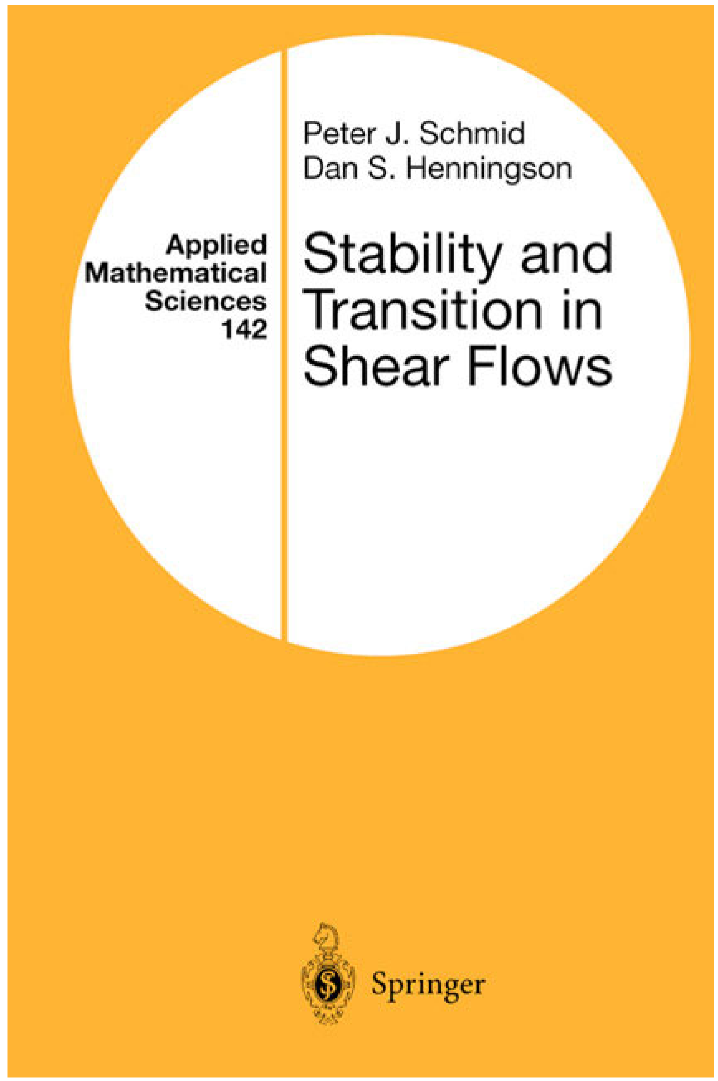
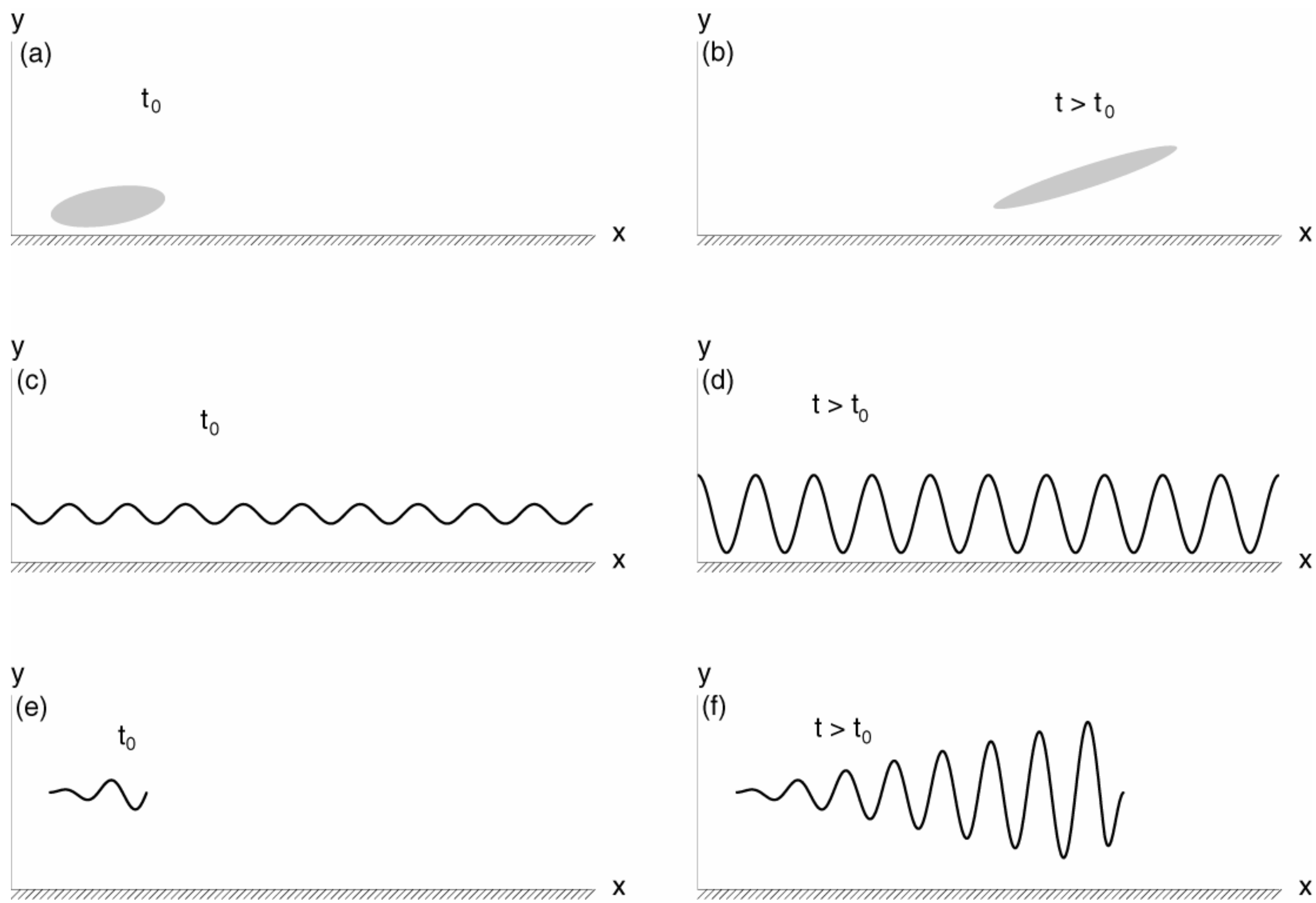
Course on Stability and Transition: Spatial Stability



