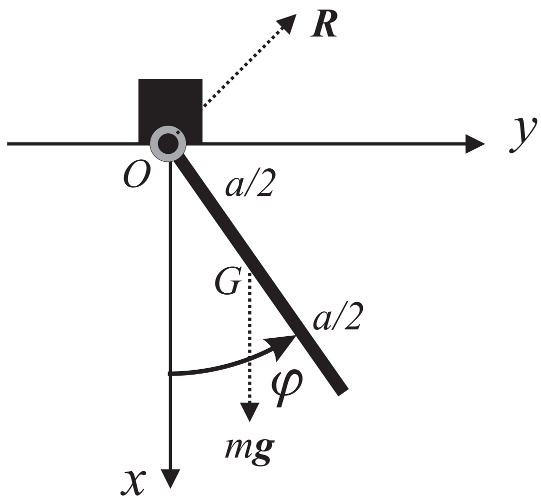
Rigid Body Dynamics, SG2150
Solutions to Exam, 2011 10 22

Calculational problems

Problem 1: A slender homogeneous rod of mass m and length a can rotate in a vertical plane about a fixed smooth horizontal axis through one endpoint. Find the equation of motion 1) using $\dot{\boldsymbol{L}} = \boldsymbol{M}$ and cylindrical coordinates and 2) using Lagrange method. 3) Find the angular frequency for small amplitude motion.



Solution 1: 1) We have $\boldsymbol{L} = J_O\dot{\varphi}\,\boldsymbol{e}_z$ where $J_O = ma^2/3$. The moment of force comes from gravity alone since the reaction force in the rotation axis is at O. Thus

$$\boldsymbol{M} = \overline{OG} \times mg\,\boldsymbol{e}_x = (a/2)\boldsymbol{e}_\rho(\varphi) \times mg\,\boldsymbol{e}_x = -(a/2)mg\sin\varphi\,\boldsymbol{e}_z.$$

The equation of motion $\dot{\boldsymbol{L}} = \boldsymbol{M}$ thus gives,

$$\frac{ma^2}{3}\ddot{\varphi}\,\boldsymbol{e}_z = -\frac{mga}{2}\sin\varphi\,\boldsymbol{e}_z.$$

The z-component of this vector equation is (**Answer 1**):

$$\ddot{\varphi} = -\frac{3g}{2a}\sin\varphi.$$

2) The kinetic energy is $T = \frac{1}{2}J_O\dot{\varphi}^2$ and the potential energy is $V = -mg(a/2)\cos\varphi$. The Lagrange function is therefore,

$$L = T - V = \tfrac{1}{2}\frac{ma^2}{3}\dot{\varphi}^2 + \frac{mga}{2}\cos\varphi$$

The Lagrange equation of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = 0$, becomes (**Answer 2:**)

$$\ddot{\varphi} + \frac{3g}{2a}\sin\varphi = 0,$$

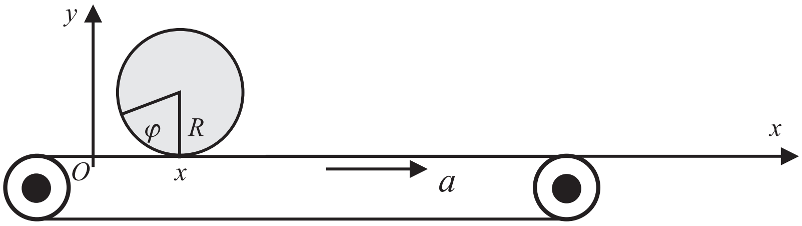
i.e. the same as in 1).
3) For small amplitude motion we have, $\sin\varphi \approx \varphi$, so that the equation of motion is

$$\ddot{\varphi} = -\frac{3g}{2a}\varphi.$$

This means that the angular frequency squared is $\omega^2 = \frac{3g}{2a}$. Thus (**Answer 3:**)

$$\omega = \sqrt{\frac{3g}{2a}}.$$

Problem 2: A straight circular cylinder of mass m and radius R is at rest on a rough horizontal conveyor belt. The axis of the cylinder is perpendicular to the direction of motion of the belt. The conveyor belt is then given a constant acceleration a. Find the Lagrangian that determines the motion of the center of mass of the cylinder. Find its translational acceleration from the Lagrange equation of motion.



