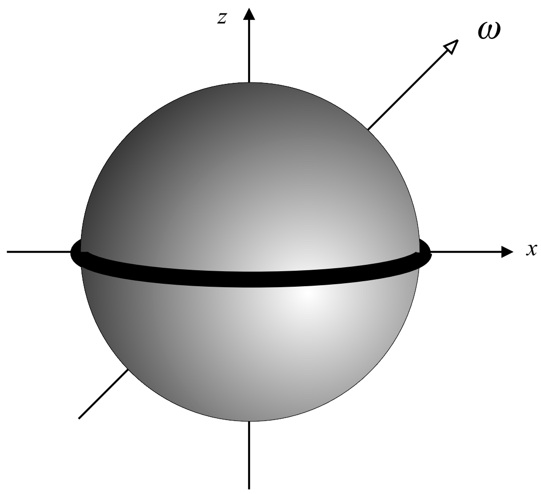
Rigid Body Dynamics, SG2150

Solutions to Exam, 2007 10 23, kl 09.00-13.00

Calculational problems

Problem 1: A spherical shell of mass $3m$ and radius R is rotating freely in space with angular velocity $\boldsymbol{\omega} = \frac{\omega}{\sqrt{2}}(\boldsymbol{e}_x + \boldsymbol{e}_z)$. A thin ring of radius R and mass $2m$, with its center coinciding with that of the shell, is initially at rest in the xy-plane. The ring suddenly becomes attached to the spherical shell. Find the angular velocity $\boldsymbol{\omega}'$ of the resulting rigid body immediately after the attachment.



Solution 1:
Since there are no external moments of force on the system of spherical shell plus ring, the total angular momentum $\boldsymbol{L}$ is a conserved vector. Initially when only the shell is moving (rotating) we have that,

$$\boldsymbol{L} = J_s \boldsymbol{\omega},$$

where,

$$J_s = \frac{2}{3}(3m)R^2 = 2mR^2,$$

is the moment of inertia of the spherical shell with respect to any axis through its center.

When the ring becomes attached to the shell we have a new rigid body with inertia tensor components given by,

$$\mathsf{J} = J_s \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2mR^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = mR^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The new expression for the angular momentum is then,

$$\boldsymbol{L} = \mathsf{J}\,\boldsymbol{\omega}'.$$

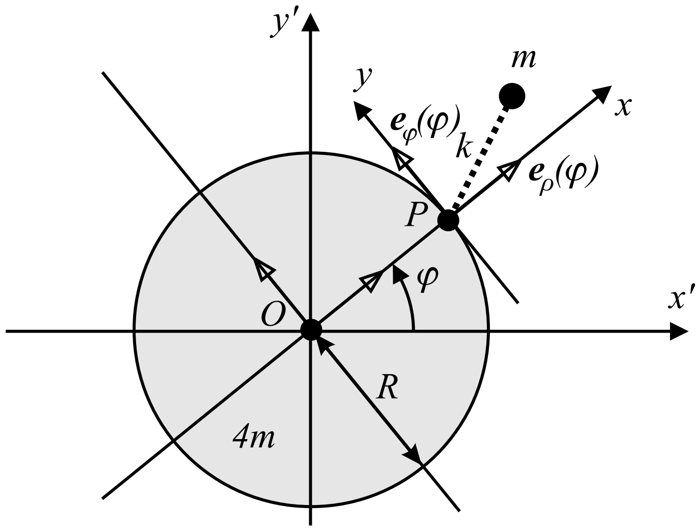
Combining the two expression for $\boldsymbol{L}$ we find for the **Answer:**,

$$\boldsymbol{\omega}' = \mathsf{J}^{-1} J_s \boldsymbol{\omega} = \frac{1}{mR^2} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} 2mR^2 \frac{\omega}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{\omega}{\sqrt{2}} \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

Problem 2: A particle of mass m can slide on a smooth horizontal plane (the $x'y'$- or, equivalently, the xy-plane). A circular homogeneous horizontal disc of mass $4m$ and radius R can rotate freely about a vertical axis through its center O, just above the smooth plane. An elastic band, with stiffness k attaches the particle to the disc and is such that it has its natural length when the particle is just below the point P on the periphery of the disc.