Evolution of disturbances in shear flows



Model problem

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}$$

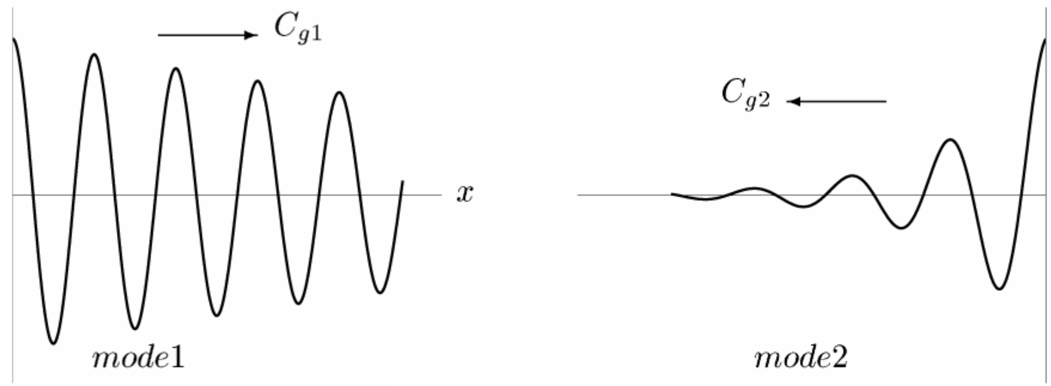
spatial evolution: ω is real valued

$$u = \hat{u}e^{-i\omega t} \qquad \Rightarrow \qquad -i\omega\hat{u} + U\frac{\partial \hat{u}}{\partial x} \;=\; \epsilon\frac{\partial^2 \hat{u}}{\partial x^2}$$

$$\hat{v} = \frac{\partial \hat{u}}{\partial x} \qquad \Rightarrow \qquad \frac{\partial}{\partial x}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} \;=\; \begin{pmatrix}0 & 1\\ -i\omega/\epsilon & U/\epsilon\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} \qquad \text{IVP in x}$$

$$\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} \sim \; e^{i\alpha x} \qquad \Rightarrow \qquad -\det\begin{pmatrix}-i\alpha & 1\\ -\frac{i\omega}{\epsilon} & \frac{U}{\epsilon} - i\alpha\end{pmatrix} \;=\; \alpha^2 + \frac{iU}{\epsilon}\alpha - \frac{i\omega}{\epsilon} = 0$$

Burger's eq., cont.



$$
\begin{aligned}
\alpha &= -\frac{iU}{2\epsilon} \pm \sqrt{\frac{i\omega}{\epsilon} - \frac{U^2}{4\epsilon^2}} \\
&= \frac{iU}{2\epsilon} \pm \frac{iU}{2\epsilon}\left[1 - \frac{2i\omega\epsilon}{U^2} + \frac{2\omega^2\epsilon^2}{U^4} + \mathcal{O}(\epsilon^3)\right] \\
&= \begin{cases} \dfrac{\omega}{U} + i\dfrac{\omega^2}{U^3}\epsilon & c_g = \dfrac{\partial \omega}{\partial \alpha} = U \qquad mode1 \\ -\dfrac{\omega}{U} - i\dfrac{U}{\epsilon} & c_g = \dfrac{\partial \omega}{\partial \alpha} = -U \qquad mode2 \end{cases}
\end{aligned}
$$

Gaster's transformation

$$D(\alpha, \omega, \mathrm{Re}) = 0 \quad \Rightarrow \quad dD = \frac{\partial D}{\partial \alpha} d\alpha + \frac{\partial D}{\partial \omega} d\omega + \frac{\partial D}{\partial \mathrm{Re}} d\mathrm{Re} = 0$$

$$\begin{aligned}
d\omega|_{\alpha_0} &= -d\mathrm{Re}\, \frac{\partial D}{\partial \mathrm{Re}} \Big/ \frac{\partial D}{\partial \omega} \\
d\alpha|_{\omega_0} &= -d\mathrm{Re}\, \frac{\partial D}{\partial \mathrm{Re}} \Big/ \frac{\partial D}{\partial \alpha} \\
d\omega|_{\mathrm{Re}_0} &\equiv \left.\frac{\partial \omega}{\partial \alpha}\right|_{\mathrm{Re}_0} d\alpha = -d\alpha\, \frac{\partial D}{\partial \alpha} \Big/ \frac{\partial D}{\partial \omega}
\end{aligned}$$

$$\Rightarrow \qquad d\omega|_{\alpha_0} = d\alpha|_{\omega_0} \frac{\partial D/\partial \alpha}{\partial D/\partial \omega} = \left(- \left.\frac{\partial \omega}{\partial \alpha}\right|_{\mathrm{Re}_0} \right) d\alpha|_{\omega_0} = -c_g d\alpha|_{\omega_0}$$

model problem: $\qquad d\omega_i = -c_g d\alpha_i = -\frac{\omega^2 \epsilon}{U^2} = -\alpha^2 \epsilon$

Spatial OS-SQ system

ω, β given, non-linear eigenvalue problem in α

$$
\begin{aligned}
\left[(-i\omega + i\alpha U)(D^2 - \alpha^2 - \beta^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - \alpha^2 - \beta^2)^2\right]\tilde{v} &= 0 \\
\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(D^2 - \alpha^2 - \beta^2)\right]\tilde{\eta} &= -i\beta U'\tilde{v}
\end{aligned}
$$

$\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} \tilde{V} \\ \tilde{E} \end{pmatrix} \exp(-\alpha y)$ reduces order in α

$$
\begin{aligned}
(i\omega - i\alpha U)(D^2 - 2\alpha D - \beta^2)\hat{V} + i\alpha U''\hat{V} + \frac{1}{\mathrm{Re}}(D^2 - 2\alpha D - \beta^2)^2\hat{V} &= 0 \\
(i\omega - i\alpha U)\hat{E} - i\beta U' V + \frac{1}{\mathrm{Re}}(D^2 - 2\alpha D - \beta^2)\hat{E} &= 0.
\end{aligned}
$$

