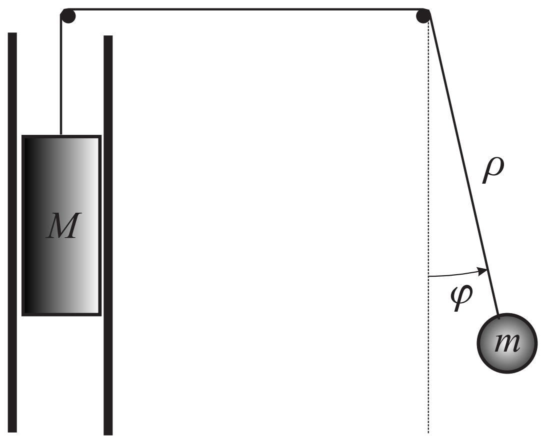
Rigid Body Dynamics, SG2150

Solutions to Exam, 2009 10 20, kl 09.00-13.00

Calculational problems

Problem 1: Find 1) the Lagrangian and 2) the Euler-Lagrange equations of motion for the system called the *swinging Atwood's machine* shown in the figure below. Use cylindrical coordinates (ρ, φ) for the particle with mass m as generalized coordinates. The particle moves only in the vertical plane and is attached by a stretched, inflexible (of fixed length) string to a weight of mass M that can move only in the vertical direction. The light string moves with negligible friction over the small pulleys.



Solution 1: The height of the particle of mass m is $h_m = -\rho\cos\varphi$. If the length of the string is L and the distance between the small pulleys is ℓ the height of the top of the weight M is $h_M = -(L - \ell - \rho)$. The potential energy is thus $V = mgh_m + Mgh_M = g\rho(M - m\cos\varphi)$+const. The kinetic energy of the weight is $T_M = M\dot{\rho}^2/2$. The kinetic energy of the particle is $T_m = (1/2)m\boldsymbol{v}^2$ and $\boldsymbol{v} = \dot{\rho}\,\boldsymbol{e}_\rho + \rho\dot{\varphi}\,\boldsymbol{e}_\varphi$ in cylinder coordinates. Thus we get the Lagrangian (**Answer 1:**),

$$L = T - V = \tfrac{1}{2}(m + M)\dot{\rho}^2 + \tfrac{1}{2}m\rho^2\dot{\varphi}^2 - g\rho(M - m\cos\varphi).$$

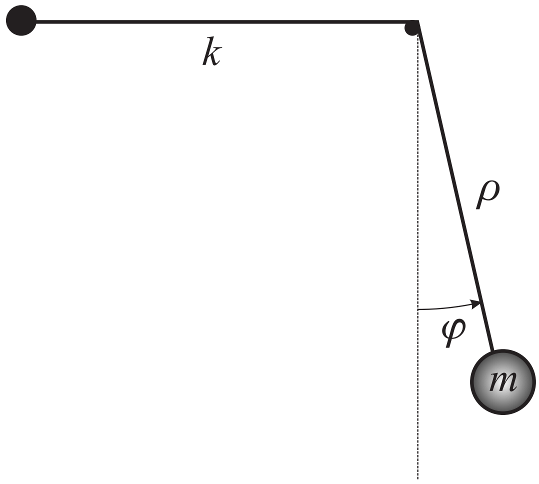
The equations of motion become (**Answer 2:**):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\rho}} - \frac{\partial L}{\partial\rho} = 0 \;\Leftrightarrow\; (m + M)\ddot{\rho} - m\rho\dot{\varphi}^2 + g(M - m\cos\varphi) = 0,$$

and,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = 0 \;\Leftrightarrow\; m(\rho^2\ddot{\varphi} + 2\rho\dot{\rho}\dot{\varphi}) + g\rho m\sin\varphi = 0.$$

Problem 2: Fix the mass M in the previous problem and replace the string with an elastic band of stiffness k. Find the frequencies for small oscillations of the particle of mass m about the equilibrium position.



