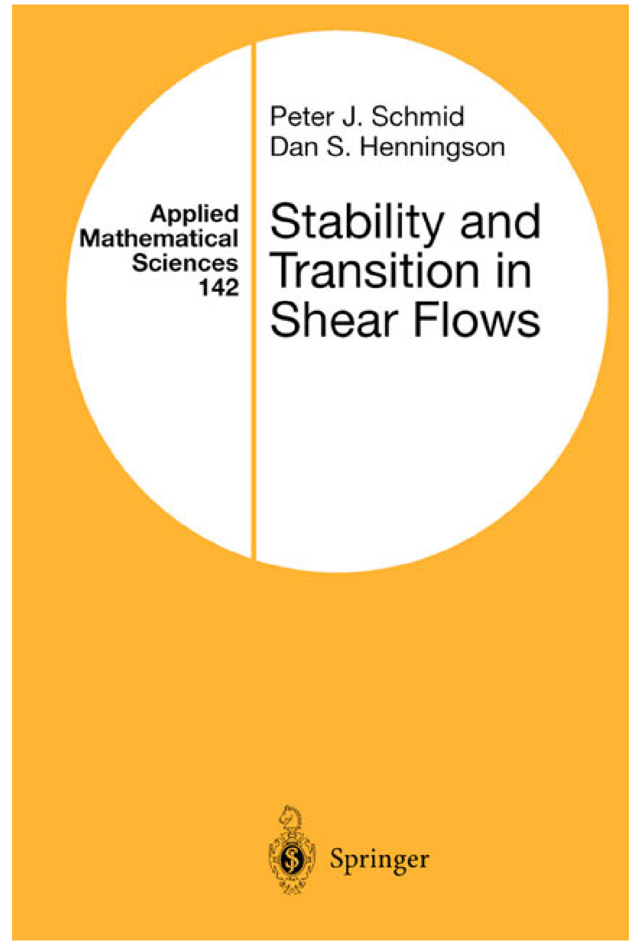
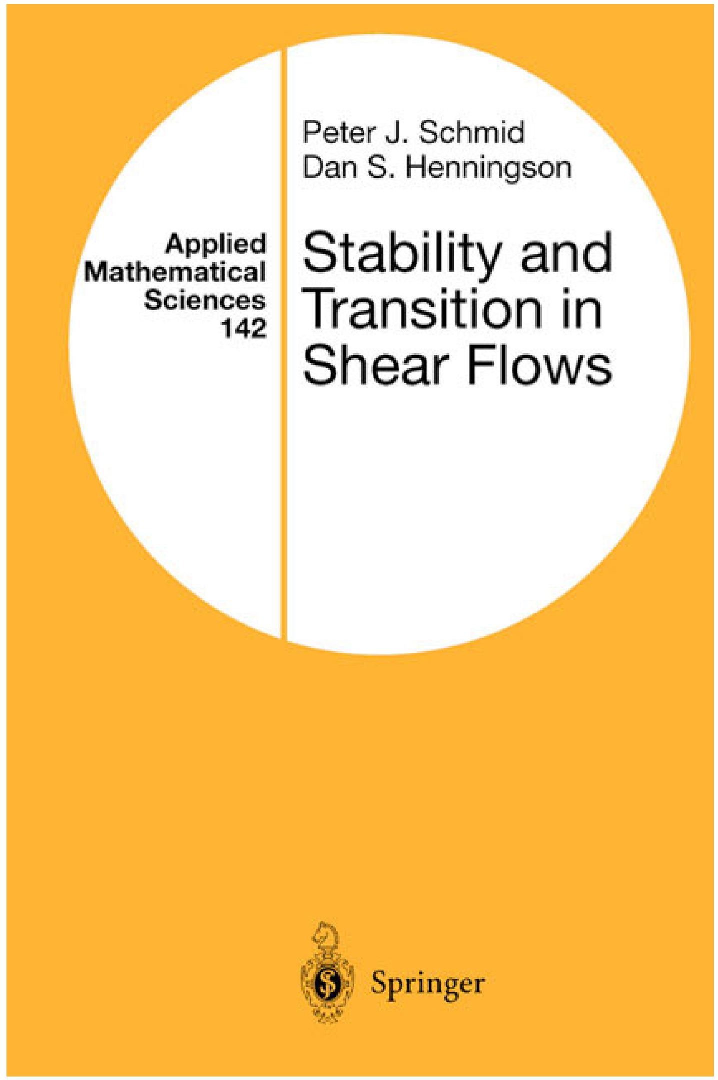
Hydrodynamic stability 2004

- Part of graduate student course, Department of Mechanics
- Lectures: 16/9 8-12, K52
  30/9 8-12, L44
  6/10 8-12, K51
- Projects: 10/11 8-12, K52
  17/11 8-12, E36

- Book: Stability and Transition in Shear Flows
- Further info on web
  www2.mech.kth.se/~henning

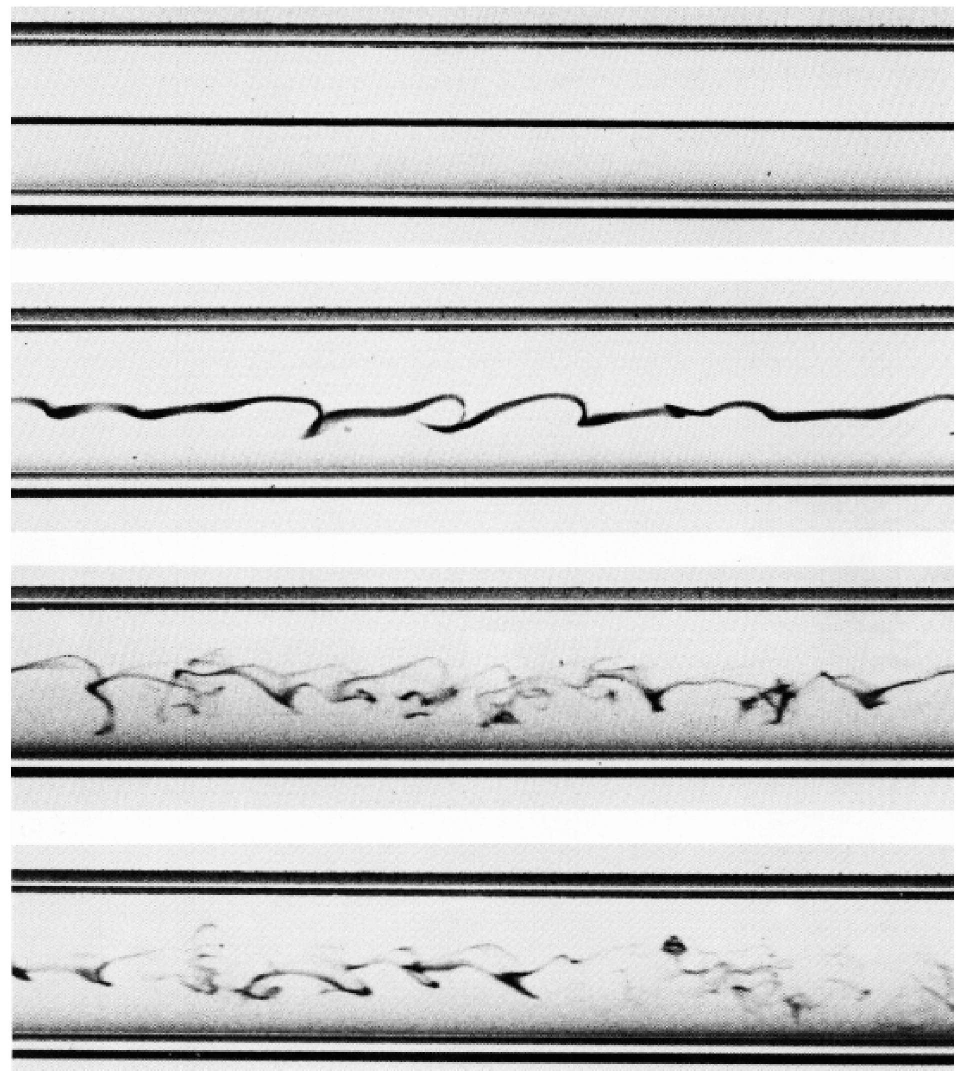


Course on Stability and Transition



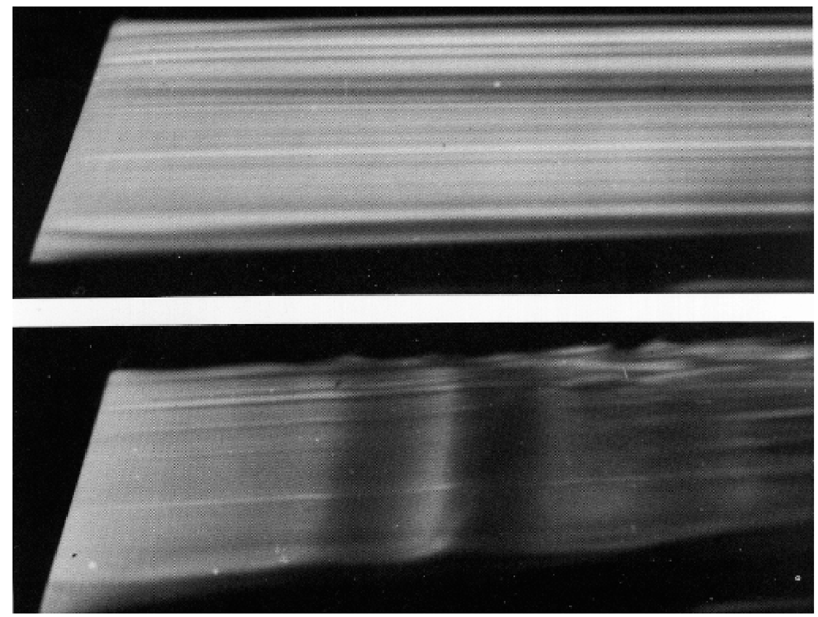
Reynolds pipe flow experiment

- *Original 1883 apparatus*
- *Dye into center of pipe*
- *Critical Re=13.000*
- *Lower today due to traffic*