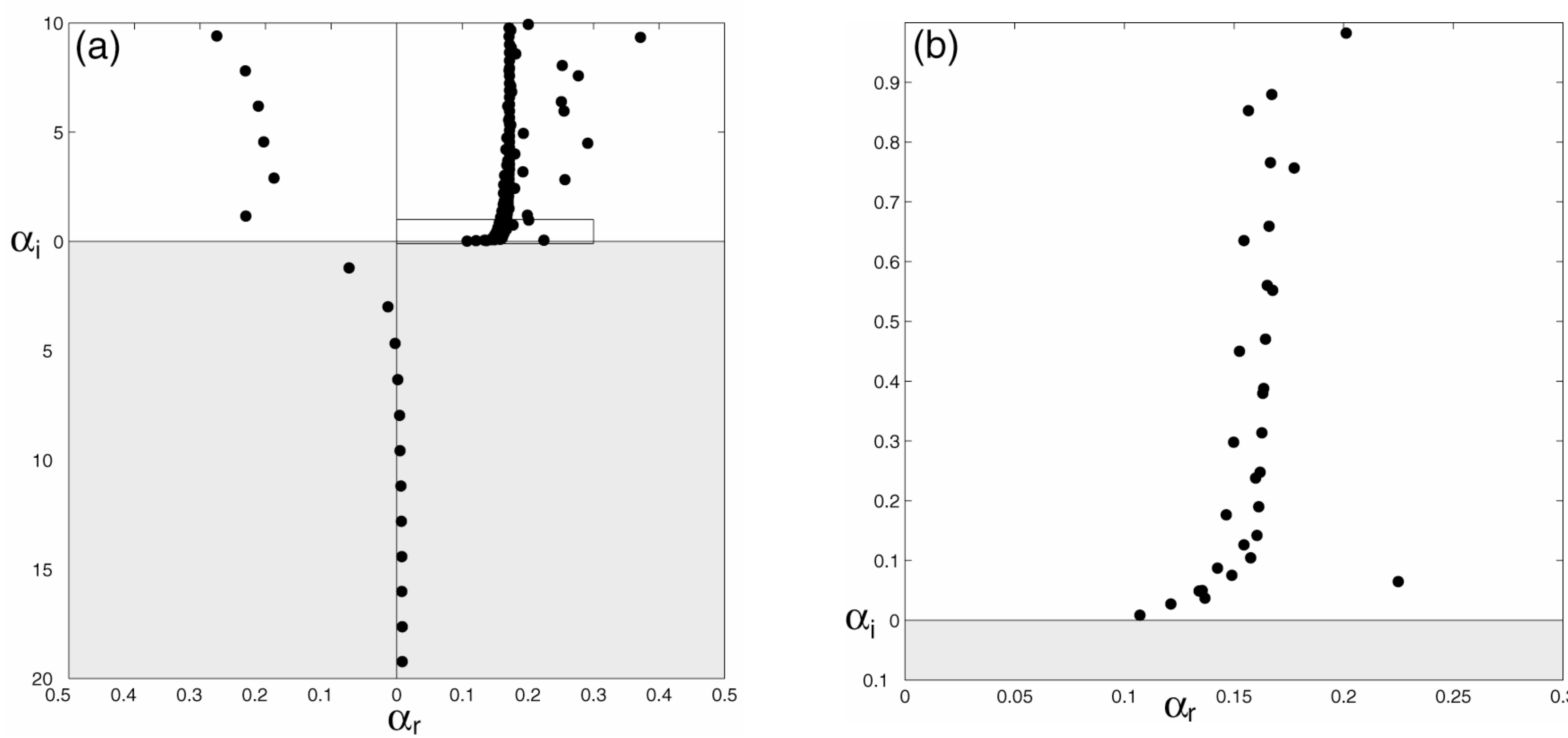
Spatial OS-SQ system, cont.

$$\tilde{\mathrm{q}} = (\alpha\hat{V}, \hat{V}, \hat{E})^T$$

$$\begin{pmatrix} -R_1 & -R_0 & 0 \\ I & 0 & 0 \\ 0 & -S & -T_0 \end{pmatrix} \begin{pmatrix} \alpha\hat{V} \\ \hat{V} \\ \hat{E} \end{pmatrix} = \alpha \begin{pmatrix} R_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_1 \end{pmatrix} \begin{pmatrix} \alpha\hat{V} \\ \hat{V} \\ \hat{E} \end{pmatrix}$$

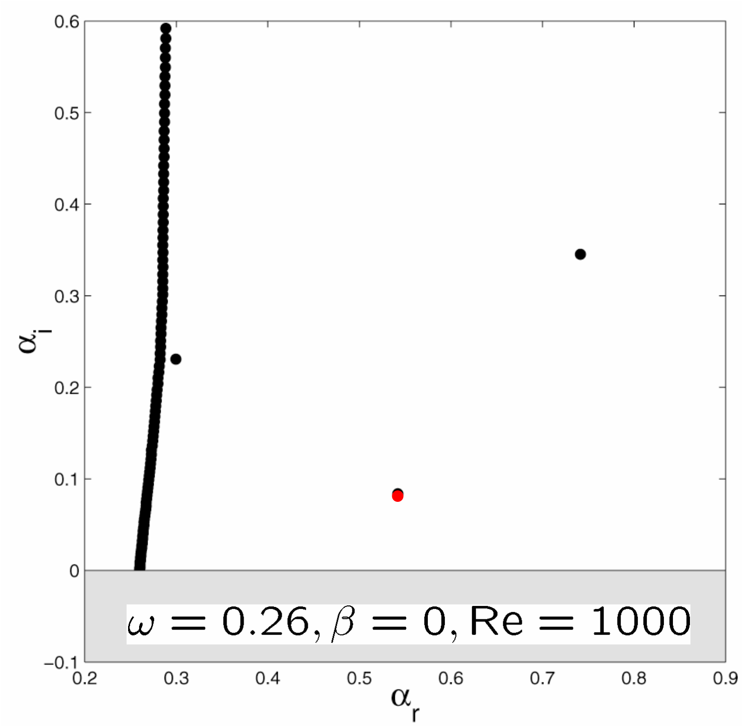
$$\begin{aligned}
R_2 &= \frac{4}{\mathrm{Re}} D^2 + 2iUD \\
R_1 &= -2i\omega D - \frac{4}{\mathrm{Re}} D^3 + \frac{4}{\mathrm{Re}} \beta^2 D - iUD^2 + iU\beta^2 + iU'' \\
R_0 &= i\omega D^2 - i\omega\beta^2 + \frac{1}{\mathrm{Re}} D^4 - \frac{2}{\mathrm{Re}} \beta^2 D^2 + \frac{1}{\mathrm{Re}} \beta^4 \\
T_1 &= \frac{2}{\mathrm{Re}} D + iU \\
T_0 &= -i\omega - \frac{1}{\mathrm{Re}} D^2 + \frac{1}{\mathrm{Re}} \beta^2 \\
S &= i\beta U'
\end{aligned}$$