Solution 2: The Lagrangian is now

$$L = T - V = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + mg\rho\cos\varphi - \frac{k}{2}(\rho - \rho_1)^2.$$

Here ρ_1 is the value of ρ when the rubber band is relaxed. The equilibrium position is found from:

$$\frac{\partial V}{\partial \rho} = mg\cos\varphi - k(\rho - \rho_1) = 0,$$

$$\frac{\partial V}{\partial \varphi} = -mg\rho\sin\varphi = 0.$$

The second of these equations show that $\varphi_0 = 0$ is the equilibrium value of φ. Use of this in the first equation shows that the equilibrium value of ρ is $\rho_0 = \rho_1 + mg/k$. We now find the approximate Lagrangian quadratic in φ and $u = \rho - \rho_0$ and their time derivatives. The kinetic energy gives $T \approx \frac{1}{2}m(\dot{u}^2 + \rho_0^2\dot{\varphi}^2)$. The potential energy becomes,

$$V \approx -mg(u + \rho_0)\left(1 - \tfrac{1}{2}\varphi^2 + \ldots\right) + \frac{k}{2}\left(u + \frac{mg}{k}\right)^2,$$

after expansion of the cosine to quadratic terms. Doing the multiplications and simplifying gives:

$$V \approx \tfrac{1}{2}(ku^2 + mg\rho_0\varphi^2) + \text{const.}$$

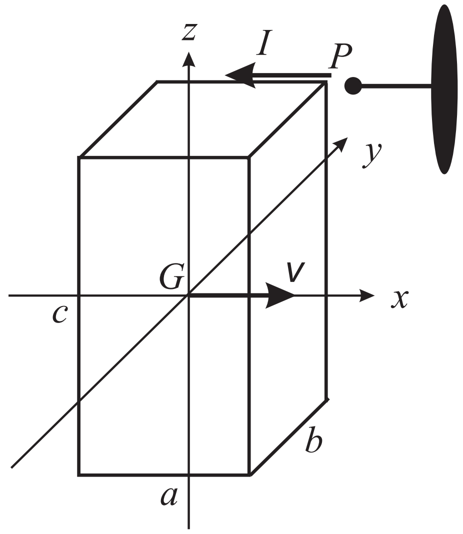
The mass and the stiffness matrices are thus both diagonal. The equations of motion corresponding to small oscillations near the equilibrium are:

$$m\ddot{u} = -ku, \quad m\rho_0^2\ddot{\varphi} = -mg\rho_0\varphi$$

The angular frequencies corresponding to these simple harmonic oscillator equations are thus:

Answer: $\omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{g/\rho_0}.$

Problem 3: A rectangular homogeneous solid with side lengths a, b, and, c and mass m impacts with a smooth knob at one of the corners P. Initially the solid has pure translation parallel to the side of length a, see Fig. on this page. The initial center of mass velocity is $\boldsymbol{v} = v\,\boldsymbol{e}_x$ and the impulse delivered at impact can be assumed to be in the negative x-direction $\boldsymbol{I} = -I\,\boldsymbol{e}_x$. 1) Find the center of mass velocity $\boldsymbol{v}_G$ and angular velocity $\boldsymbol{\omega}$ after impact. 2) Assume that the impact is elastic (energy conserving) and calculate the size of the impulse I.



Solution 3: The impulse equation gives,

$$\boldsymbol{p}(t_f) - \boldsymbol{p}(t_i) = \boldsymbol{I} \;\Leftrightarrow\; m(\boldsymbol{v}_G - v\,\boldsymbol{e}_x) = -I\,\boldsymbol{e}_x.$$

For the angular impulse $\hat{J}[\boldsymbol{\omega}(t_f) - \boldsymbol{\omega}(t_i)] = \boldsymbol{H} = \overline{GP} \times \boldsymbol{I}$, where $\overline{GP} = \frac{1}{2}(a\,\boldsymbol{e}_x + b\,\boldsymbol{e}_y + c\,\boldsymbol{e}_z)$, we get,

$$J_x\omega_x\boldsymbol{e}_x + J_y\omega_y\boldsymbol{e}_y + J_z\omega_z\boldsymbol{e}_z = (I/2)(-c\,\boldsymbol{e}_y + b\,\boldsymbol{e}_z),$$

since the initial angular velocity is zero and the inertia tensor is diagonal. So we find the three equations,

$$\omega_x = 0, \quad \omega_y = -\frac{cI}{2J_y}, \quad \omega_z = \frac{bI}{2J_z}.$$

Since $J_y = (m/12)(a^2 + c^2),\ J_z = (m/12)(a^2 + b^2)$, we find **Answer 1:**

$$\boldsymbol{v}_G = (v - I/m)\boldsymbol{e}_x, \quad \boldsymbol{\omega} = \frac{6I}{m}\left(-\frac{c}{a^2 + c^2}\boldsymbol{e}_y + \frac{b}{a^2 + b^2}\boldsymbol{e}_z\right),$$

for the center of mass velocity and the angular velocity just after impact.