When the particle deviates from P in the smooth plane there is thus a linear force pulling it back to P.

Find the Lagrangian in terms of generalized coordinates φ, x, and y, where φ is the angle of rotation of the disc with respect to the fixed x'-axis, and where x, y are Cartesian coordinates of the particle with respect to xy-axes rotating with the disc and with origin at P and find conserved quantities of the system.



Solution 2: We have that,

$$T = \tfrac{1}{2}J\dot{\varphi}^2 + \tfrac{1}{2}m\boldsymbol{v}^2,$$

where $J = \frac{1}{2}(4m)R^2 = 2mR^2$ and where,

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(R\boldsymbol{e}_\rho + x\boldsymbol{e}_\rho + y\boldsymbol{e}_\varphi) = (\dot{x} - y\dot{\varphi})\boldsymbol{e}_\rho + (R\dot{\varphi} + x\dot{\varphi} + \dot{y})\boldsymbol{e}_\varphi.$$

Since the potential energy is, $V = \frac{1}{2}k(x^2 + y^2)$, we thus find the **Answer:**

$$L = T - V =$$

$$\tfrac{1}{2}m\left\{\left[2R^2 + (R+x)^2 + y^2\right]\dot{\varphi}^2 + \left(\dot{x}^2 + \dot{y}^2\right) + 2\left[(R+x)\dot{y} - y\dot{x}\right]\dot{\varphi}\right\} - \tfrac{1}{2}k\left(x^2 + y^2\right),$$

for the Lagrangian.

One conserved quantity is the energy and the other is the generalized coordinate $p_\varphi = \partial L/\partial\dot{\varphi}$, since the coordinate φ does not appear in the Lagrangian.

Answer:

$$E = \tfrac{1}{2}m\left\{\left[2R^2 + (R+x)^2 + y^2\right]\dot{\varphi}^2 + \left(\dot{x}^2 + \dot{y}^2\right) + 2\left[(R+x)\dot{y} - y\dot{x}\right]\dot{\varphi}\right\} + \tfrac{1}{2}k\left(x^2 + y^2\right),$$

and,

$$p_\varphi = m\left\{[2R^2 + (R+x)^2 + y^2]\dot{\varphi} + (R+x)\dot{y} - y\dot{x}\right\}$$

are constants of the motion.

Problem 3: Assume that the generalized coordinates x, y, and generalized velocities $\dot{\varphi}, \dot{x}, \dot{y}$, of Problem 2, are all small. Find the normal mode frequencies of the system.

Solution 3: Putting $q = (x, y, \varphi) = 0$ in the $g_{ab}(q)$ of the Lagrangian found in Solution 2 we get the quadratic expression:

$$L \approx \tfrac{1}{2}m\left(3R^2\dot{\varphi}^2 + \dot{x}^2 + \dot{y}^2 + 2R\dot{y}\dot{\varphi}\right) - \tfrac{1}{2}k\left(x^2 + y^2\right).$$

This can be written as,

$$L \approx \tfrac{1}{2}\left(\dot{\varphi}\ \ \dot{x}\ \ \dot{y}\right)\begin{pmatrix} 3mR^2 & 0 & mR \\ 0 & m & 0 \\ mR & 0 & m \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \end{pmatrix} - \tfrac{1}{2}\left(\varphi\ \ x\ \ y\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}\begin{pmatrix} \varphi \\ x \\ y \end{pmatrix},$$

making the mass and the stiffness matrices evident.