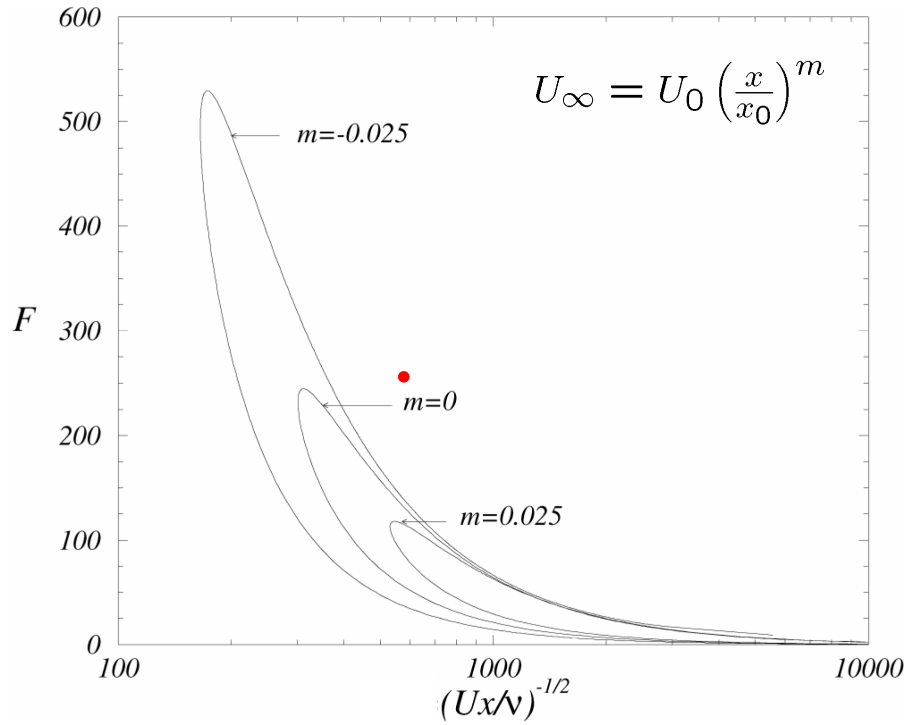
Spatial PPF spectra



$$\omega = 0.3, \beta = 0, \mathrm{Re} = 2000$$

Boundary layer flow





$$F = 10^6 \omega\nu / U_\infty^2 = 10^6 \omega / \mathrm{Re}$$

$$\mathrm{Re} = 1.72\sqrt{Ux/\nu}$$

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Spatial IVP

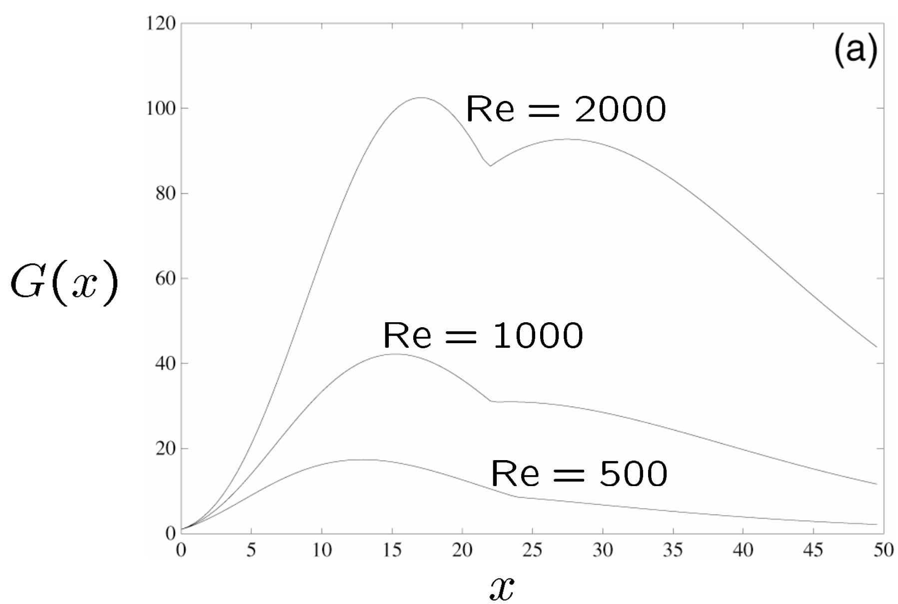
$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix} = \begin{pmatrix} i\alpha D/k^2 & -i\beta/k^2 \\ 1 & 0 \\ i\beta D/k^2 & i\alpha \end{pmatrix} \begin{pmatrix}\tilde{V}\\ \tilde{E}\end{pmatrix} e^{-\alpha y}$$

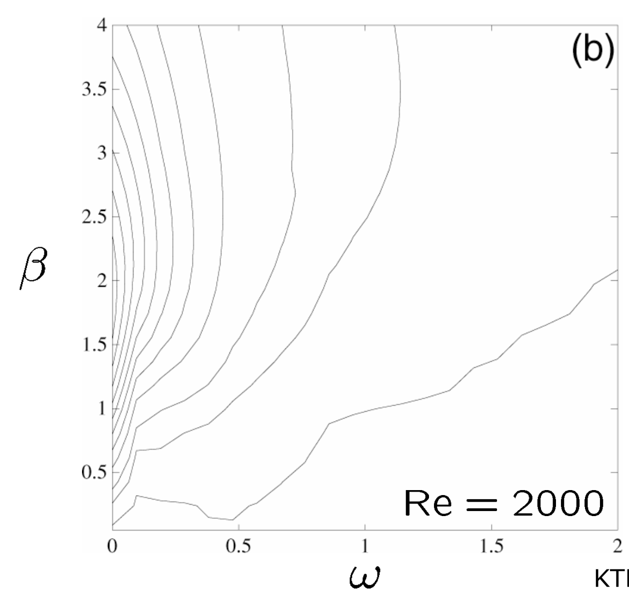
$$\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{w}\end{pmatrix} = \sum_{j=1}^{N} \kappa_j(x) \begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix}_j = \sum_{j=1}^{N} \kappa_j^0(x) \begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{w}\end{pmatrix}_j e^{i\alpha_j x}$$

$$\begin{aligned} \kappa^T &= (\kappa_1, \kappa_2, \ldots, \kappa_N)^T \\ \Lambda &= \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\} \\ \frac{d\kappa}{dx} &= i\Lambda\kappa, \quad \kappa(0) = \kappa^0 \quad \Rightarrow \quad \kappa = e^{i\Lambda x}\kappa^0 \end{aligned}$$