Assuming that the (kinetic) energy is conserved we have,

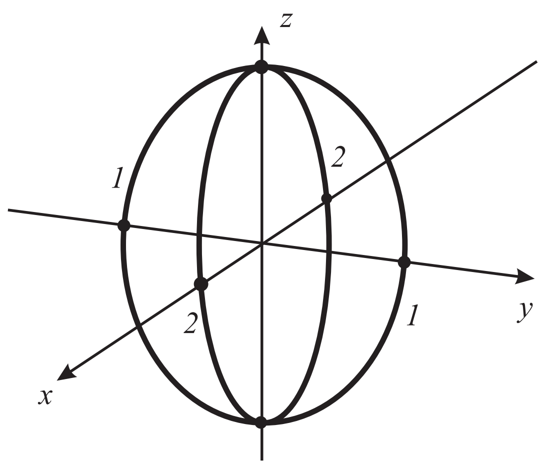
$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(v - I/m)^2 + \tfrac{1}{2}(J_y\omega_y^2 + J_z\omega_z^2).$$

Inserting the results above and simplifying we find an equation for I which has two roots. One of them is zero. The other root gives the correct **Answer 2:**

$$I = \frac{2mv}{1 + 3[c^2/(a^2 + c^2) + b^2/(a^2 + b^2)]}.$$

Idea problems:

Problem 4: A rigid body is constructed by welding together two identical metal rings, each of mass m and radius R, so that the rings are perpendicular and concentric, see Fig. above. 1) Calculate the inertia tensor with respect to the origin and axes given in the figure. 2) Rotate the coordinate system $\pi/4$ radians about the x-axis and find the inertia tensor with respect to the new axes.



Solution 4: For the ring in the yz-plane we have $J_x^1 = mR^2$ and since the ring is flat one must have $J_y^1 = J_z^1 = \frac{1}{2}mR^2$. The ring in the xz-plane has $J_y^2 = mR^2$ and $J_x^2 = J_z^2 = \frac{1}{2}mR^2$. The total inertia tensor is diagonal and its elements are the sum of the contribution from the two rings (**Answer 1:**)

$$J_x = \frac{3}{2}mR^2, \quad J_y = \frac{3}{2}mR^2, \quad J_z = mR^2.$$

The new inertia tensor in a rotated basis is given in formula (4.11), page 60 of Dynamics of Bodies. We find in the case of a 45 degree rotation about the x-axis that the new matrix in the rotated system becomes (**Answer 2:**)

$$\begin{aligned}\mathbf{J'} &= \mathbf{R}\,\mathbf{J}\,\widetilde{\mathbf{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= mR^2 \begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 5/4 & -1/4 \\ 0 & -1/4 & 5/4 \end{pmatrix}. \end{aligned} \tag{1}$$

Problem 5: Lagrangians for a class of three degree of freedom problems are given by,

$$L(q_1, \dot{q}_1, \dot{q}_2, \dot{q}_3) = \frac{1}{2}\left[a\,\dot{q}_1^2 + b\,\dot{q}_2^2 + f(q_1)\,\dot{q}_3^2 + 2\,g(q_1)\,\dot{q}_2\dot{q}_3\right] - V(q_1)$$

where a and b are constants, and $f(q_1)$ and $g(q_1)$ are arbitrary positive functions of the generalized coordinate q_1. For these systems the number of constants of the motion (conserved quantities) is the same as the number of degrees of freedom. Such systems are called integrable. 1) Find the three constants of the motion. 2) Find the effective potential for the q_1-motion by eliminating the generalized velocities $\dot{q}_2, \dot{q}_3$ from the Lagrangian using the constants of the motion.