History of shear flow stability and transition

- *Reynolds pipe flow experiment (1883)*
- *Rayleigh's inflection point criterion (1887)*
- *Orr (1907) Sommerfeld (1908) viscous eq.*
- *Heisenberg (1924) viscous channel solution*
- *Tollmien (1931) Schlichting (1933) viscous BL solution*
- *Schubauer & Skramstad (1947) experimental TS-wave verification*
- *Klebanoff, Tidstrom & Sargent (1962) 3D breakdown*



Bypass transition

- *High and low speed streaks in the streamwise direction*
- *Transition due to free-stream turbulence*
- *Klebanoff (1977) modes, Tu > 0.5% in BL*
- *Subcritical transition in Poiseuille and Couette flows*

Disturbance equations

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0 \quad \text{on solid boundaries}$$

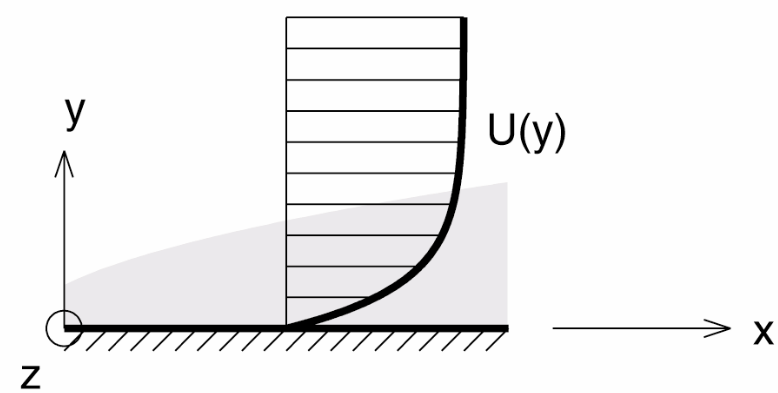
$$\mathrm{Re} = U_\infty \delta_* / \nu$$

$$u_i = U_i + u_i'$$

$$p = P + p' \quad \text{drop primes}$$

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

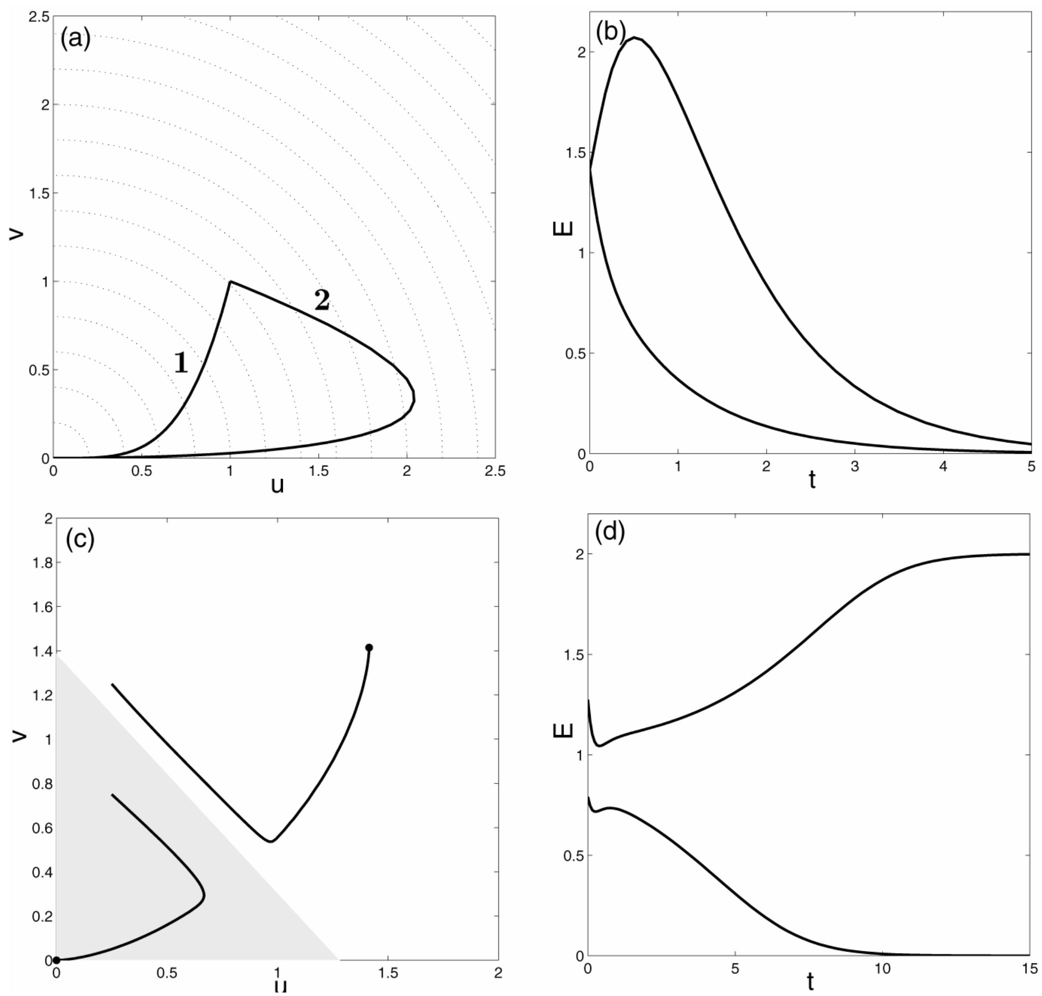


Stability definitions

$$E_V = \frac{1}{2}\int_V u_i u_i \; dV.$$

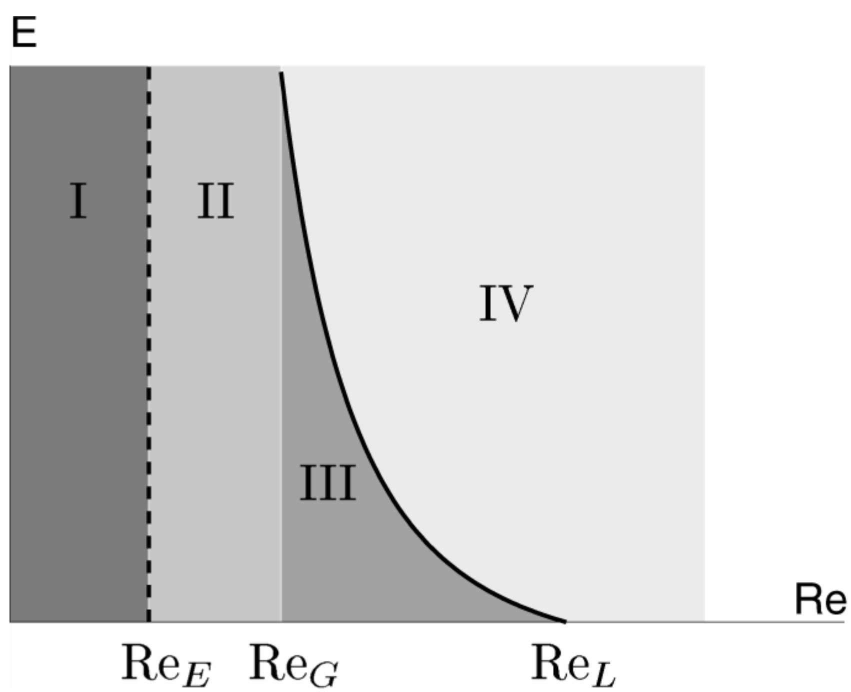
Stable:	$\lim_{t\to\infty} \frac{E_V(t)}{E_V(0)} \to 0$
Conditionally stable:	$\exists \;\; \delta : E(0) < \delta \;\; \Rightarrow$ stable
Globally stable:	Conditionally stable with $\delta \to \infty$
Monotonically stable	$\frac{dE}{dt} \leq 0 \; \forall \;\; t > 0$

Stability definitions



Critical Reynolds numbers

Re_E :	$\mathrm{Re} < \mathrm{Re}_E$	flow monotonically stable
Re_G :	$\mathrm{Re} < \mathrm{Re}_G$	flow globally stable
Re_L :	$\mathrm{Re} > \mathrm{Re}_L$	flow linearly unstable ($\delta \to 0$)



Critical Reynolds numbers

Flow	Re_E	Re_G	Re_T	Re_L
Hagen-Poiseuille	81.5	—	2000	∞
Plane Poiseuille	49.6	—	1000	5772
Plane Couette	20.7	125	360	∞

Critical Reynolds numbers for a number of wall-bounded shear flows compiled from the literature.

Reynolds-Orr equation

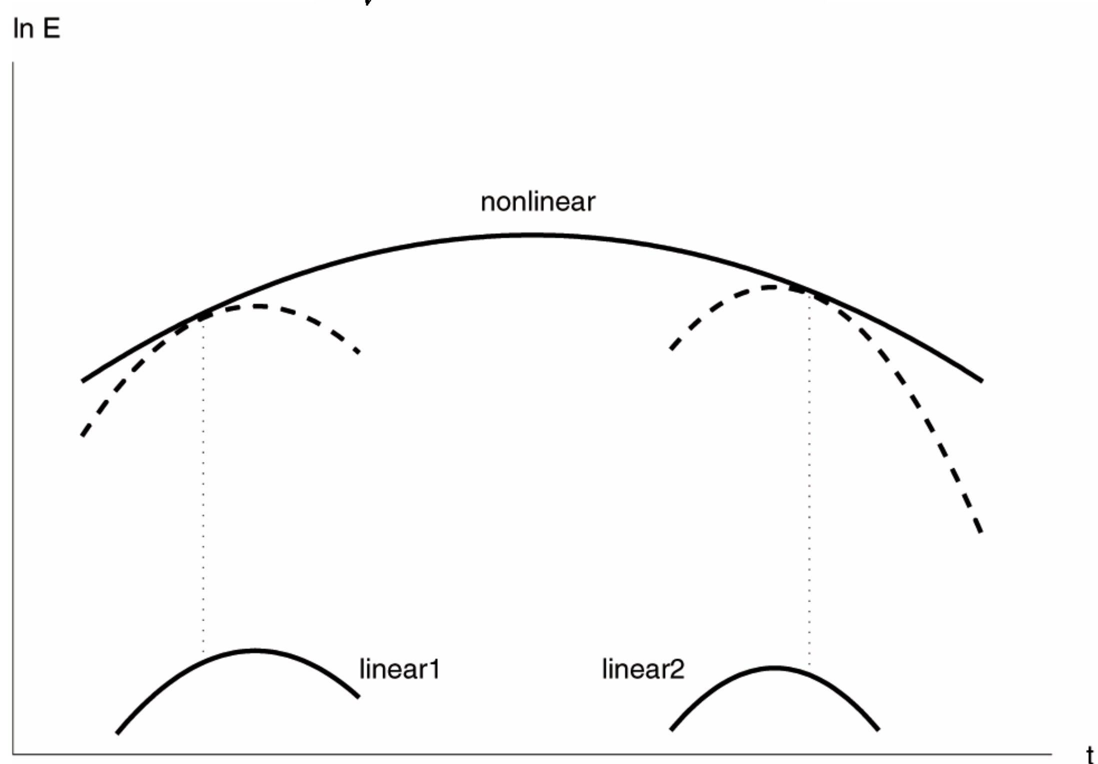
$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{\mathrm{Re}} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j} \left[-\frac{1}{2} u_i u_i U_j - \frac{1}{2} u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{\mathrm{Re}} u_i \frac{\partial u_i}{\partial x_j} \right] \\
&\Rightarrow \\
\frac{dE_V}{dt} &= -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{\mathrm{Re}} \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV
\end{aligned}
$$