Solution 2: The velocity of the conveyor belt surface is $v = at$, to the right in the figure. The lowest point on the cylinder must have this velocity, since the surface is rough. The connection formula for velocities in a rigid body then gives

$$\dot{x} + R\dot{\varphi} = at.$$

We now find that the Lagrangian is given by,

$$L = T - V = T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_G\dot{\varphi}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{mR^2}{2}\frac{(at-\dot{x})^2}{R^2},$$

so we find that (**Answer:**)

$$L = \frac{3}{4}m\dot{x}^2 - \frac{1}{2}mat\dot{x} + \frac{1}{4}ma^2t^2.$$

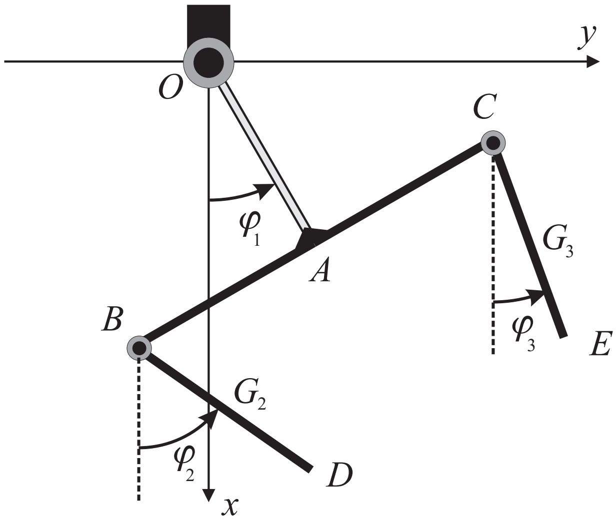
The equation of motion is given by $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$ and this gives us,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}m\dot{x} - \frac{1}{2}mat\right) = 0.$$

From this one immediately finds the **Answer:** $\ddot{x} = a/3$.

Problem 3: Four slender homogeneous rods OA, BC, BD, and CE, constitute a planar mechanism. OA has length a and is light. It is fixed to BC at a right angle so that A is at the midpoint of BC. BD and CE each have mass m and length a while BC has mass $2m$ and length $2a$. There are smooth joints at B and C connecting the rods. Due to a smooth joint at O, OA can rotate about a fixed horizontal axis. The mechanism moves in a vertical plane.

Find the Lagrangian of the system. Find the approximation for small amplitude motion about the equilibrium and determine the M and the K-matrix. Find at least one angular eigen frequency of the system.



Solution 3: The kinetic energy is given by:

$$T = \frac{1}{2}J_O\dot{\varphi}_1^2 + \frac{1}{2}m(\boldsymbol{v}_{G_2}^2 + \boldsymbol{v}_{G_3}^2) + \frac{1}{2}J_{G_2}(\dot{\varphi}_2^2 + \dot{\varphi}_3^2).$$

Here J_O is the moment of inertia of the body $OBAC$ with respect to a z-axis through O. Since the mass of OA is negligible, and the mass of BC is $2m$, we find $J_O = J_A + 2ma^2$ from Steiner's theorem. This gives $J_O = \frac{1}{12}(2m)(2a)^2 + 2ma^2 = \frac{8}{3}ma^2$. We also have that $J_{G_2} = J_{G_3} = \frac{ma^2}{12}$. The position vectors of the centers of mass of BD and CE are, respectively,

$$\boldsymbol{r}_{G_2} = \sqrt{2}a\,\boldsymbol{e}_\rho(\varphi_1 - \pi/4) + (a/2)\boldsymbol{e}_\rho(\varphi_2),\ \boldsymbol{r}_{G_3} = \sqrt{2}a\,\boldsymbol{e}_\rho(\varphi_1 + \pi/4) + (a/2)\boldsymbol{e}_\rho(\varphi_3).$$

The velocities are then,

$$\boldsymbol{v}_{G_2} = \sqrt{2}a\dot{\varphi}_1\,\boldsymbol{e}_\varphi(\varphi_1 - \pi/4) + (a/2)\dot{\varphi}_2\,\boldsymbol{e}_\varphi(\varphi_2),\ \boldsymbol{v}_{G_3} = \sqrt{2}a\dot{\varphi}_1\,\boldsymbol{e}_\varphi(\varphi_1 + \pi/4) + (a/2)\dot{\varphi}_3\,\boldsymbol{e}_\varphi(\varphi_3).$$

