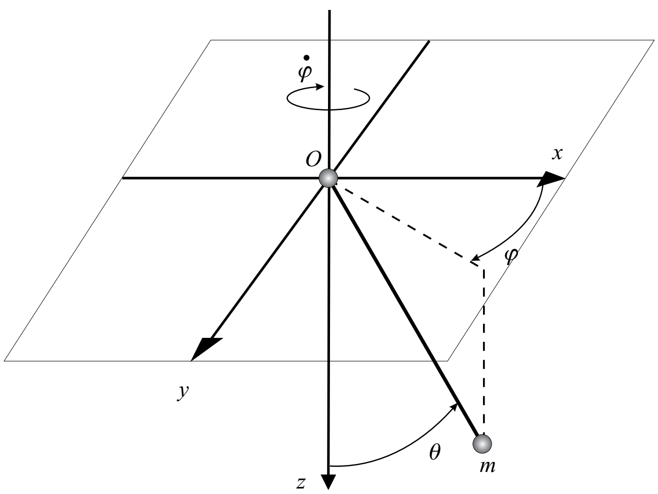
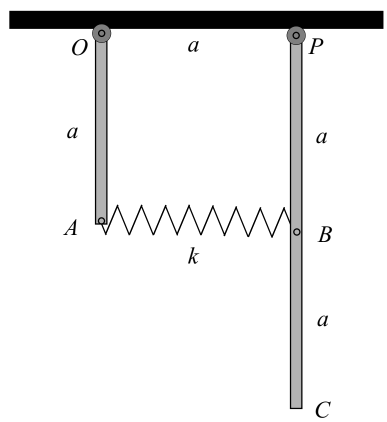
Rigid Body Dynamics, SG2150

Exam, 2008 10 21, kl 08.00-12.00

Calculational problems

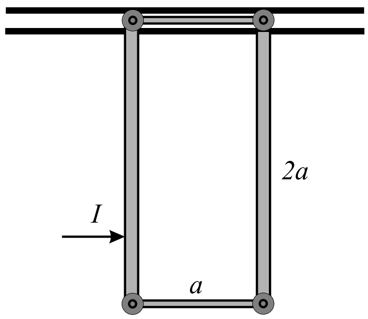
Problem 1: Consider the spherical pendulum: a mass m suspended at the end of a light rod of length ℓ, the other end of which is hinged at a fixed ball and socket joint. Use a downward directed z-axis (and horizontal xy-plane) and the spherical angles θ and φ as generalized coordinates. a) Find the Lagrangian of the system. b) Find the conserved quantities. c) Using one of these, eliminate the generalized velocity $\dot{\varphi}$ from the energy expression. d) Assume the initial conditions, $\theta(0) = \pi/2, \dot{\theta}(0) = 0$, $\varphi(0) = 0, \dot{\varphi}(0) = \sqrt{2g/\ell}$, and find an equation for the turning points of the θ-motion. e) Solve it.

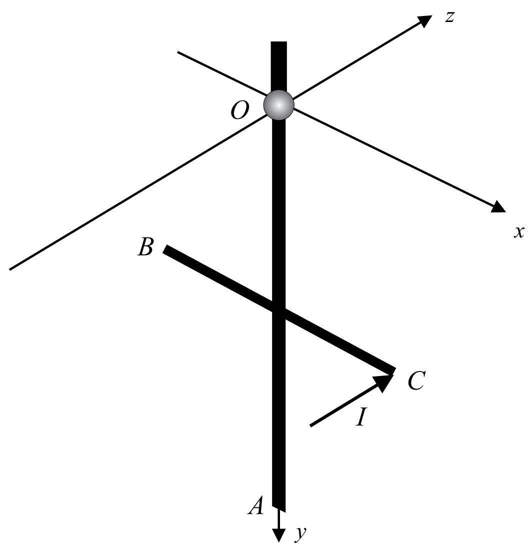




Problem 2: Two rods OA and PC, of length a and $2a$, and mass m and $2m$ respectively, are suspended in hinges of negligible friction so that they can swing in the same vertical plane. A spring of natural length a connects the end point A of rod OA with the midpoint B of PC. The distance $|OP| = a$ so that both rods are vertical at equilibrium. Find the angular frequencies of the system.