Theorem: Linear mechanisms required for energy growth

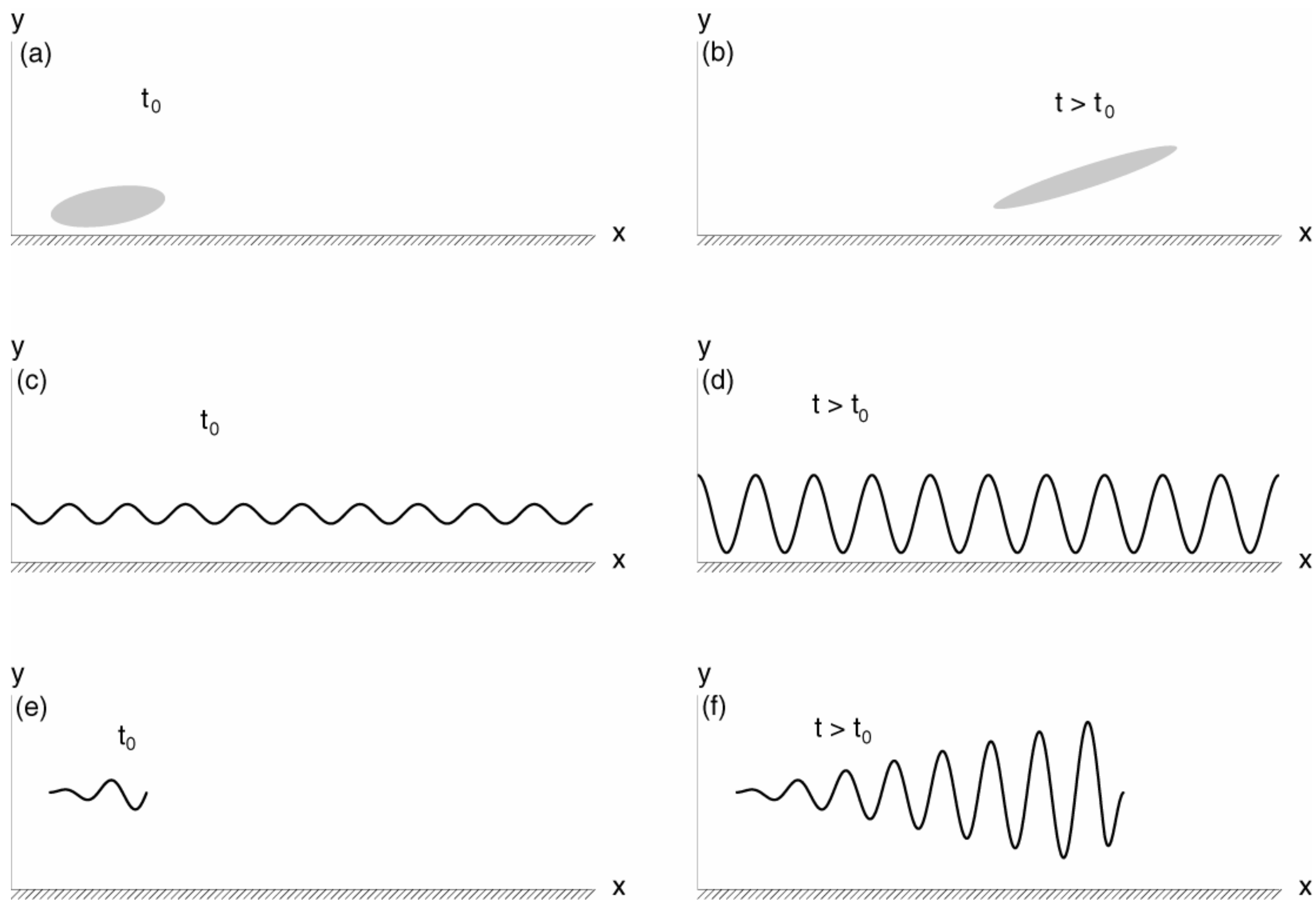
Proof: $\frac{1}{E_V} \frac{dE_V}{dt}$ independent of disturbance amplitude

Linear growth mechanisms

$$\frac{1}{E_V}\frac{dE_V}{dt} = \frac{d}{dt}\ln E_V$$



Evolution of disturbances in shear flows



Parallel shear flows: $U_i = U(y)\delta_{1i}$

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u\\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v\\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}
$$

divergence of the momentum equations gives $\nabla^2 p = -2U'\frac{\partial v}{\partial x}$

elliminate pressure in v-equation $\Rightarrow$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4\right]v = 0$$

Parallel shear flows, cont

normal vorticity describes horizontal flow

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\text{Re}}\nabla^2\right]\eta = -U'\frac{\partial v}{\partial z}$$

with the boundary conditions

$v = v' = \eta = 0$ at a solid wall and in the far field

Orr-Sommerfeld and Squire equations

Assume wavelike solutions: $v(x, y, z, t) = \tilde{v}(y)\ e^{i(\alpha x + \beta z - \omega t)} \quad \Rightarrow$

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - k^2)^2\right]\tilde{v} \ = \ 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(D^2 - k^2)\right]\tilde{\eta} \ = \ -i\beta U'\tilde{v}$$

Orr-Sommerfeld modes: $\left\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\right\}_{n=1}^{N}$

Squire modes: $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^{M}$

Interpretation of modal results

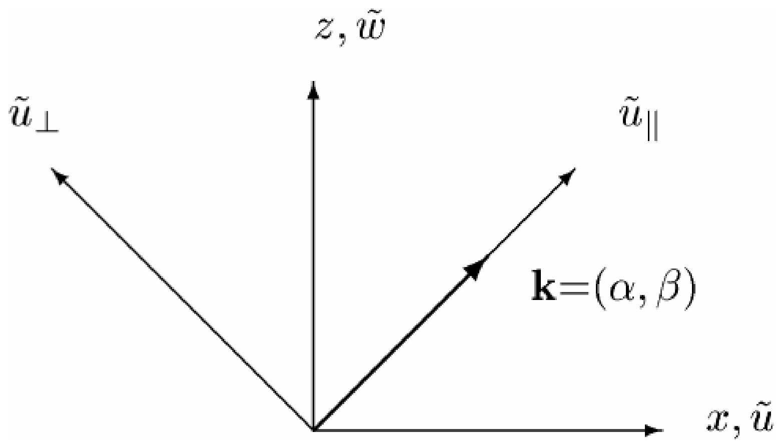
$$\omega = \alpha c$$

$$v = \text{Real}\{|\tilde{v}(y)|\ e^{i\phi(y)}\ e^{i[\alpha x+\beta z-\alpha(c_r+ic_i)t]}\}$$

$$= |\tilde{v}(y)|\ e^{\alpha c_i t}\ \cos[\alpha(x-c_r t)+\beta z+\phi(y)]$$

ω	angular frequency
c_r	phase speed
c_i	temporal growthrate
α	streamwise wavenumber
β	spanwise wavenumber

Interpretation of modal results, cont.



$$\tilde{u}_{||} = \frac{1}{k}\begin{pmatrix}\alpha & \beta\end{pmatrix}\cdot\begin{pmatrix}\tilde{u}\\ \tilde{w}\end{pmatrix} = \frac{1}{k}(\alpha\tilde{u}+\beta\tilde{w}) = -\frac{1}{ik}\frac{d\tilde{v}}{dy}$$

$$\tilde{u}_{\perp} = \frac{1}{k}\begin{pmatrix}-\beta & \alpha\end{pmatrix}\cdot\begin{pmatrix}\tilde{u}\\ \tilde{w}\end{pmatrix} = \frac{1}{k}(\alpha\tilde{w}-\beta\tilde{u}) = -\frac{1}{ik}\tilde{\eta}$$

Squire's transformation

3D and 2D Orr-Sommerfeld equation with $\omega = \alpha c$

$$(U-c)(D^2-k^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha \mathrm{Re}}(D^2-k^2)^2\tilde{v} = 0$$

$$(U-c)(D^2-\alpha_{2D}^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha_{2D} \mathrm{Re}_{2D}}(D^2-\alpha_{2D}^2)^2\tilde{v} = 0$$

$$\begin{aligned} \alpha_{2D} &= k = \sqrt{\alpha^2+\beta^2} \\ \alpha_{2D}\mathrm{Re}_{2D} &= \alpha \mathrm{Re} \\ &\Rightarrow \\ \mathrm{Re}_{2D} &= \mathrm{Re}\frac{\alpha}{k} < \mathrm{Re} \end{aligned}$$

Squire's theorem

Each 3D Orr-Sommerfeld mode corresponds a 2D Orr-Sommerfeld mode at a *lower* Re, *i.e.*

$$\mathrm{Re}_{2D} = \mathrm{Re}\frac{\alpha}{k} < \mathrm{Re}$$

$$\Rightarrow$$

$$\mathrm{Re}_c \equiv \min_{\alpha,\beta} \mathrm{Re}_L(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_L(\alpha,0)$$

since growth rate increases with Reynolds number.