When squaring these velocity vectors we use the scalar product $\boldsymbol{e}_\varphi(\varphi_1 - \pi/4) \cdot \boldsymbol{e}_\varphi(\varphi_2) = \cos(\varphi_2 - \varphi_1 + \pi/4)$, and similarly for the other one. Algebra then gives (**Answer:**),

$$T = ma^2\left\{\frac{10}{3}\dot{\varphi}_1^2 + \frac{1}{6}(\dot{\varphi}_2^2 + \dot{\varphi}_3^2) + \frac{1}{\sqrt{2}}\left[\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1 + \pi/4) + \dot{\varphi}_1\dot{\varphi}_3\cos(\varphi_3 - \varphi_1 - \pi/4)\right]\right\}.$$

The potential energy is,

$$V = -2mga\cos\varphi_1 - mga(\cos\varphi_1 + \sin\varphi_1 + \tfrac{1}{2}\cos\varphi_2) - mga(\cos\varphi_1 - \sin\varphi_1 + \tfrac{1}{2}\cos\varphi_3)$$

That is, $V = -mga(4\cos\varphi_1 + \frac{1}{2}\cos\varphi_2 + \frac{1}{2}\cos\varphi_3)$, and the Lagrangian is $L = T - V$.

Assuming small amplitude motion we keep the quadratic terms in the Lagrangian and get (**Answer:**),

$$L = \frac{ma^2}{2}\left[\frac{20}{3}\dot{\varphi}_1^2 + \frac{1}{3}(\dot{\varphi}_2^2 + \dot{\varphi}_3^2) + \dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_1\dot{\varphi}_3\right] - \frac{mga}{2}\left[4\varphi_1^2 + \frac{1}{2}\left(\varphi_2^2 + \varphi_3^2\right)\right]$$

Below we skip the common factor ma of L. From this we can read off the M- and K-matrices:

$$\mathbf{M} = a\begin{pmatrix} \frac{20}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \quad \mathbf{K} = g\begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Putting $\omega^2 = x$ we now find the secular equation,

$$\det(-\mathbf{M}x + \mathbf{K}) = \begin{vmatrix} -a\frac{20}{3}x + g4 & -a\frac{1}{2}x & -a\frac{1}{2}x \\ -a\frac{1}{2}x & -a\frac{1}{3}x + g\frac{1}{2} & 0 \\ -a\frac{1}{2}x & 0 & -a\frac{1}{3}x + g\frac{1}{2} \end{vmatrix} = 0,$$

which gives,

$$\left(-a\frac{20}{3}x + g4\right)\left(-a\frac{1}{3}x + g\frac{1}{2}\right)^2 - \left(-a\frac{1}{2}x\right)^2\left(-a\frac{1}{3}x + g\frac{1}{2}\right) - \left(-a\frac{1}{2}x\right)^2\left(-a\frac{1}{3}x + g\frac{1}{2}\right) = 0.$$

It is obvious that one root is (**Answer:**)

$$\omega^2 = x_1 = \frac{3g}{2a}.$$

Note that this is the same as in Problem 1 and corresponds to the two rods BD and CE oscillating in opposite directions, while $OBAC$ is at rest.

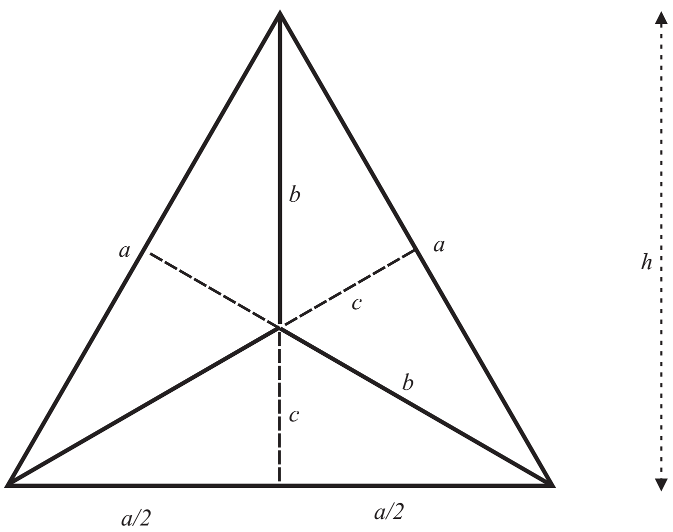
The other two roots are

$$x_{2,3} = \frac{6}{31}(7 \pm 3\sqrt{2})\frac{g}{a},$$

i.e. approximately, $x_2 = 2.176\,\frac{g}{a}$, and $x_3 = 0.534\,\frac{g}{a}$.