The secular equation for the normal mode frequencies ω is then,

$$\begin{vmatrix} -3mR^2z & 0 & -mRz \\ 0 & -mz+k & 0 \\ -mRz & 0 & -mz+k \end{vmatrix} = 0,$$

where $z = \omega^2$. This gives,

$$-3mR^2z(-mz+k)^2 - (-mRz)^2(-mz+k) = 0$$

After noting the trivial root $z_1 = 0$ we get a quadratic equation which gives the other two roots, $z_2 = \frac{k}{m}, z_3 = \frac{3k}{2m}$.

Thus we get the **Answer:**

$$\omega_1 = 0,\ \ \omega_2 = \sqrt{\frac{k}{m}},\ \ \omega_3 = \sqrt{\frac{3k}{2m}}.$$

Here the first root corresponds to the approximately conserved quantity,

$$p_\varphi \approx mR\left(3R\dot{\varphi} + \dot{y}\right),$$

the second to the x-motion, and the third to an out phase motion of φ and y.

Idea problems:

Problem 4: Spherical coordinates (r, ϑ, φ) can be defined by the transformation formulas,

$$x = r\sin\vartheta\,\cos\varphi, \quad y = r\sin\vartheta\,\sin\varphi, \quad z = r\cos\vartheta,$$

to Cartesian coordinates. The appropriate moving basis vectors are given by,

$$\begin{aligned}
\boldsymbol{e}_r = &\quad \sin\vartheta\,\cos\varphi\,\boldsymbol{e}_x + \sin\vartheta\,\sin\varphi\,\boldsymbol{e}_y + \cos\vartheta\,\boldsymbol{e}_z,\\
\boldsymbol{e}_\vartheta = &\quad \cos\vartheta\,\cos\varphi\,\boldsymbol{e}_x + \cos\vartheta\,\sin\varphi\,\boldsymbol{e}_y - \sin\vartheta\,\boldsymbol{e}_z,\\
\boldsymbol{e}_\varphi = &\quad -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y.
\end{aligned}$$

Assume that the angles, ϑ, φ, depend on time and that at a given moment the angular velocities are $\dot{\vartheta}, \dot{\varphi}$. Find the components of the angular velocity vector $\boldsymbol{\omega}(\vartheta, \varphi, \dot{\vartheta}, \dot{\varphi})$ of the moving basis $\boldsymbol{e}_r, \boldsymbol{e}_\vartheta, \boldsymbol{e}_\varphi$ with respect to the fixed Cartesian basis. The components of $\boldsymbol{\omega}$ should be with respect to the moving basis ($\boldsymbol{\omega} = \omega_r\boldsymbol{e}_r + \omega_\vartheta\boldsymbol{e}_\vartheta + \omega_\varphi\boldsymbol{e}_\varphi$).

Solution 4: Graphical solution is recommended. The result should be,

$$\boldsymbol{\omega} = \dot{\vartheta}\,\boldsymbol{e}_\varphi + \dot{\varphi}\,\boldsymbol{e}_z.$$

One then has to express $\boldsymbol{e}_z$ in terms of the moving basis vectors, which gives,

$$\boldsymbol{e}_z = \cos\vartheta\,\boldsymbol{e}_r - \sin\vartheta\,\boldsymbol{e}_\vartheta.$$

Finally thus we get the **Answer:**

$$\boldsymbol{\omega} = \dot{\varphi}\,\cos\vartheta\,\boldsymbol{e}_r - \dot{\varphi}\,\sin\vartheta\,\boldsymbol{e}_\vartheta + \dot{\vartheta}\,\boldsymbol{e}_\varphi,$$

for the angular velocity vector.

Problem 5: Show that the kinetic energy of a system can be written,

$$T(q, \dot{q}) = \sum_{a=1}^{n}\sum_{b=1}^{n}\frac{1}{2}g_{ab}(q)\dot{q}_a\dot{q}_b,$$

assuming that the constraints are holonomic and time independent.
Solution 5: See the textbook.

Problem 6: Assume that the Lagrangian of a system is given by,

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q).$$

We know that L itself is not a conserved quantity. By considering the time derivative of L and the equations of motion one can find a conserved quantity that corresponds to the energy. Do that!
Solution 6: See the textbook.

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.
Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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