Inviscid disturbances

$$\text{Re} \rightarrow \infty \Rightarrow \text{Rayleigh equation}$$

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U''\right]\tilde{v} = 0$$

$$\omega = \alpha c \Rightarrow$$

$$\left(D^2 - k^2 - \frac{U''}{U - c}\right)\tilde{v} = 0$$

Rayleigh's inflection point criterion

Theorem: A *necessary* condition for *invicid* instability is an *inflection* point in $U(y)$

$$-\int_{-1}^{1} \tilde{v}^* \left(D^2\tilde{v} - k^2\tilde{v} - \frac{U''}{U-c}\tilde{v} \right) dy \;=$$

$$\int_{-1}^{1} |D\tilde{v}|^2 + k^2|\tilde{v}|^2 \; dy + \int_{-1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 \; dy \;=\; 0$$

$$\text{Im}\left\{ \int_{1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 \; dy \right\} = \int_{-1}^{1} \frac{U'' c_i |\tilde{v}|^2}{|\; U-c\; |^2} \; dy = 0$$

Inviscid algebraic instability

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{\eta} = -i\beta U'\hat{v} \qquad \text{where} \qquad -i\omega \rightarrow \frac{\partial}{\partial t}$$

$$\hat{\eta}(t=0) = \hat{\eta}_0$$

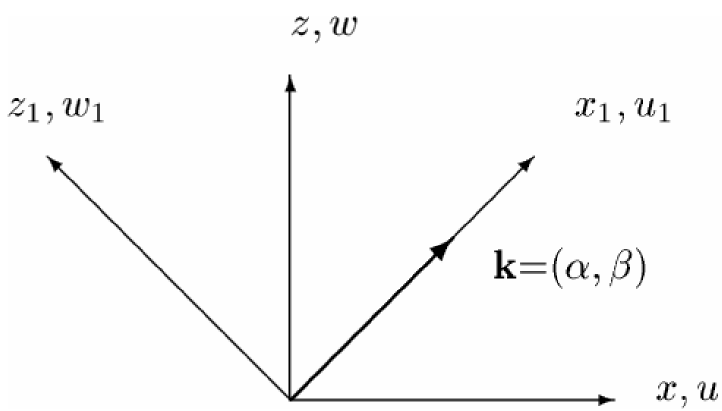
$$\Rightarrow$$

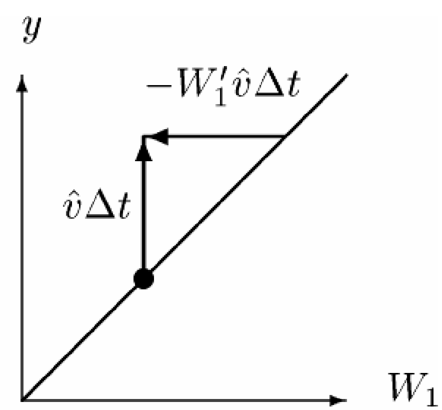
$$\hat{\eta} = \hat{\eta}_0 e^{-i\alpha U t} - i\beta U' e^{-i\alpha U t} \int_0^t \hat{v}(y,t')\, e^{i\alpha U t'} dt'$$

for $\alpha = 0 \quad \Rightarrow \quad \hat{v} = const \quad \Rightarrow$

$$\hat{\eta} = \hat{\eta}_0 - i\beta U' \hat{v}_0 t$$

Lift-up effect





$$W_1 = -\frac{\beta}{k}U$$

$$\hat{w}_1 = \frac{i}{k}\hat{\eta}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + i\alpha U$$

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{\eta} = -i\beta U'\hat{v}$$

$$\frac{D\hat{w}_1}{Dt} = -W_1'\hat{v}$$

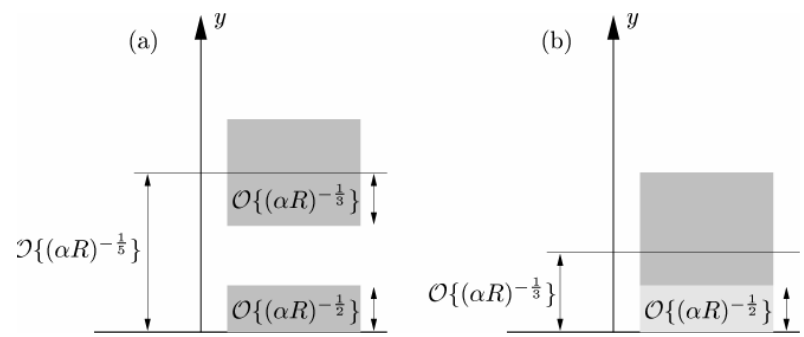
$$\Delta\hat{w}_1 \approx -W_1'\hat{v}\Delta t$$

Critical Layer

Regular singular point in Rayleigh equation at $y_c : U(y_c) = c$
Frobenius expansion around $y = y_c \quad \Rightarrow$

$$\tilde{v}_1 = (y - y_c) P_1(y)$$

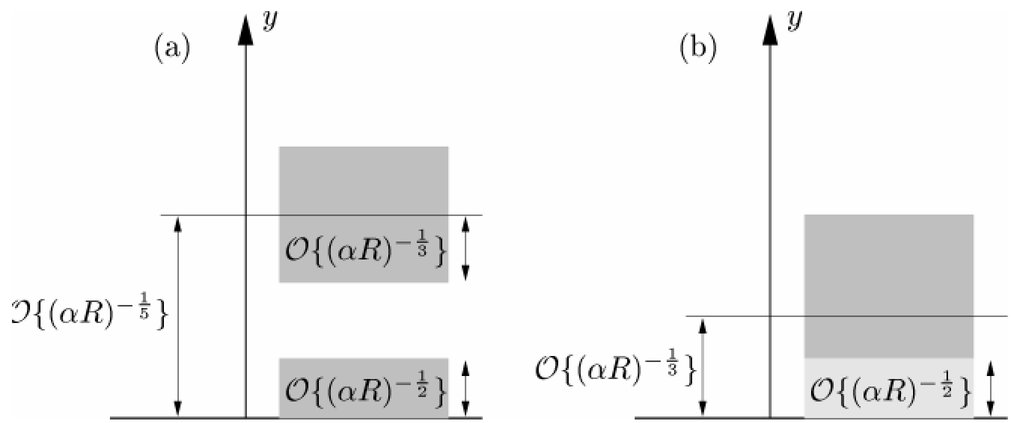
$$\tilde{v}_2 = P_2(y) + \frac{U_c''}{U_c'} \tilde{v}_1(y) \ln(y - y_c)$$



Viscous effects important in two regions

1. boundaries where BC not satisfied
2. critical layer where inviscid is singular

Thickness of critical layer



$U - c \approx U_c'(y - y_c)$ gives leading terms in OS eq.

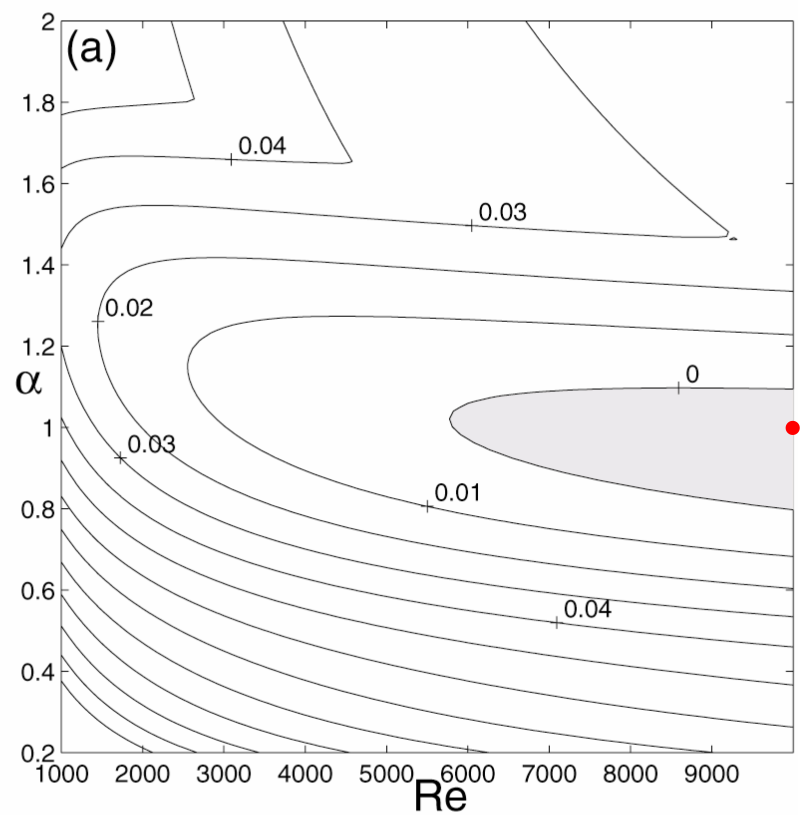
$$\frac{1}{i\alpha \mathrm{Re}} D^4\tilde{v} = U_c'(y - y_c)D^2\tilde{v}$$