Idea problems:

Problem 4: Six identical slender homogeneous rods, each of mass m and length a are welded together at the endpoints so that they constitute the edges of a regular tetrahedron. Find the moment of inertia of this body with respect to an axis through the midpoint.



Solution 4:
The figure above shows the tetrahedron as seen from above at rest on a horizontal surface. Now imagine the lines projected to the plane of the base (horizontal) triangle. We see that the height of the base triangle is $h = b + c$ so Pythagoras' gives that $h^2 + (a/2)^2 = a^2$. This gives $h = (\sqrt{3}/2)a$. Since the center of mass of the triangle is $(1/3)h$ above the base we find that $c = h/3 = a/(2\sqrt{3})$ and that $b = 2h/3 = a/\sqrt{3}$.

The moment of inertia with respect to a vertical axis through the midpoint is now,

$$J = 3\left(m\frac{1}{12}a^2 + mc^2\right) + 3\left(m\frac{1}{3}b^2\right).$$

Here the first parenthesis gives the contribution from one of the edge bars in the base triangle. The second parenthesis is due to a bar that goes from the top of the tetrahedron to a corner in the base triangle. The contribution from such a bar is the same as for a bar perpendicular to the vertical axis but of length b. Algebra now gives the **Answer:**

$$J = \frac{5}{6}ma^2.$$

Since the tetrahedron is symmetric all moments of inertia for axes through the midpoint are the same.

Problem 5: Use the equation $\dot{\boldsymbol{L}} = \boldsymbol{M}$ to find a simple approximation for the precession angular velocity ($\Omega = \dot{\psi}$) of the heavy fast symmetric top. Hint: use $\dot{\boldsymbol{e}} = \boldsymbol{\Omega} \times \boldsymbol{e}$ and assume $\boldsymbol{L}$ parallel to the axis of the top.

Solution 5:
See Section 5.2.3, pages 82-83, in *Dynamics of Bodies.* Equations (5.62) to (5.65) constitute a derivation.

Here is a slightly simplified derivation. From $\dot{\boldsymbol{L}} = \boldsymbol{M}$ one finds

$$\dot{\boldsymbol{L}} \approx h\,\boldsymbol{e}_z^B \times (-mg\,\boldsymbol{e}_z^O) = mgh\,\boldsymbol{e}_z^O \times \boldsymbol{e}_z^B.$$

But for the fast top we have $\boldsymbol{L} \approx J\omega\,\boldsymbol{e}_z^B$ where $\omega = \dot{\varphi} \approx$constant. This means that $\dot{\boldsymbol{L}} \approx J\omega\,\dot{\boldsymbol{e}}_z^B$. Combining we get,

$$J\omega\,\dot{\boldsymbol{e}}_z^B \approx mgh\,\boldsymbol{e}_z^O \times \boldsymbol{e}_z^B,$$

which means that

$$\dot{\boldsymbol{e}}_z^B = \left(\frac{mgh}{J\omega}\,\boldsymbol{e}_z^O\right) \times \boldsymbol{e}_z^B.$$

We see that the result is, **Answer:**

$$\boldsymbol{\Omega} = \Omega\,\boldsymbol{e}_z^O = \frac{mgh}{J\omega}\,\boldsymbol{e}_z^O.$$

This is thus the angular velocity of precession of the fast heavy symmetric top. (Here h is the distance from the point in contact with the table to the center of mass, m is the mass, J is the moment of inertia with respect to the symmetry axis.)

Problem 6: Find the motion of the free symmetric top in terms of suitable Euler angles. Discuss the difference between prolate and oblate bodies.
Solution 6:
This is done in Section 5.2.1, pages 78-79, in *Dynamics of Bodies.*

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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