Problem 3: A four bar linkage is constructed of two bars of length a and mass m and two bars of length $2a$ and mass $4m$. The hinges are such that the two joined bars can rotate about an axis perpendicular to both bars; i.e. the linkage is coplanar. One of the small bars can slide along a horizontal track and the linkage is suspended vertically below this track. The linkage is hit by a hammer at a point $3a/2$ below the track and receives an impulse of magnitude I in the horizontal direction in the plane of the linkage. Calculate the velocity of the bar that slides along the track just after the hammer blow, assuming that the system was initially at rest. (Figure on next page.)





Idea problems

Problem 4: Two rods OA and BC, both of length $2a$ and mass m, have been welded together at their midpoints, perpendicular to each other, to make a cross. The cross is suspended, at rest, from a ball and socket joint at O. It is then struck at C so that it receives an impulse of magnitude I and direction perpendicular to the plane of the cross (z-direction in the figure). Find its angular velocity immediately after the impact.

Problem 5: If the Lagrange function $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$ for a conservative system does not depend explicitly on time the quantity,

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

is a constant of the motion. Show this.

Problem 6: Study the motion of the free symmetric top and solve it. Explain the motion and which qualitative features of the body that determines its motion.

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3 F; 4-5 FX; 6 E; 7-9 D; 10-12 C; 13-15 B; 16-18 A.

The grade can be raised, maximum one step, by doing a major project after the course. Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

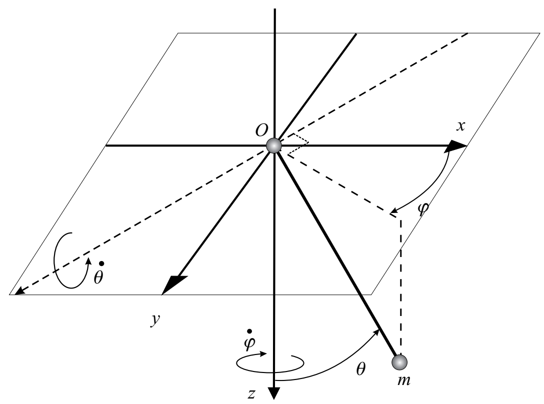
HE 08 10 21

Rigid Body Dynamics, SG2150

Solutions to Exam, 2008 10 21, kl 08.00-12.00

Calculational problems

Problem 1: Consider the spherical pendulum: a mass m suspended at the end of a light rod of length ℓ, the other end of which is hinged at a fixed ball and socket joint. Use a downward directed z-axis (and horizontal xy-plane) and the spherical angles θ and φ as generalized coordinates. a) Find the Lagrangian of the system. b) Find the conserved quantities. c) Using one of these, eliminate the generalized velocity $\dot{\varphi}$ from the energy expression. d) Assume the initial conditions, $\theta(0) = \pi/2, \dot{\theta}(0) = 0, \varphi(0) = 0, \dot{\varphi}(0) = \sqrt{2g/\ell}$, and find an equation for the turning points of the θ-motion. e) Solve it.



Solution 1: The Lagrangian is,

$$L = \tfrac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) - mg\ell(1 - \cos\theta).$$

Since the system is conservative the energy

$$E = \tfrac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + mg\ell(1 - \cos\theta),$$

is conserved. Since the coordinate φ is cyclic, the generalized momentum,

$$p_\varphi = m\ell^2\dot{\varphi}\sin^2\theta,$$

is also conserved.