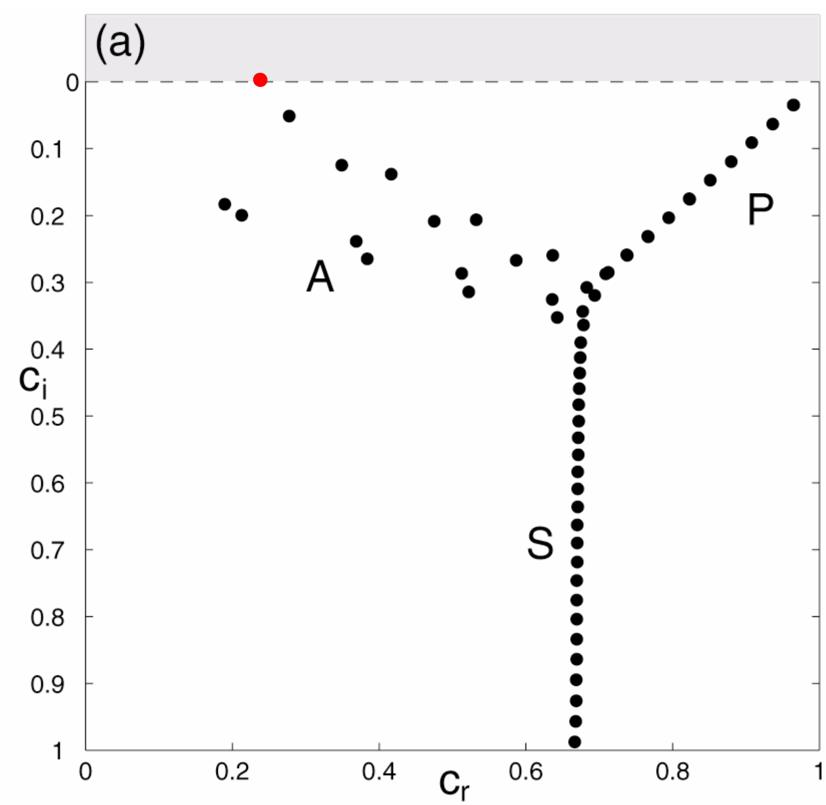
let $\tilde{v}(y) = V(\xi)$, $\xi = (y-y_c)/\epsilon$, $\epsilon = (i\alpha \mathrm{Re} U_c')^{-1/3}$ $\Rightarrow$

$$\left(\frac{d^2}{d\xi^2} - \xi\right)\frac{d^2}{d\xi^2}V = 0$$

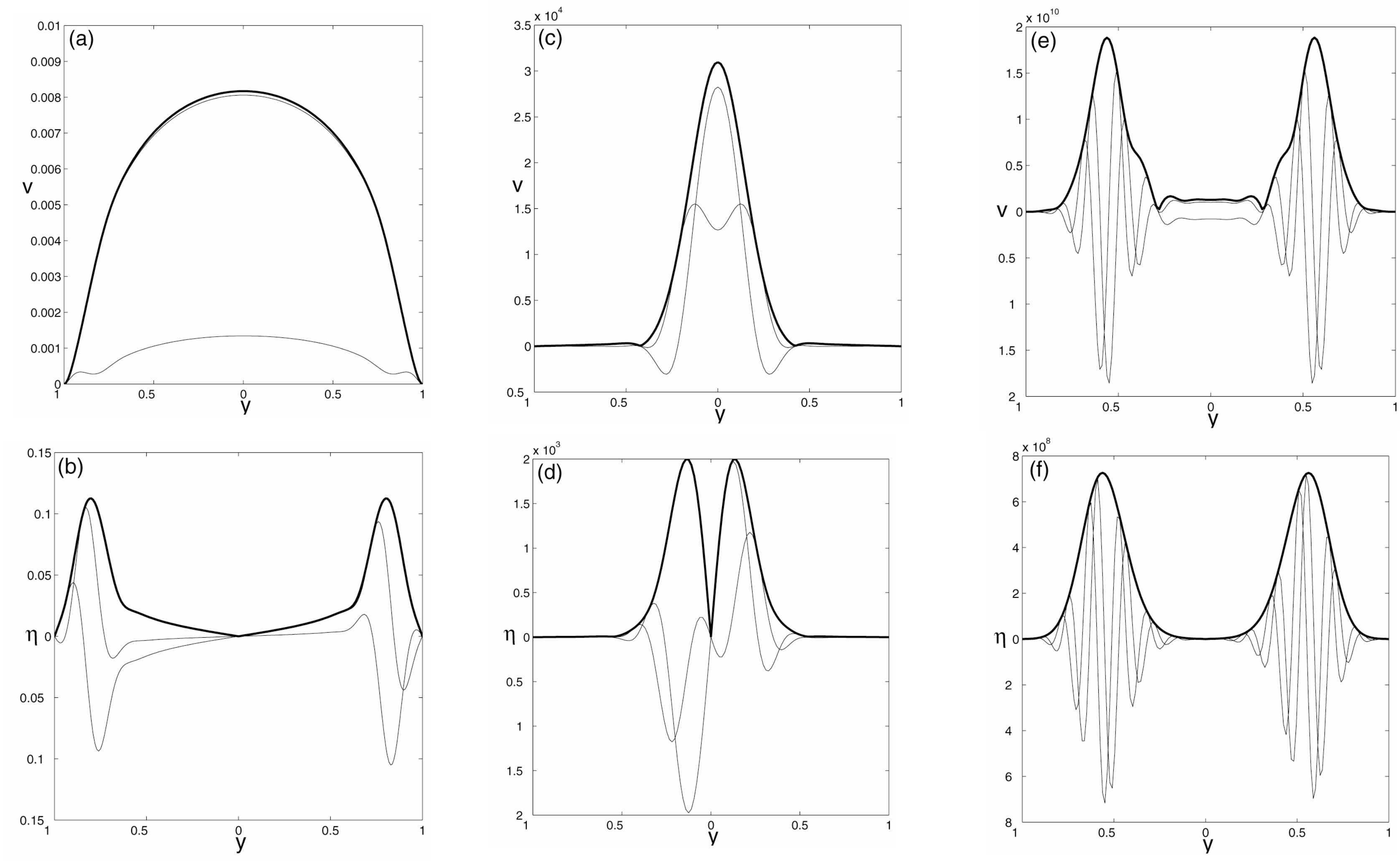
Plane Poiseuille flow

Neutral curve and spectrum

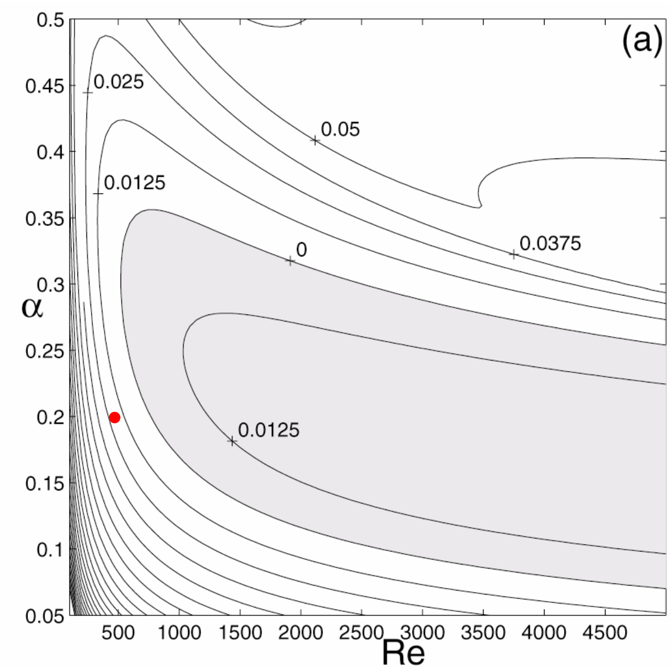


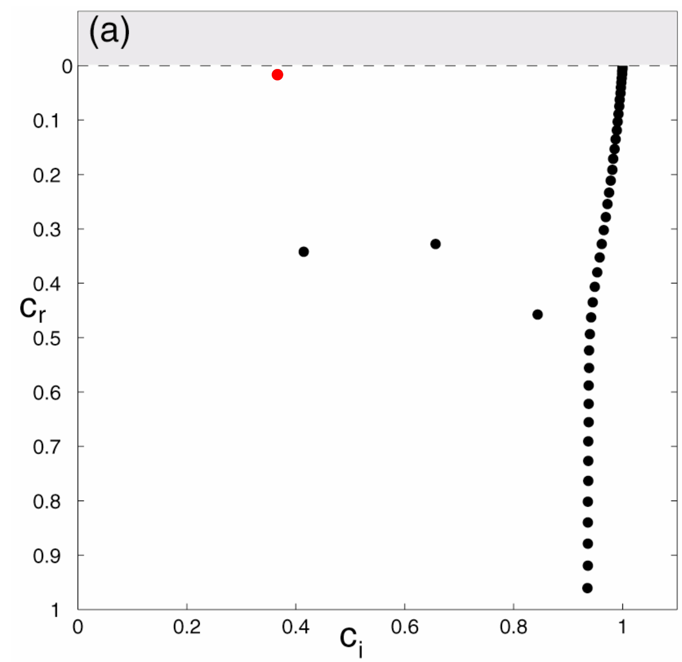


A, P, S- Eigenfunctions for PPF

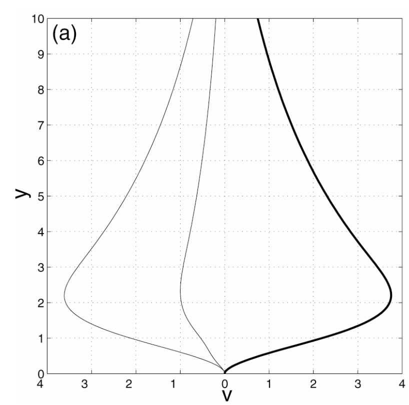


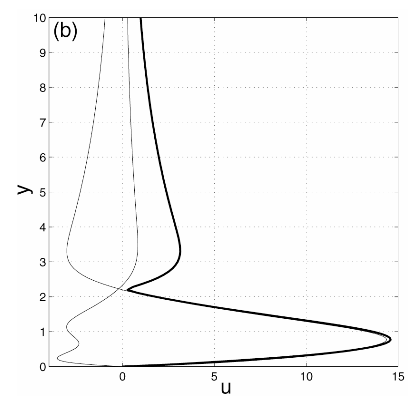
Blasius boundary layer





- Re = 500
- $\alpha = 0.2$
- TS-mode





Continuous spectrum

$$\left(D^2 - k^2\right)^2 \hat{v} = i\alpha\text{Re}\left[(U_\infty - c)\left(D^2 - k^2\right)\right]\hat{v}$$