Optimal spatial growth for PPF

$$
\begin{aligned}
E(\kappa) &= \kappa^H A\kappa = \kappa^H F^H F\kappa = \|F\kappa\|^2 \\
A_{ij} &= \frac{1}{2}\int_{-1}^{1} \left(\tilde{u}_i^* \tilde{u}_j + \tilde{v}_i^* \tilde{v}_j + \tilde{w}_i^* \tilde{w}_j\right) dy \\
G(x) &= \sup_{\kappa_0} \frac{E(\kappa)}{E(\kappa_0)} = \|F \exp(i\Lambda x) F^{-1}\|_2^2
\end{aligned}
$$





2D Parabolic Stability Equations (PSE)

$$u(x,y,t) = \tilde{u}(x,y)\,\mathrm{exp}\left(i\int_{x_0}^{x}\alpha(\xi)\;d\xi - i\omega t\right)$$

$$\begin{aligned}
-i\omega\tilde{u} + U\tilde{u}_x + i\alpha U\tilde{u} + U_x\tilde{u} + V\tilde{u}_y + U_y\tilde{v} &= -\tilde{p}_x - i\alpha\tilde{p} \\
+\frac{1}{\mathrm{Re}}\left(\tilde{u}_{xx} + 2i\alpha\tilde{u}_x + i\frac{d\alpha}{dx}\tilde{u} - \alpha^2\tilde{u} + \tilde{u}_{yy}\right) & \\
-i\omega\tilde{v} + U\tilde{v}_x + i\alpha U\tilde{v} + V_x\tilde{u} + V\tilde{v}_y + V_y\tilde{v} &= -\tilde{p}_y \\
+\frac{1}{\mathrm{Re}}\left(\tilde{v}_{xx} + 2i\alpha\tilde{v}_x + i\frac{d\alpha}{dx}\tilde{v} - \alpha^2\tilde{v} + \tilde{v}_{yy}\right) & \\
\tilde{u}_x + i\alpha\tilde{u} + \tilde{v}_y &= 0
\end{aligned}$$

multiplescale assumption: $\frac{\partial}{\partial x}, \quad V \sim \mathcal{O}\left(\frac{1}{\mathrm{Re}}\right)$

PSE, cont.

keep $\mathcal{O}(1)$ and $\mathcal{O}(1/\mathrm{Re})$ terms $\Rightarrow$

$$\frac{d}{dx}\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{p}\end{pmatrix} = \begin{pmatrix} -i\alpha & -D & 0 \\ 0 & -\frac{c_1}{U} - \frac{V_y}{U} & -\frac{D}{U} \\ -c_1 + i\alpha U - U_x & UD - U_y & -i\alpha \end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{p}\end{pmatrix}$$

$$c_1 = i\alpha U - i\omega + VD - \frac{1}{\mathrm{Re}}\left(D^2 - \alpha^2\right)$$

normalization condition determines α and keeps $\tilde{u}(x,.)$ slowly varying

$$\int_{y_{\mathrm{min}}}^{y_{\mathrm{max}}} \left(\tilde{u}^*\tilde{u}_x + \tilde{v}^*\tilde{v}_x + \tilde{p}^*\tilde{p}_x\right)\ dy = 0$$

Growthrate and wavenumber

$$\phi = \tilde{\phi}(x) e^{i(\int \alpha dx - \omega t)}$$

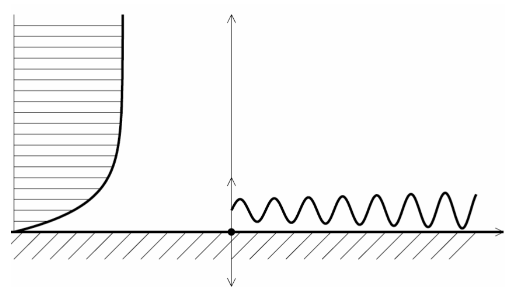
$$\begin{aligned}\bar{\alpha} &= -i\frac{1}{\phi}\frac{\partial \phi}{\partial x} \\ &= \alpha - i\frac{1}{\tilde{\phi}}\frac{\partial \tilde{\phi}}{\partial x} \\ &= \begin{cases} \alpha_r + \Im\left\{\frac{1}{\tilde{\phi}}\frac{\partial \tilde{\phi}}{\partial x}\right\} \\ \alpha_i - \Re\left\{\frac{1}{\tilde{\phi}}\frac{\partial \tilde{\phi}}{\partial x}\right\} \end{cases}\end{aligned}$$

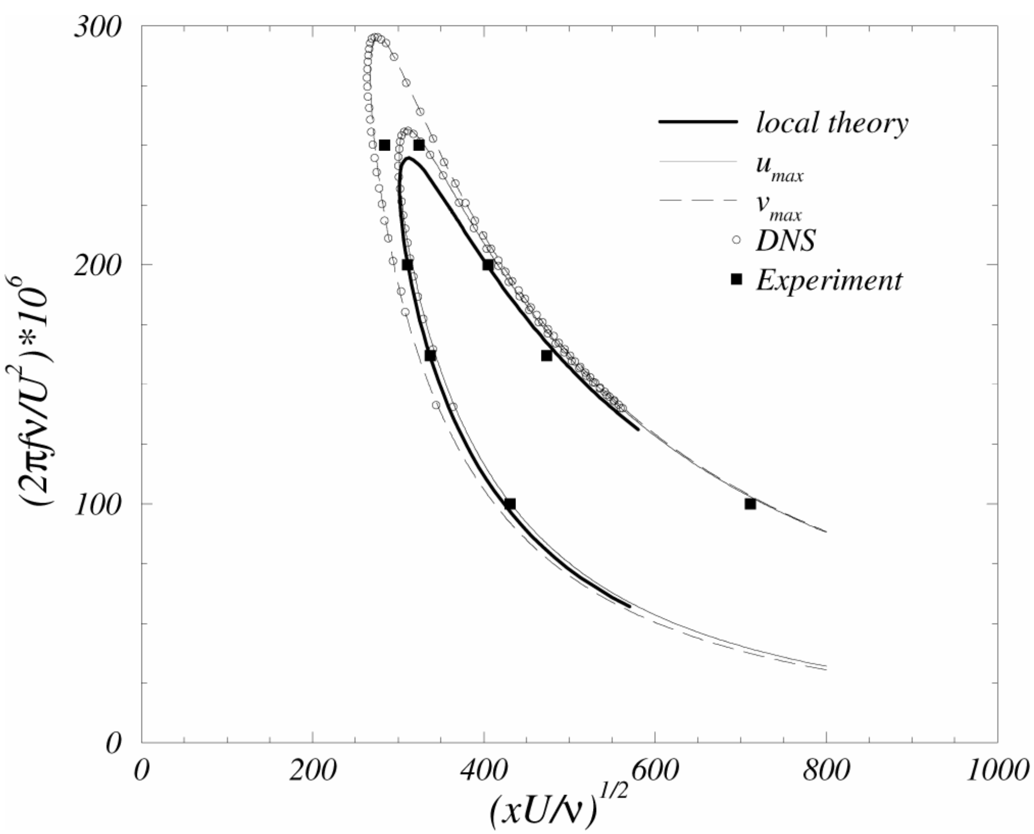
e.g. $\tilde{\phi} = \max_y \tilde{u}(x,y) \equiv \tilde{u}_m(x) \quad \Rightarrow \quad \bar{\alpha}_i(x) = \alpha_i(x) - \Re\left\{\frac{1}{\tilde{u}_m}\frac{\partial \tilde{u}_m}{\partial x}\right\}$