Using the conserved generalized momentum $p_\varphi = L_z$ we can insert $\dot{\varphi} = p_\varphi/(m\ell^2 \sin^2\theta)$ in the energy expression to get,

$$E = \tfrac{1}{2}m\ell^2\dot{\theta}^2 + \tfrac{1}{2}\frac{p_\varphi^2}{m\ell^2 \sin^2\theta} + mg\ell(1 - \cos\theta).$$

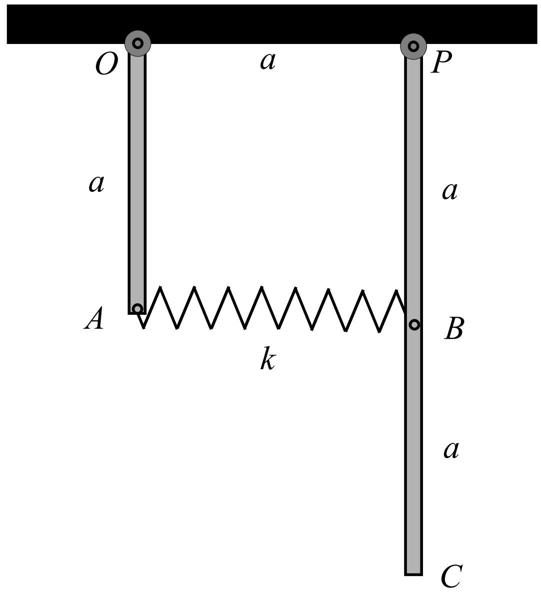
Now the initial conditions gives the values of p_φ and E,

$$p_\varphi = m\ell^2\sqrt{\frac{2g}{\ell}}, \quad E = 2mg\ell.$$

Insert these in the energy expression and put $\dot{\theta} = 0$ to get the equation, $\frac{mg\ell}{\sin^2\theta} + mg\ell(1 - \cos\theta) = 2mg\ell$, for the turning point angles. Simplifying and using the trigonometric unity this gives, $\cos\theta(\cos\theta - 1 + \cos^2\theta) = 0$. If the first factor is zero we find that $\theta = \pi/2$, if the second factor is zero we get $\cos\theta = (\sqrt{5} - 1)/2$, that is $\theta = \arccos\left((\sqrt{5} - 1)/2\right)$. Thus

$$\pi/2 = 90^\circ \;\geq\; \theta \;\geq\; \arccos\left((\sqrt{5} - 1)/2\right) \approx 51^\circ.83$$

Problem 2: Two rods OA and PC, of length a and $2a$, and mass m and $2m$ respectively, are suspended in hinges of negligible friction so that they can swing in the same vertical plane. A spring of natural length a connects the end point A of rod OA with the midpoint B of PC. The distance $|OP| = a$ so that both rods are vertical at equilibrium. Find the angular frequencies of the system, assuming small oscillations.



Solution 2: We find that,

$$T = \tfrac{1}{2}\left(\frac{1}{3}ma^2\right)\dot{\varphi}_1^2 + \tfrac{1}{2}\left(\frac{1}{3}2m(2a)^2\right)\dot{\varphi}_2^2.$$

The potential energy is,

$$V = -\left[\frac{a}{2}mg\cos\varphi_1 + a(2m)g\cos\varphi_2\right] + \tfrac{1}{2}k\left(|\boldsymbol{r}_A - \boldsymbol{r}_B| - a\right)^2 \approx \frac{a}{2}\frac{mg}{2}(\varphi_1^2 + 4\varphi_2^2) + \tfrac{1}{2}ka^2(\varphi_1 - \varphi_2)^2.$$

Introducing the dimensionless constant,

$$\epsilon \equiv \frac{ka}{mg},$$

we find the quadratic Lagrangian,

$$L = \tfrac{1}{2}ma^2\left\{\left(\frac{1}{3}\dot{\varphi}_1^2 + \frac{8}{3}\dot{\varphi}_2^2\right) - \frac{g}{a}\left[\left(\frac{1}{2} + \epsilon\right)\varphi_1^2 + (2+\epsilon)\varphi_2^2 - 2\epsilon\varphi_1\varphi_2\right]\right\}.$$

This gives us the M- and K- matrices

$$\mathbf{M} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \quad \mathbf{K} = \frac{g}{a}\begin{pmatrix} \frac{1}{2}+\epsilon & -\epsilon \\ -\epsilon & 2+\epsilon \end{pmatrix}.$$