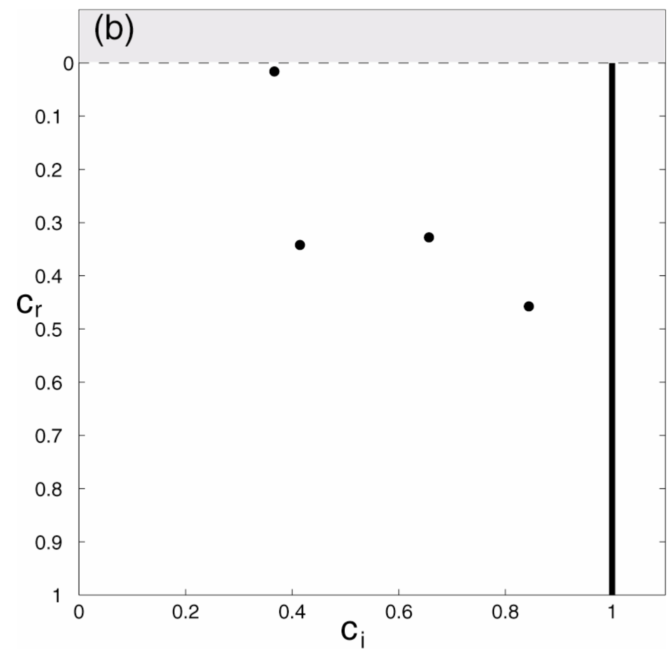
$$\hat{v}_n = \exp(\lambda_n y)$$

$$\lambda_{1,2} = \pm\sqrt{i\alpha\text{Re}(U_\infty - c) + k^2}, \quad \lambda_{3,4} = \pm k$$

$\hat{v}, D\hat{v}$ bounded as $y \to \infty \quad \Rightarrow$

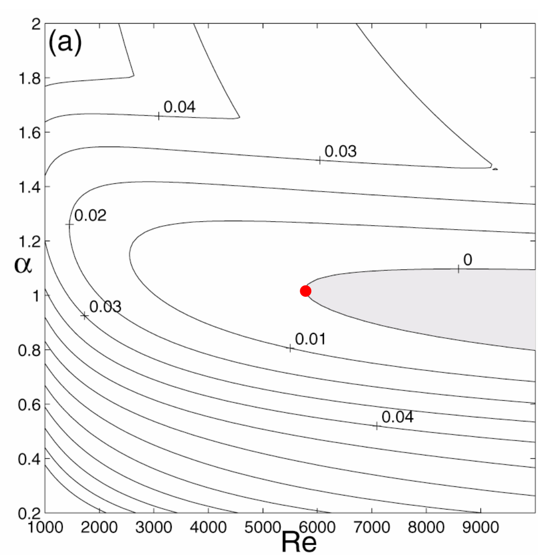
$$\alpha\text{Re}c_i + k^2 < 0 \quad \alpha\text{Re}(U_\infty - c_r) = 0 \quad \Rightarrow$$

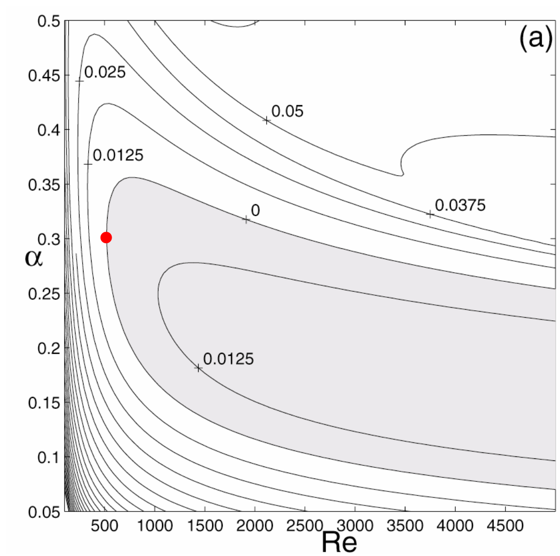
$$c = U_\infty - i(1 + \xi^2)\frac{k^2}{\alpha\text{Re}}$$



Critical Reynolds numbers

Flow	α_{crit}	Re_{crit}	$c_r\vert_{crit}$
Plane Poiseuille flow	1.02	5772	0.264
Blasius boundary layer flow	0.303	519.4	0.397





The intial value problem (IVP)

$$v(x,y,z,t) = \hat{v}(y,t)\ e^{i(\alpha x+\beta z)}$$

$$\eta(x,y,z,t) = \hat{\eta}(y,t)\ e^{i(\alpha x+\beta z)}$$

$$\begin{aligned}\left[(\frac{\partial}{\partial t}+i\alpha U)(D^2-k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2-k^2)^2\right]\hat{v} &= 0\\ \left[(\frac{\partial}{\partial t}+i\alpha U) - \frac{1}{\mathrm{Re}}(D^2-k^2)\right]\hat{\eta} &= -i\beta U'\hat{v}\end{aligned}$$

$$\begin{aligned}\hat{v} = D\hat{v} = \hat{\eta} &= 0\\ \hat{v}(t=0) &= \hat{v}_0\\ \hat{\eta}(t=0) &= \hat{\eta}_0\end{aligned}$$

A simple eigenfunction expansion

$$\hat{v} = \tilde{v}\, e^{-i\alpha c t} \qquad \text{single OS-mode}$$

$$\hat{\eta} = \sum_j C_j \tilde{\eta}_j\, e^{-i\alpha\sigma_j t} + \sum_k D_k \tilde{\eta}_k\, e^{-i\alpha c t} \quad \Rightarrow$$

$$\sum_k D_k[U - c - \frac{1}{i\alpha \mathrm{Re}}(D^2 - k^2)]\tilde{\eta}_k = -\frac{\beta}{\alpha} U' \tilde{v}$$

$$\sum_k D_k[\sigma_k - c] \underbrace{\int_{-1}^{1} \tilde{\eta}_k \tilde{\eta}_j dy}_{\delta_{kj}} = -\frac{\beta}{\alpha} \int_{-1}^{1} U' \tilde{v}\, \tilde{\eta}_j dy$$

$$\hat{\eta}(t=0) = 0 \;\Rightarrow \qquad \hat{\eta} = \sum_j \beta \int_{-1}^{1} U' \tilde{v}\, \tilde{\eta}_j\, dy \frac{e^{-i\alpha c t} - e^{-i\alpha\sigma_j t}}{\alpha c - \alpha\sigma_j} \tilde{\eta}_j$$

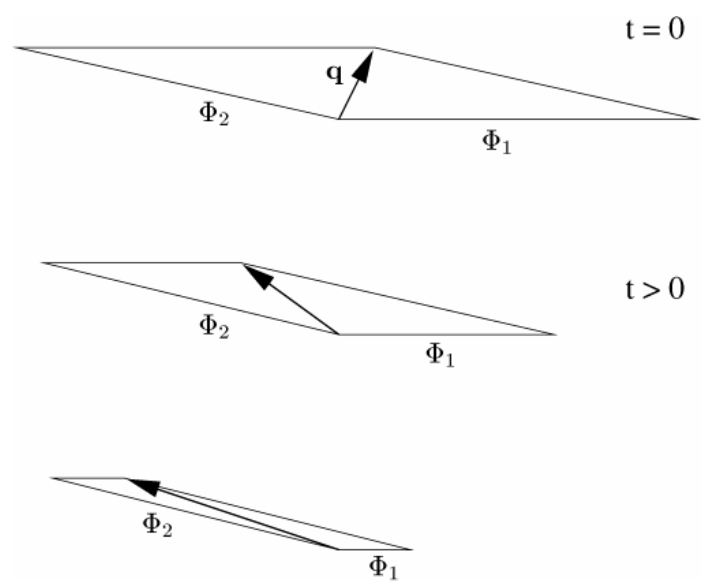
Transient growth

$$\alpha \to 0 \quad \Rightarrow \qquad \omega^{OS} = \alpha c = -i\mu/\text{Re}$$

$$\omega_j^{SQ} = \alpha\sigma_j = -i\nu_j/\text{Re}$$

$$\begin{aligned}
\hat{\eta} &= \sum_j \frac{i\beta\text{Re}\int_{-1}^{1} U'\tilde{v}\ \tilde{\eta}_j\ dy}{\mu - \nu_j} \left(e^{-\mu t/\text{Re}} - e^{-\nu_j t/\text{Re}}\right) \tilde{\eta}_j \\
&= \underbrace{\sum_j \tilde{\eta}_j \int_{-1}^{1} U'\tilde{v}\ \tilde{\eta}_j\ dy}_{U'\tilde{v}} \left[-i\beta t + O(t^2/\text{Re})\right] \\
&= -i\beta U'\tilde{v}_0 t + O(t^2/\text{Re})
\end{aligned}$$

Model with non-orthogonal eigenfunctions



$$\frac{d}{dt}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\text{Re}} & 0 \\ 1 & -\frac{2}{\text{Re}} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}$$

$$\begin{pmatrix} v \\ \eta \end{pmatrix} = v_0 \begin{pmatrix} 1 \\ \text{Re} \end{pmatrix} e^{-t/\text{Re}} + (\eta_0 - v_0 \text{Re}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/\text{Re}}$$

$$\eta(t) = \eta_0 e^{-2t/\text{Re}} + \underbrace{\text{Re}\ v_0 \left(e^{-t/\text{Re}} - e^{-2t/\text{Re}}\right)}_{v_0 t - \frac{3 v_0}{\text{Re}} t^2 + \cdots}$$

General formulation of viscous IVP

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix} -D^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}}\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{L}_{OS} & 0 \\ -i\beta U' & \mathcal{L}_{SQ} \end{pmatrix}}_{\mathbf{L}}\underbrace{\begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix}}_{\hat{\mathbf{q}}}$$

$$\mathcal{L}_{OS} = -i\alpha U(k^2-D^2) - i\alpha U'' - \frac{1}{\text{Re}}(k^2-D^2)^2$$

$$\mathcal{L}_{SQ} = -i\alpha U - \frac{1}{\text{Re}}(k^2-D^2).$$

$$\frac{\partial}{\partial t}\mathbf{M}\hat{\mathbf{q}} = \mathbf{L}\hat{\mathbf{q}}$$

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{L}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}}$$