e.g. $\tilde{\phi}^2 = \int_0^\infty (|\tilde{u}|^2 + |\tilde{v}|^2) dy \equiv E \quad \Rightarrow \quad \bar{\alpha}_i(x) = \alpha_i(x) - \Re\left\{\frac{\partial}{\partial x} \ln \sqrt{E}\right\}$

Non-parallel neutral curve for Blasius BL

Spatially developing TS-wave



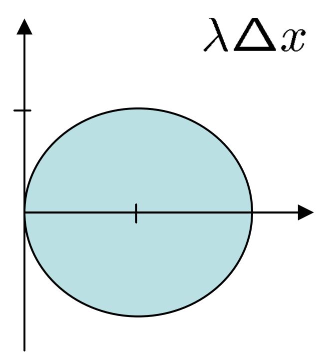


Step-size restriction

$$U = 1, V = 0, \frac{\partial}{\partial y} \sim i\eta \quad \Rightarrow \quad \frac{d}{dx}\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{p}\end{pmatrix} = \begin{pmatrix} -i\alpha & -i\eta & 0 \\ 0 & -c_1 & -i\eta \\ -c_1 + i\alpha & i\eta & -i\alpha \end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\\ \tilde{p}\end{pmatrix}$$

$$\det\begin{pmatrix} -i\alpha - \lambda & -i\eta & 0 \\ 0 & -c_1 - \lambda & -i\eta \\ -c_1 + i\alpha & i\eta & -i\alpha - \lambda \end{pmatrix} =$$

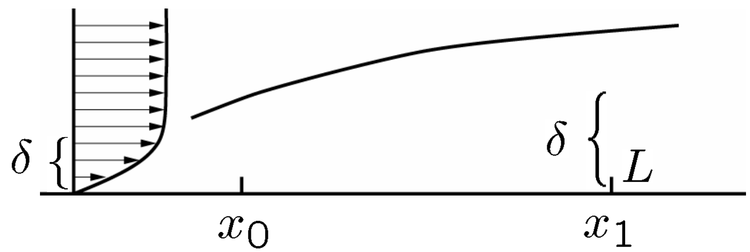
$$(-c_1 - \lambda)(\lambda - \eta + i\alpha)(\lambda + \eta + i\alpha) = 0$$



$$\text{Backward Euler} \quad \Rightarrow \quad \frac{1}{|1 - \lambda\Delta x|} < 1$$

$$\Rightarrow \quad \Delta x > \frac{1}{|\alpha_r|}$$

Optimal disturbances in spatial BL



BL scalings: $\delta = \sqrt{\frac{\nu L}{U_\infty}}, \quad \mathrm{R}_L = \frac{U_\infty L}{\nu} = \left(\frac{U_\infty \delta}{\nu}\right)^2 = \mathrm{Re}^2$

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{\delta} = \frac{y^*}{L}\mathrm{Re}, \quad z = \frac{z^*}{L}\mathrm{Re}, \quad p = \frac{p^* \mathrm{R}_L}{\rho U_\infty}$$

$$u = \frac{u^*}{U_\infty}, \quad v = \frac{v^*}{U_\infty}\mathrm{Re}, \quad w = \frac{w^*}{U_\infty}\mathrm{Re}$$

$$E(\mathbf{u}) = ||\mathbf{u}||^2 = (\mathbf{u}, \mathbf{u}) = \int_0^\infty \left(u^2 + \frac{v^2}{\mathrm{Re}^2} + \frac{w^2}{\mathrm{Re}^2}\right) dy$$

Spatial optimals, cont.

Linearize stationary BL equations around $(U(x,y), V(x,y), 0)$

$$(u,v,w,p) = (\widehat{u}\cos\beta z, \widehat{v}\cos\beta z, \widehat{w}\sin\beta z, \widehat{p}\cos\beta z)$$

$$\begin{aligned}
\widehat{u}_x + \widehat{v}_y + \beta\widehat{w} &= 0 \\
(U\widehat{u})_x + V\widehat{u}_y + U_y\widehat{v} &= \widehat{u}_{yy} - \beta^2\widehat{u} \\
(V\widehat{u} + U\widehat{v})_x + (2V\widehat{v})_y + \beta V\widehat{w} + p_y &= \widehat{v}_{yy} - \beta^2\widehat{v} \\
(U\widehat{w})_x + (V\widehat{w})_y - \beta\widehat{p} &= \widehat{w}_{yy} - \beta^2\widehat{w}.
\end{aligned}$$

$$\begin{aligned}
\widehat{u} = \widehat{v} = \widehat{w} = 0 \qquad &\text{at} \qquad y = 0 \\
\widehat{u} = \widehat{w} = \widehat{p} = 0 \qquad &\text{as} \qquad y \to \infty
\end{aligned}$$

