot{x} + \frac{J_G}{R^2}\Omega\dot{y} = 0 \tag{8}$$
$$\left(m + \frac{J_G}{R^2}\right)\ddot{y} - \frac{J_G}{R^2}\Omega\dot{x} = 0 \tag{9}$$

This gives the two equations,

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x},$$

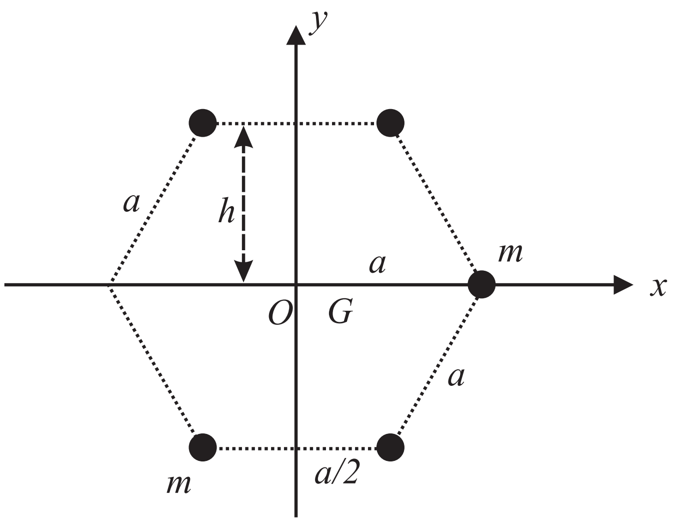
where we have put,

$$\omega = \Omega/(1 + mR^2/J_G) = (2/7)\Omega.$$

The specific value for ω is for the ball $J_G = (2/5)mR^2$. The solutions are circles traversed with angular velocity ω and radius and position given by the initial conditions.

Idea problems:

Problem 4: Five particles, each of mass m, are placed at the corners of a regular hexagon of side length a. Find the position of the center of mass G in the coordinate system of the figure. Find the inertia matrix $\mathbf{J}_G$, with respect to the same axes, but with origin at the center of mass.



Solution 4:
The distance h in the figure is given by $h^2 = (3/4)a^2$ according to a theorem by Pythagoras. One the immediately sees that:

$$J_x = 4mh^2 = 3ma^2$$

$$J_y = 4m(a/2)^2 + ma^2 = 2ma^2$$

$$J_z = J_x + J_y = 5ma^2$$

and this gives the inertia matrix $\mathbf{J}_O$ since $J_{xy} = J_{xz} = J_{yz} = 0$. The position of the center of mass is on the x-axis and the coordinate is given by,

$$x_G = \frac{4m0 + ma}{4m + m} = \frac{a}{5},$$

since the four symmetrically placed particles constitute a body with center of mass at O.

To get the inertia matrix with the origin at G we use the parallel axis theorem for J_y and J_z. J_x will not be affected. This gives

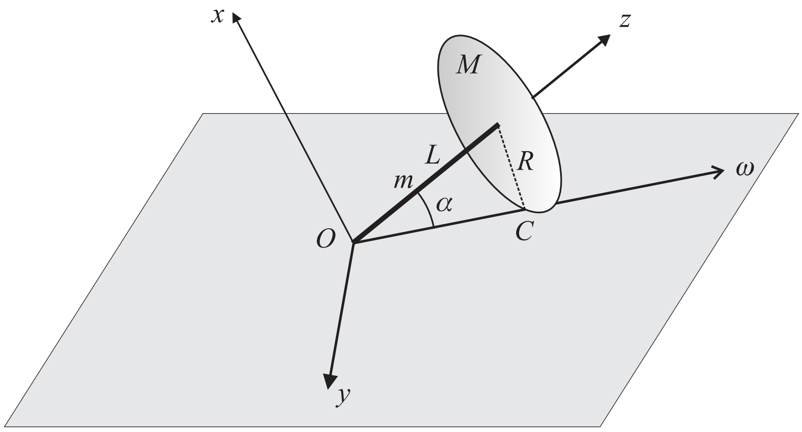
$$J'_x = 3ma^2$$

$$J'_y + 5m(a/5)^2 = 2ma^2 \Rightarrow J'_y = (9/5)ma^2$$

$$J'_z + 5m(a/5)^2 = 5ma^2 \Rightarrow J'_z = (24/5)ma^2$$

There are still no off-diagonal terms since the vector between O and G only has one non-zero component (along x).

Problem 5: A rigid body is made from a thin circular disc of radius R and mass M with a slender rod of length L and mass m attached at the center on one side of the disc. The rod is perpendicular to the plane of the disc. This body is placed on a rough horizontal plane so that the free end of the rod and a point on the periphery of the disc is in contact with the table. It is then set in rolling motion with absolute value of angular velocity given by ω. Find its kinetic energy.



Solution 5:
The solution to this problem is the same as for the first problem in *A Collection of Problems in Rigid Body and Analytical Mechanics*, except that the moments of inertia are different. First note that the angular velocity vector $\boldsymbol{\omega}$ must go through both O and C, the points of contact with the table, since these points both must have zero velocity. If α is the angle between the z-axis (symmetry axis) and the angular velocity, we find that,

$$\boldsymbol{\omega} = \omega(-\sin\alpha\, \boldsymbol{e}_x + \cos\alpha\, \boldsymbol{e}_z)$$

The moments of inertia are given by $J_x = J_y = (1/3)mL^2 + ML^2 + (1/4)MR^2$ and $J_z = (1/2)MR^2$. One finds the kinetic energy

$$T = \frac{1}{2}(J_x\omega_x^2 + J_z\omega_z^2) = \frac{1}{2}\omega^2(J_x \sin^2\alpha + J_z\cos^2\alpha)$$

We also have that: $\cos\alpha = L/\sqrt{L^2+r^2}$, and $\sin\alpha = R/\sqrt{L^2+r^2}$. This can now be put together to find the **answer**:

$$T = \frac{1}{2}\frac{MR^2\omega^2}{L^2+R^2}\left[\left(\frac{3}{2}+\frac{m}{3M}\right)L^2 + \frac{1}{4}R^2\right]$$

Problem 6: Derive Lagrange's equations of motion for one particle, either from a variational principle, or by projecting Newton's second law on the tangent vectors of the allowed motions.
Solution 6:
Various answers can be found in the text on Lagrange method of this course.

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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