Disturbance measure

$$E_V = \int_\alpha \int_\beta E \; d\alpha \; d\beta$$

$$\begin{aligned} E &= \frac{1}{2} \int_{-1}^{1} \left(|\hat{u}|^2 + |\hat{v}|^2 + |\hat{u}|^2 \right) \; dy \\ &= \frac{1}{2k^2} \int_{-1}^{1} \left(|D\hat{v}|^2 + k^2 |\hat{v}|^2 + |\hat{\eta}|^2 \right) \; dy \end{aligned}$$

$$= \frac{1}{2k^2} \int_{-1}^{1} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix}^H \begin{pmatrix} -D^2 - k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\eta} \end{pmatrix} \; dy$$

$$2k^2 E = \int_{-1}^{1} \hat{\mathbf{q}}^H \mathbf{M} \hat{\mathbf{q}} \; dy = (\hat{\mathbf{q}}, \hat{\mathbf{q}}) = \|\hat{\mathbf{q}}\|^2$$

Adjoint OS-SQ system

$$
\begin{aligned}
(\tilde{\mathbf{q}}^+, \mathbf{L}_1 \tilde{\mathbf{q}}) &= \int_{-1}^{1} \tilde{\mathbf{q}}^{+H} \mathbf{M} \mathbf{L}_1 \tilde{\mathbf{q}} dy \\
&= \int_{-1}^{1} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix}^H \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ i\beta U' & \mathcal{L}_{SQ} \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} dy \\
&= \int_{-1}^{1} [\tilde{\xi}^* \mathcal{L}_{OS} \tilde{v} + i\beta U' \tilde{\zeta}^* \tilde{v} + \tilde{\zeta}^* \mathcal{L}_{SQ} \tilde{\eta}] dy = \{\text{integration by parts}\} \\
&= \int_{-1}^{1} [(\mathcal{L}_{OS}^+ \tilde{\xi})^* \tilde{v} - (i\beta U' \tilde{\zeta})^* \tilde{v} + (\mathcal{L}_{SQ}^+ \tilde{\zeta})^* \tilde{\eta}] dy \\
&= \int_{-1}^{1} \left[\begin{pmatrix} \mathcal{L}_{OS}^+ & -i\beta U' \\ 0 & \mathcal{L}_{SQ}^+ \end{pmatrix} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix} \right]^H \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} dy \\
&= \int_{-1}^{1} [\mathbf{M} \mathbf{L}_1^+ \tilde{\mathbf{q}}^+]^H \tilde{\mathbf{q}} dy \\
&= (\mathbf{L}_1^+ \tilde{\mathbf{q}}^+, \tilde{\mathbf{q}})
\end{aligned}
$$

Biorthogonality

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}}, \quad \hat{\mathbf{q}} = \tilde{\mathbf{q}}\; e^{\lambda t} \quad \Rightarrow$$

$$0 = \left(\tilde{\mathbf{q}}^+, (\mathbf{L}_1 - \lambda\mathbf{I})\tilde{\mathbf{q}}\right) = \left((\mathbf{L}_1^+ - \lambda^*\mathbf{I})\tilde{\mathbf{q}}^+, \tilde{\mathbf{q}}\right)$$

$$\begin{aligned} 0 &= \left(\tilde{\mathbf{q}}_n^+, \mathbf{L}_1\tilde{\mathbf{q}}_m\right) - \left(\mathbf{L}_1^+\tilde{\mathbf{q}}_n^+, \tilde{\mathbf{q}}_m\right) \\ &= \left(\tilde{\mathbf{q}}_n^+, \lambda_m\tilde{\mathbf{q}}_m\right) - \left(\lambda_n^*\tilde{\mathbf{q}}_n^+, \tilde{\mathbf{q}}_m\right) \\ &= (\lambda_m - \lambda_n)\underbrace{\left(\tilde{\mathbf{q}}_n^+, \tilde{\mathbf{q}}_m\right)}_{\delta_{mn}} \end{aligned}$$

Component form of adjoint

$$\lambda^* \underbrace{\begin{pmatrix} -D^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{L}_{OS}^{+} & i\beta U' \\ 0 & \mathcal{L}_{SQ}^{+} \end{pmatrix}}_{\mathbf{L}^{+}} \underbrace{\begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix}}_{\hat{\mathbf{q}}^{+}}$$

$$\mathcal{L}_{OS}^{+} = i\alpha U(k^2 - D^2) - i\alpha 2U'D - \frac{1}{\mathrm{Re}}(k^2 - D^2)^2$$

$$\mathcal{L}_{SQ}^{+} = i\alpha U - \frac{1}{\mathrm{Re}}(k^2 - D^2).$$

Adjoint Orr-Sommerfeld modes: $\left\{\tilde{\xi}_n, \tilde{\zeta} = 0, \omega_n\right\}_{n=1}^{N}$

Adjoint Squire modes: $\left\{\tilde{\xi}_m^p, \tilde{\zeta}_m, \omega_m\right\}_{m=1}^{M}$

Solution of IVP using eigenfunction expansions

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{L}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}} \qquad \hat{\mathbf{q}}(t=0) = \hat{\mathbf{q}}_0 \\ \|\hat{\mathbf{q}}\|^2 = (\hat{\mathbf{q}}, \hat{\mathbf{q}}) = \displaystyle\int_{-1}^{1} \hat{\mathbf{q}}^H \mathbf{M}\hat{\mathbf{q}}\ dy \end{cases}$$

$$\hat{\mathbf{q}} = \sum_{n=1}^{\infty} \kappa_n^0 \tilde{\mathbf{q}}_n e^{\lambda_n t} \quad \Rightarrow$$

$$(\tilde{\mathbf{q}}_m^+, \hat{\mathbf{q}}_0) = \left(\tilde{\mathbf{q}}_m^+, \sum_{n=1}^{\infty} \kappa_n^0 \tilde{\mathbf{q}}_n\right) = \sum_{n=1}^{\infty} \kappa_n^0 (\tilde{\mathbf{q}}_m^+, \tilde{\mathbf{q}}_n) = \kappa_m^0$$

Discrete formulation

Project solution on $S^N = \text{span}\{\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \ldots, \tilde{\mathbf{q}}_N\}$

$$\hat{\mathbf{q}} = \sum_{n=1}^{N} \kappa_n^0 \tilde{\mathbf{q}}_n e^{\lambda_n t} = \sum_{n=1}^{N} \kappa_n(t) \tilde{\mathbf{q}}_n \qquad \hat{\mathbf{q}} \in S^N$$

$$\kappa = \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_N \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{pmatrix} \begin{pmatrix} \kappa_1^0 \\ \vdots \\ \kappa_N^0 \end{pmatrix} = e^{\Lambda t} \kappa^0$$

Discrete formulation, cont.

$$\|\hat{\mathrm{q}}\|^2 = (\hat{\mathrm{q}}, \hat{\mathrm{q}}) = \sum_{m=1}^{N} \sum_{n=1}^{N} \kappa_m \kappa_n^* (\tilde{\mathrm{q}}_n, \tilde{\mathrm{q}}_m) \qquad \hat{\mathrm{q}} \in S^N$$

$$= \begin{pmatrix} \kappa_1^* & \dots & \kappa_N^* \end{pmatrix} \begin{pmatrix} (\tilde{\mathrm{q}}_1, \tilde{\mathrm{q}}_1) & (\tilde{\mathrm{q}}_1, \tilde{\mathrm{q}}_2) & \\ & \ddots & \\ & & (\tilde{\mathrm{q}}_N, \tilde{\mathrm{q}}_N) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_N \end{pmatrix}$$

$$= \kappa^H A \kappa$$

$$= \kappa^H F^H F \kappa \qquad F^H F = A \quad \text{Hermitian}$$

$$= \|F\kappa\|_2^2 \qquad \text{2-norm, sum of squares}$$

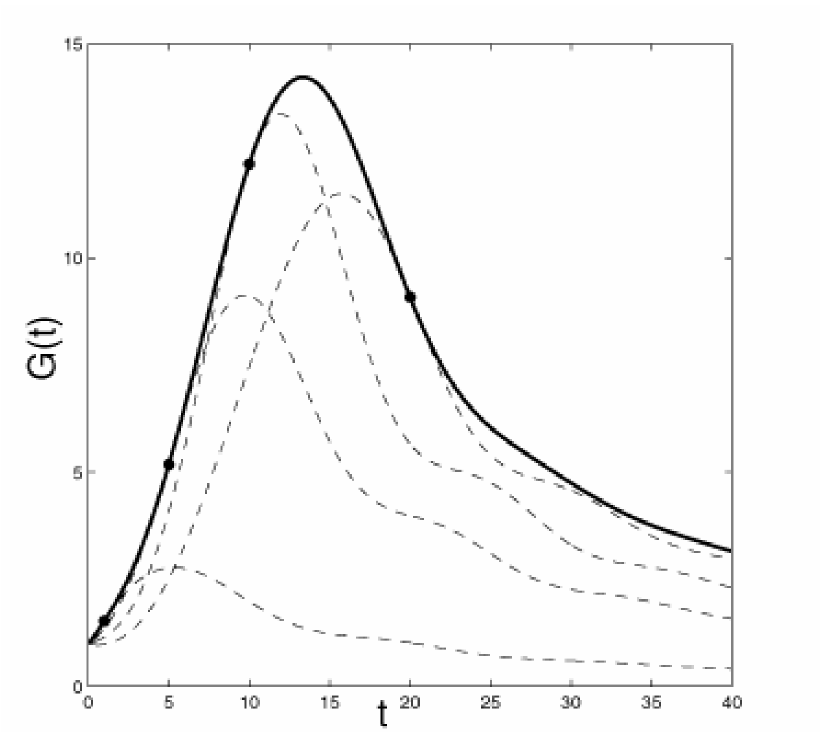
$$= \|\kappa\|_E^2 \qquad \text{energy norm}$$