Solution 5: The three conserved quantities are the generalized momenta corresponding to the two cyclic coordinates q_2, q_3, and, since there is no explicit time dependence, the energy. The generalized momenta in question are (**Answer 1:**)

$$p_2 \equiv \frac{\partial L}{\partial \dot{q}_2} = b\,\dot{q}_2 + g\,\dot{q}_3,$$

$$p_3 \equiv \frac{\partial L}{\partial \dot{q}_3} = f\,\dot{q}_3 + g\,\dot{q}_2,$$

and the energy is,

$$E = \frac{1}{2}\left[a\,\dot{q}_1^2 + b\,\dot{q}_2^2 + f(q_1)\,\dot{q}_3^2 + 2\,g(q_1)\,\dot{q}_2\dot{q}_3\right] + V(q_1).$$

If we solve the equations for the generalized momenta in terms of the generalized velocities we get,

$$\dot{q}_2 = \frac{g\,p_3 - f\,p_2}{g^2 - bf},$$

$$\dot{q}_3 = \frac{g\,p_2 - b\,p_3}{g^2 - bf}.$$

When this is inserted in the Lagrangian one finds,

$$L = \tfrac{1}{2}a\,\dot{q}_1^2 - V_{\text{eff}}(q_1; p_2, p_3),$$

where (**Answer 2:**)

$$V_{\text{eff}} = \frac{1}{2}\left[b\left(\frac{g\,p_3 - f\,p_2}{g^2 - bf}\right)^2 + f\left(\frac{g\,p_2 - b\,p_3}{g^2 - bf}\right)^2 + 2\,g\,\frac{g\,p_3 - f\,p_2}{g^2 - bf}\frac{g\,p_2 - b\,p_3}{g^2 - bf}\right] + V.$$

This can be simplified to,

$$V_{\text{eff}}(q_1) = \frac{1}{2[b\,f(q_1) - g^2(q_1)]}\left[f(q_1)\,p_2^2 + b\,p_3^2 - 2g(q_1)\,p_2p_3\right] + V(q_1).$$

Problem 6: A heavy symmetric top is spinning with $\omega = \dot{\varphi}$ about its symmetry axis. The symmetry axis is precessing with $\Omega = \dot{\psi}$ about the vertical. The angle θ between the symmetry axis and the vertical is constant. The moments of inertia are J with respect to the symmetry axis and $J+N$ with respect to the two perpendicular axes through the point of contact with the ground. The distance along the symmetry axis from the ground to the center of mass is h. 1) Assuming $|\Omega| \ll |\omega|$, show that,

$$\Omega = \frac{1}{2}\left[\frac{J\omega}{N\cos\theta} - \sqrt{\left(\frac{J\omega}{N\cos\theta}\right)^2 - \frac{4mgh}{N\cos\theta}}\,\right].$$

2) Using this, show that a good approximation is given by $\Omega = \frac{mgh}{J\omega}$. Find the first order correction to this approximation.

Solution 6: The equation,

$$\dot{\boldsymbol{L}} = \frac{{}^{\mathrm{B}}\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} + {}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} \times \boldsymbol{L} = \boldsymbol{M},$$

with

$$\boldsymbol{L} = \hat{J}({}^{\mathrm{O}}\boldsymbol{\omega}^{\mathrm{B}} + {}^{\mathrm{B}}\boldsymbol{\omega}^{\mathrm{A}}) = (J+N)\Omega\,\sin\theta\,\boldsymbol{e}_2^{\mathrm{B}} + J(\Omega\cos\theta + \omega)\boldsymbol{e}_3^{\mathrm{B}},$$

and,

$$\boldsymbol{M} = mgh\,\sin\theta\boldsymbol{e}_1^{\mathrm{B}},$$

gives only one component (along $\boldsymbol{e}_1^{\mathrm{B}}$). This gives

$$[J - (J+N)]\Omega^2\sin\theta\,\cos\theta + J\Omega\omega\,\sin\theta = mgh\,\sin\theta$$

Divide this by $\sin\theta$ and solve the quadratic for Ω to get the desired result as the relevant root.

The result can be rewritten in the form,

$$\Omega = \frac{1}{2}\frac{J\omega}{N\cos\theta}\left[1 - \sqrt{1 - \frac{4mghN\,\cos\theta}{(J\omega)^2}}\,\right].$$

The square root can be expanded so one finds that,

$$1 - \sqrt{1-x} = \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$$

Using $x = 4mghN\,\cos\theta/(J\omega)^2$ we find **Answer 2:**

$$\Omega = \frac{mgh}{J\omega}\left(1 + \frac{mgh}{J\omega}\frac{N\cos\theta}{J\omega} + \ldots\right)$$

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.

Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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