The optimization problem

$$\hat{\mathbf{u}}_1 = \mathcal{A}\hat{\mathbf{u}}_0 \qquad \hat{\mathbf{u}}_0 = \begin{pmatrix} \widehat{u}_0(y) \\ \widehat{v}_0(y) \\ \widehat{w}_0,(y) \end{pmatrix} \qquad \hat{\mathbf{u}}_1 = \begin{pmatrix} \widehat{u}_1(x,y) \\ \widehat{v}_1(x,y) \\ \widehat{w}_1(x,y) \end{pmatrix}$$

$$G(x_1) = \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{\|\hat{\mathbf{u}}_1\|^2}{\|\hat{\mathbf{u}}_0\|^2} = \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{\|\mathcal{A}\hat{\mathbf{u}}_0\|^2}{\|\hat{\mathbf{u}}_0\|^2}$$

$$= \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{(\mathcal{A}\hat{\mathbf{u}}_{in}, \mathcal{A}\hat{\mathbf{u}}_0)}{(\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0)} = \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{(\mathcal{A}^*\mathcal{A}\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0)}{(\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0)} = \lambda_{max}$$

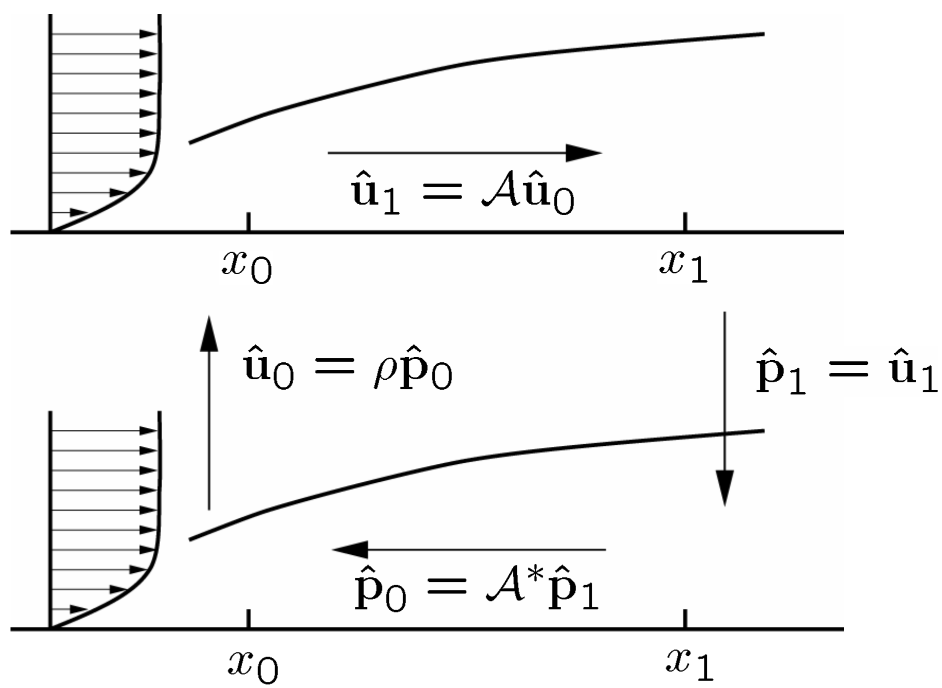
λ_{max} largest eigenvalue of $\quad \mathcal{A}^*\mathcal{A}\hat{\mathbf{u}}_0 = \lambda\hat{\mathbf{u}}_0$

found by power iterations $\quad \hat{\mathbf{u}}_0^{n+1} = \rho_n \mathcal{A}^*\mathcal{A}\hat{\mathbf{u}}_0^n$

Optimization algorithm

$$\hat{\mathbf{u}}_0^{n+1} = \rho_n \mathcal{A}^* \mathcal{A} \hat{\mathbf{u}}_0^n$$

$\mathcal{A}^*$ solves the adjoint equations backward



Derivation of the action of the adjoint

$$\hat{\mathbf{u}}_x - \mathcal{L}_0 \hat{\mathbf{u}} - \mathcal{L}_1 \hat{\mathbf{u}}_y - \mathcal{L}_2 \hat{\mathbf{u}}_{yy} = 0$$

$$\begin{aligned}
0 &= \int_{x_0}^{x_1} \int_0^\infty \hat{\mathbf{p}} \left[\hat{\mathbf{u}}_x - \mathcal{L}_0 \hat{\mathbf{u}} - \mathcal{L}_1 \hat{\mathbf{u}}_y - \mathcal{L}_2 \hat{\mathbf{u}}_{yy} \right] dy dx \\
&= \int_{x_0}^{x_1} \int_0^\infty \hat{\mathbf{u}} \underbrace{\left[-\hat{\mathbf{p}}_x - \mathcal{L}_0^* \hat{\mathbf{p}} - (\mathcal{L}_1^* \hat{\mathbf{p}})_y - (\mathcal{L}_2^* \hat{\mathbf{p}})_{yy} \right]}_{=0} dy dx + \\
&\quad \underbrace{\int_0^\infty \hat{\mathbf{p}}_1 \hat{\mathbf{u}}_1 dy}_{(\hat{\mathbf{p}}_1, \mathcal{A} \hat{\mathbf{u}}_0)} - \underbrace{\int_0^\infty \hat{\mathbf{p}}_0 \hat{\mathbf{u}}_0 dy}_{(\mathcal{A}^* \hat{\mathbf{p}}_1, \hat{\mathbf{u}}_0)}
\end{aligned}$$