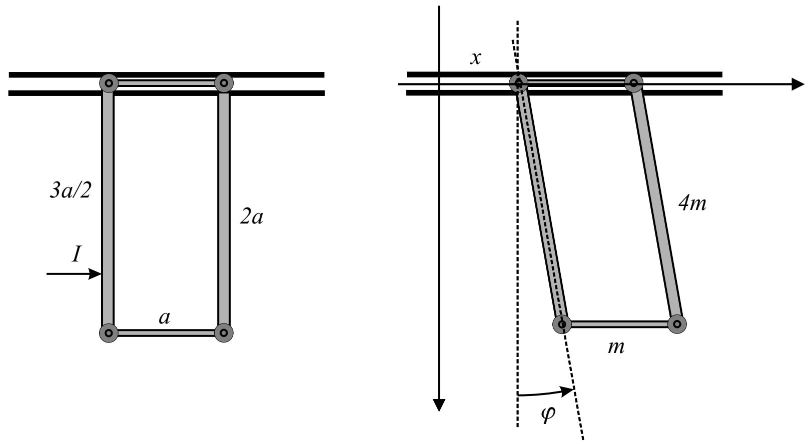
The secular equation, $\det(-\mathbf{M}x + \mathbf{K}) = 0$, becomes, after division by a^2,

$$\frac{8}{9}x^2 - (2+3\epsilon)\frac{g}{a}x + \left(1 + \frac{5}{2}\epsilon\right)\left(\frac{g}{a}\right)^2 = 0.$$

The roots, which are the eigen frequencies squared can be written, using $\epsilon = \frac{ka}{mg}$,:

$$x_{1,2} = \omega_{1,2}^2 = \frac{3}{16}\left[(6+9\epsilon) \pm \sqrt{4 + 28\epsilon + 81\epsilon^2}\right]\frac{g}{a}.$$

Problem 3: A four bar linkage is constructed of two bars of length a and mass m and two bars of length $2a$ and mass $4m$. The hinges are such that the two joined bars can rotate about an axis perpendicular to both bars; i.e. the linkage is coplanar. One of the small bars can slide along a horizontal track and the linkage is suspended vertically below this track. The linkage is hit by a hammer at a point $3a/2$ below the track and receives an impulse of magnitude I in the horizontal direction in the plane of the linkage. Calculate the velocity of the bar that slides along the track just after the hammer blow, assuming that the system was initially at rest.



Solution 3: We only need the kinetic energy very near the initial position since in an impact problem like this the system only moves infinitesimally between the initial time and the final time of the impact. As generalized coordinates we take x and φ as shown in the figure. One then finds that,

$$T = \tfrac{1}{2}m\dot{x}^2 + 2\left\{\tfrac{1}{2}\left[\frac{1}{12}4m(2a)^2\right]\dot{\varphi}^2 + \tfrac{1}{2}4m\left(\dot{x} + a\dot{\varphi}\right)^2\right\} + \tfrac{1}{2}m\left(\dot{x} + 2a\dot{\varphi}\right)^2,$$

where the first term is the kinetic energy of the top horizontal bar, the next two terms in the bracket (multiplied by 2) represent the rotational and translational kinetic energy of a vertical bar. The final term is the translational kinetic energy of the bottom horizontal bar.

Collecting terms here we thus find,

$$T = 5m\dot{x}^2 + \frac{22}{3}ma^2\dot{\varphi}^2 + 10ma\dot{x}\dot{\varphi}.$$

The generalized momenta are thus,

$$\begin{aligned} p_x &= 10m(\dot{x} + a\dot{\varphi}), \\ p_\varphi &= 10ma(\dot{x} + \frac{44}{30}a\dot{\varphi}). \end{aligned}$$

