> $with(LinearAlgebra):$

The potential energy is:

> $V := \frac{1}{2} \cdot k \cdot u1^2 + \frac{1}{2} \cdot 4 \cdot k \cdot (u1 - u2)^2;$

$$V := \frac{1}{2}\, k\, u1^2 + 2\, k\, (u1 - u2)^2 \quad (1)$$

> $V := simplify(V);$

$$V := \frac{5}{2}\, k\, u1^2 - 4\, k\, u1\, u2 + 2\, k\, u2^2 \quad (2)$$

> $K := k \cdot Matrix([[5, -4], [-4, 4]]);$

$$K := \begin{bmatrix} 5\,k & -4\,k \\ -4\,k & 4\,k \end{bmatrix} \quad (3)$$

> $U := Vector([u1, u2]);$

$$U := \begin{bmatrix} u1 \\ u2 \end{bmatrix} \quad (4)$$

> $Ut := U^{\%T};$

$$Ut := \begin{bmatrix} u1 & u2 \end{bmatrix} \quad (5)$$

Check that the K-matrix really gives the correct potential energy.

> $V1 := \frac{1}{2} \cdot Ut.K.U;$

$$V1 := \left(\frac{5}{2}\, u1\, k - 2\, u2\, k\right) u1 + (-2\, u1\, k + 2\, u2\, k)\, u2 \quad (6)$$

> $simplify(V - V1);$

$$0 \quad (7)$$

The mass matrix:

> $M := \frac{1}{2} \cdot m \cdot Matrix([[1, 0], [0, 1]]);$

$$M := \begin{bmatrix} \frac{1}{2}\, m & 0 \\ 0 & \frac{1}{2}\, m \end{bmatrix} \quad (8)$$

> $Sc := Determinant(-x \cdot M + K);$

$$Sc := \frac{1}{4}\, x^2\, m^2 - \frac{9}{2}\, x\, m\, k + 4\, k^2 \quad (9)$$

> $omeg2 := solve(Sc = 0, x);$

$$omeg2 := \frac{2\left(\frac{9}{2} + \frac{1}{2}\sqrt{65}\right) k}{m}, \frac{2\left(\frac{9}{2} - \frac{1}{2}\sqrt{65}\right) k}{m} \quad (10)$$

The roots are the squares of the characteristic frequencies (eigen frequencies):

> $\omega_1 := \mathrm{sqrt}(simplify(omeg2[2]));$

$$\omega_1 := \sqrt{\frac{(9 - \sqrt{65})\, k}{m}} \quad (11)$$

> $\omega_2 := \mathrm{sqrt}(simplify(omeg2[1]));$

$$\omega_2 := \sqrt{\frac{\left(9+\sqrt{65}\right)k}{m}} \tag{12}$$

Now find the a-vectors corresponding to these roots:
first insert x=omega_1^2 in the matrix -Mx+K:

> $Sm1 := -M\cdot\omega_1^2 + K;$

$$Sm1 := \begin{bmatrix} -\frac{1}{2}\left(9-\sqrt{65}\right)k+5k & -4k \\ -4k & -\frac{1}{2}\left(9-\sqrt{65}\right)k+4k \end{bmatrix} \tag{13}$$

> $a1 := Vector([a11, 1]);$

$$a1 := \begin{bmatrix} a11 \\ 1 \end{bmatrix} \tag{14}$$

> $Sma1 := Sm1.a1;$

$$Sma1 := \begin{bmatrix} \left(-\frac{1}{2}\left(9-\sqrt{65}\right)k+5k\right)a11-4k \\ -4k\,a11-\frac{1}{2}\left(9-\sqrt{65}\right)k+4k \end{bmatrix} \tag{15}$$

Determine the value of one component of the a1-vector so that it becomes zero when multiplied by the matrix -M(omega_1^2)+K:

> $R1 := solve(\{Sma1[1]=0, Sma1[2]=0\}, \{a11\});$

$$R1 := \left\{a11 = -\frac{1}{8}+\frac{1}{8}\sqrt{65}\right\} \tag{16}$$

> $assign(R1);$

> $a1;$

$$\begin{bmatrix} -\frac{1}{8}+\frac{1}{8}\sqrt{65} \\ 1 \end{bmatrix} \tag{17}$$

Now repeat this for the second root to get its a-vector:

> $Sm2 := -M\cdot\omega_2^2 + K;$

$$Sm2 := \begin{bmatrix} -\frac{1}{2}\left(9+\sqrt{65}\right)k+5k & -4k \\ -4k & -\frac{1}{2}\left(9+\sqrt{65}\right)k+4k \end{bmatrix} \tag{18}$$

> $a2 := Vector([a21, 1]);$

$$a2 := \begin{bmatrix} a21 \\ 1 \end{bmatrix} \tag{19}$$

> $Sma2 := Sm2.a2;$

(20)

$$Sma2 := \begin{bmatrix} \left(-\frac{1}{2}\left(9+\sqrt{65}\right)k+5k\right)a21-4k \\ -4\,k\,a21-\frac{1}{2}\left(9+\sqrt{65}\right)k+4k \end{bmatrix} \tag{20}$$

> $R2 := solve(\{Sma2[1]=0, Sma2[2]=0\}, \{a21\});$

$$R2 := \left\{a21 = -\frac{1}{8}-\frac{1}{8}\sqrt{65}\right\} \tag{21}$$

> $assign(R2);$

> $a2;$

$$\begin{bmatrix} -\frac{1}{8}-\frac{1}{8}\sqrt{65} \\ 1 \end{bmatrix} \tag{22}$$

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