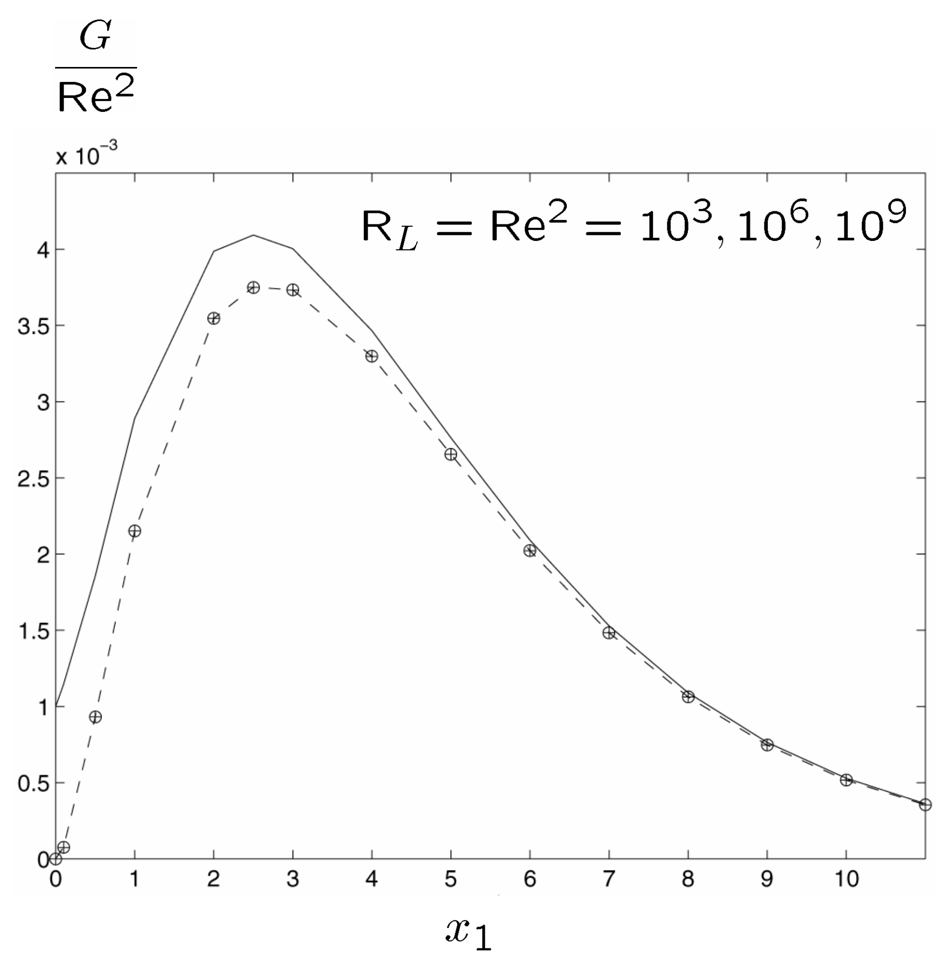
definition of the adjoint $\quad (\hat{\mathbf{p}}_1, \mathcal{A} \hat{\mathbf{u}}_0) = (\mathcal{A}^* \hat{\mathbf{p}}_1, \hat{\mathbf{u}}_0)$

Reynoldsnumber dependence

$$G(x_1) = \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{\|\hat{\mathbf{u}}_1\|^2}{\|\hat{\mathbf{u}}_0\|^2}$$

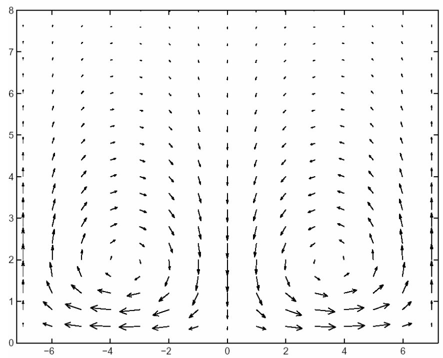
$$= \max_{\hat{\mathbf{u}}_0 \neq 0} \frac{\int_0^\infty \left(u^2 + \frac{v^2}{\mathrm{Re}^2} + \frac{w^2}{\mathrm{Re}^2}\right) dy}{\int_0^\infty \left(u^2 + \frac{v^2}{\mathrm{Re}^2} + \frac{w^2}{\mathrm{Re}^2}\right) dy}$$

$$\approx \max_{\hat{\mathbf{u}}_0 \neq 0} \mathrm{Re}^2 \frac{\int_0^\infty u^2 dy}{\int_0^\infty \left(v^2 + w^2\right) dy}$$

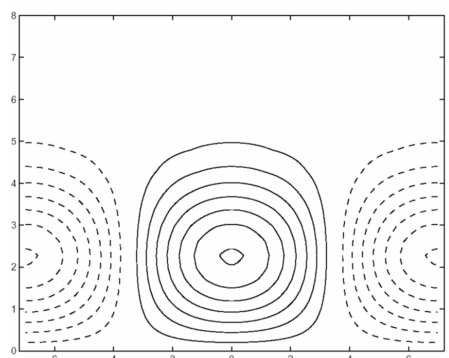


Optimal growth and disturbance

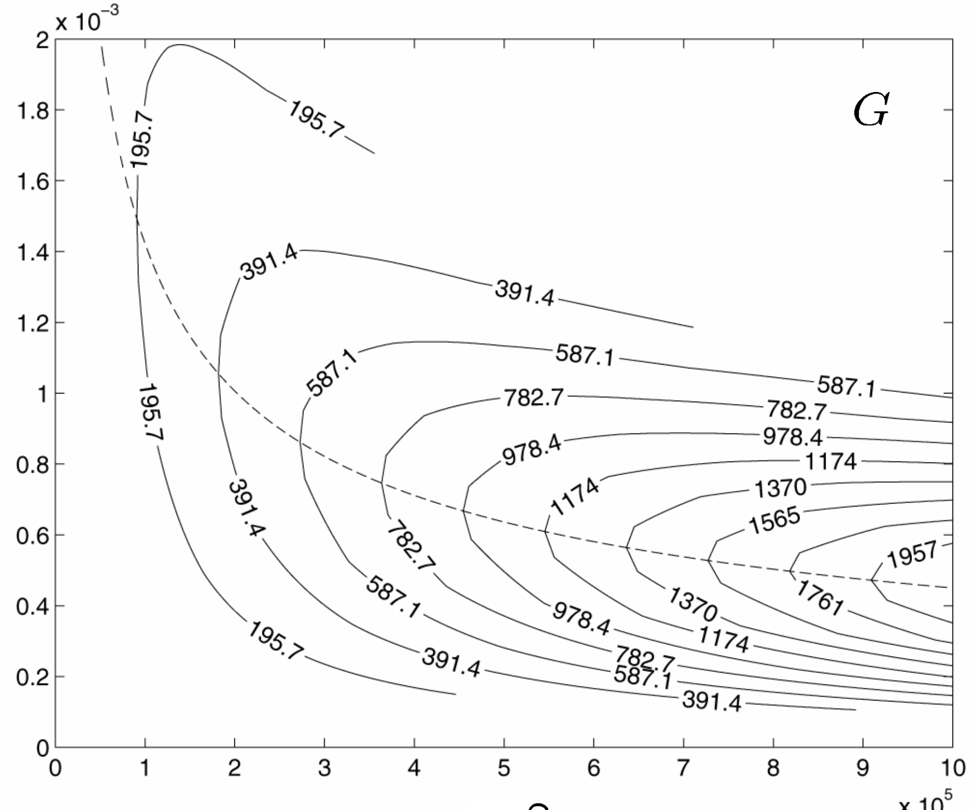
input:
streamwise vortex



output:
streamwise streak



$\bar{\beta} = \beta/\mathrm{Re}$



Re^2