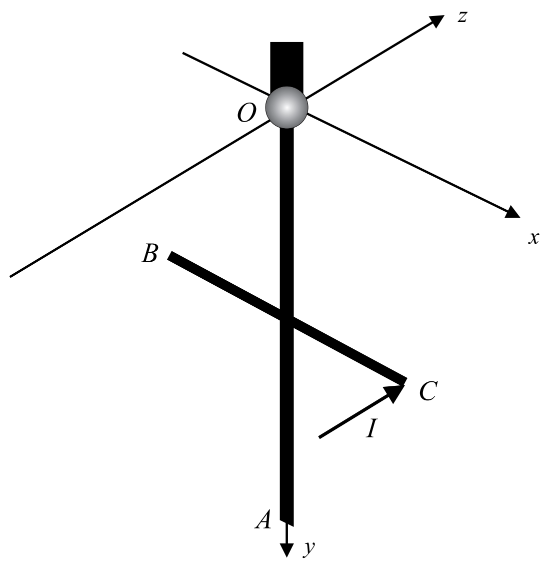
The work done by the impact force is, $\delta W = F\,\delta x + F\frac{3a}{2}\,\delta\varphi = Q_x\,\delta x + Q_\varphi\,\delta\varphi$. this means that the generalized impulses are $I_x = I,\;\; I_\varphi = \frac{3a}{2}I$. Since the system is initially at rest we get for the final velocities,

$$\begin{aligned} 10m(\dot{x} + a\dot{\varphi}) &= I, \\ 10ma(\dot{x} + \frac{44}{30}a\dot{\varphi}) &= \frac{3a}{2}I. \end{aligned}$$

Solving for $\dot{x}$ one finds the **Answer:** $\dot{x} = -\frac{I}{14\cdot 10m}$.

Idea problems:

Problem 4: Two rods OA and BC, both of length $2a$ and mass m, have been welded together at their midpoints, perpendicular to each other, to make a cross. The cross is suspended, at rest, from a ball and socket joint at O. It is then struck at C so that it receives an impulse of magnitude I and direction perpendicular to the plane of the cross (z-direction in the figure). Find its angular velocity immediately after the impact.



Solution 4: Using moments with respect to the point O we have the general equation,

$$\boldsymbol{L}(t_f) - \boldsymbol{L}(t_i) = \boldsymbol{r} \times \boldsymbol{I}.$$

Here $\boldsymbol{r}$ is the position vector of the impact. In our problem we have, $\boldsymbol{r} = \overline{OC}$, $\boldsymbol{L}(t_i) = \mathbf{0}$, $\boldsymbol{I} = I\boldsymbol{e}_z$, and $\boldsymbol{L}(t_f) = \mathbf{J}\boldsymbol{\omega}$.

Hence $\boldsymbol{r} = a(\boldsymbol{e}_x + \boldsymbol{e}_y)$ and $\boldsymbol{r} \times \boldsymbol{I} = aI(\boldsymbol{e}_x - \boldsymbol{e}_y)$. The inertia tensor is,

$$\mathbf{J} = \mathbf{J}_G + \text{Steiner} = ma^2 \begin{pmatrix} 7/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 8/3 \end{pmatrix},$$

so, if we denote the components of the angular velocity $\omega_x, \omega_y \omega_z$ we find that the three components of the vector equation $\mathbf{J}\boldsymbol{\omega} = \boldsymbol{r} \times \boldsymbol{I}$, are,

$$\begin{aligned} ma^2(7/3)\omega_x &= aI, & (1) \\ ma^2(1/3)\omega_y &= -aI, & (2) \\ ma^2(8/3)\omega_z &= 0. & (3) \end{aligned}$$

We thus find the **Answer:** $\omega_x = \frac{3I}{7ma}$, $\omega_y = -\frac{3I}{ma}$, $\omega_z = 0$.

The remaining problems are answered in the texts to this course.

Each problem gives maximum 3 points, so that the total maximum is 18. Grading: 1-3, F; 4-5, FX; 6, E; 7-9, D; 10-12, C; 13-15, B; 16-18; A.

Allowed equipment: Handbooks of mathematics and physics. One A4 size page with your own compilation of formulas.

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