Maximum amplification

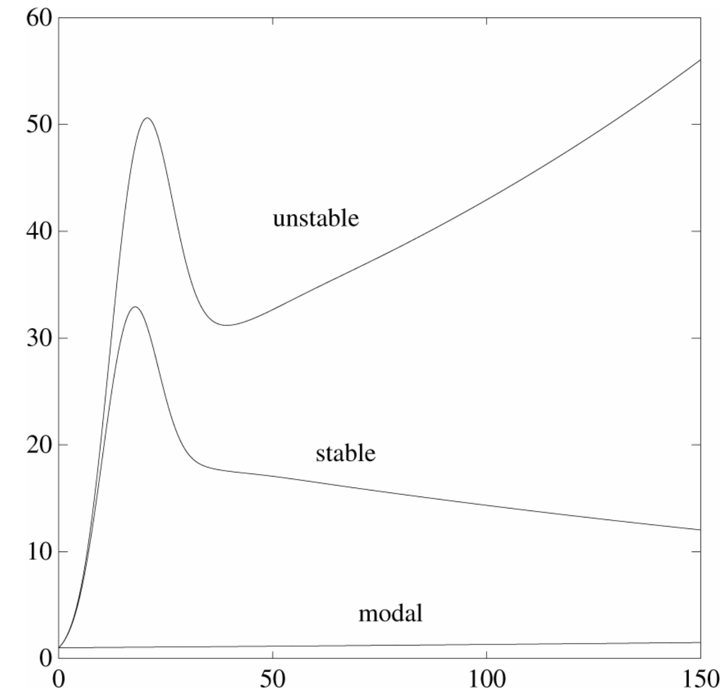
$$
\begin{aligned}
G(t) &= \max_{\hat{\mathbf{q}}_0 \neq 0} \frac{\|\hat{\mathbf{q}}(t)\|^2}{\|\hat{\mathbf{q}}_0\|^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|\kappa\|_E^2}{\|\kappa_0\|_E^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|e^{\Lambda t}\kappa_0\|_E^2}{\|\kappa_0\|_E^2} \\
&= \max_{\kappa_0 \neq 0} \frac{\|F e^{\Lambda t} F^{-1} F \kappa_0\|_2^2}{\|F\kappa_0\|_2^2} \\
&= \| \underbrace{F e^{\Lambda t} F^{-1}}_{B} \|_2^2 \\
&\leq \|F\|_2^2 \, \|F^{-1}\|_2^2 \, \|e^{\Lambda t}\|_2^2 \; = \; \text{cond}(F)^2 \; e^{2\Re\{\lambda_{max}\}t}
\end{aligned}
$$

$$\|B\|_2^2 = \lambda_{max}(B^H B) = \sigma_1^2(B) \quad \text{for} \quad F\kappa_0 = v_1$$

2D PPF: envelope and selected IC

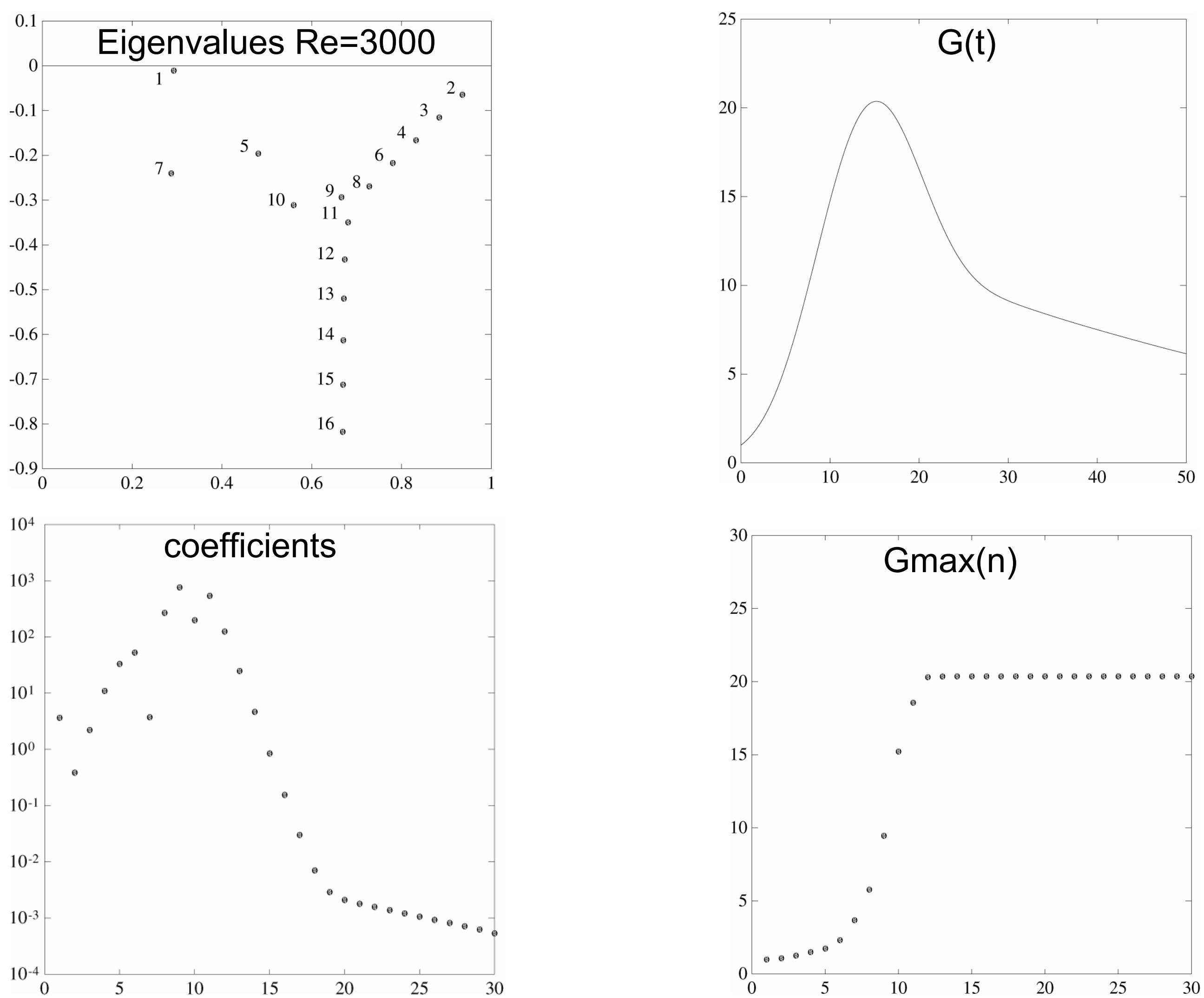


Re=1000



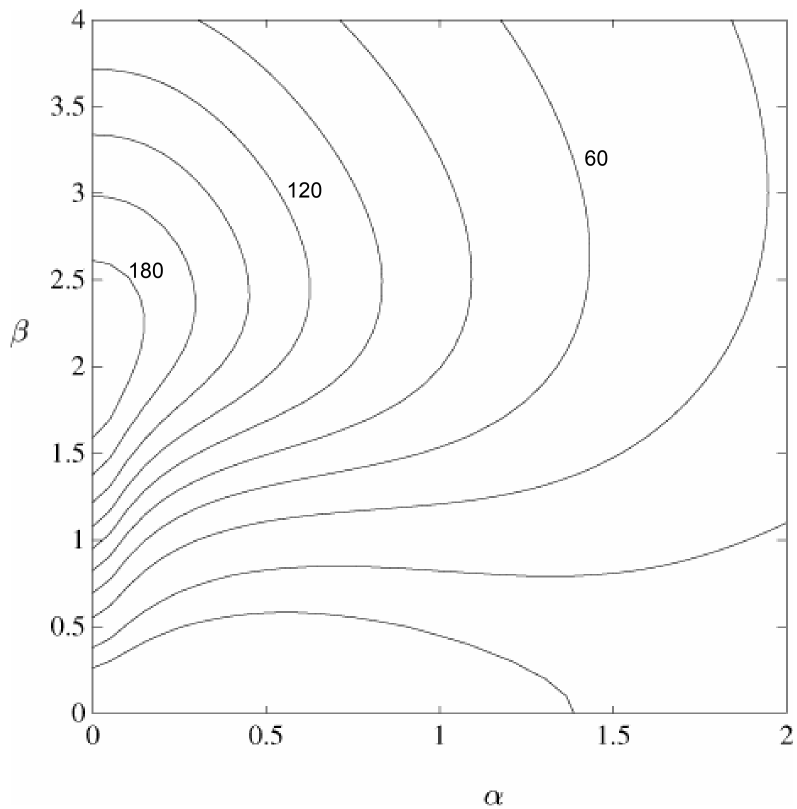
Re=5000, 8000

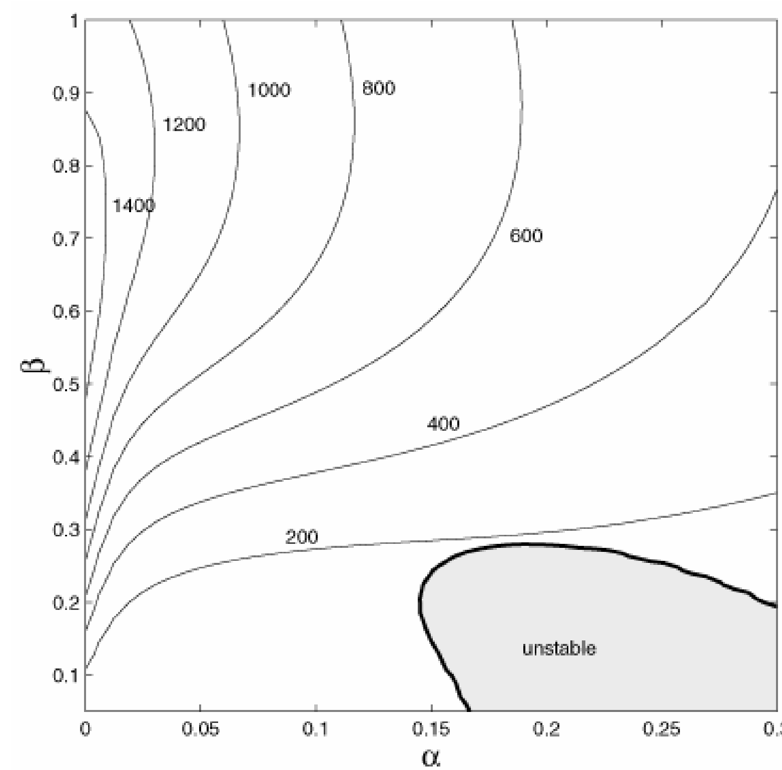
2D PPF: dependence on N



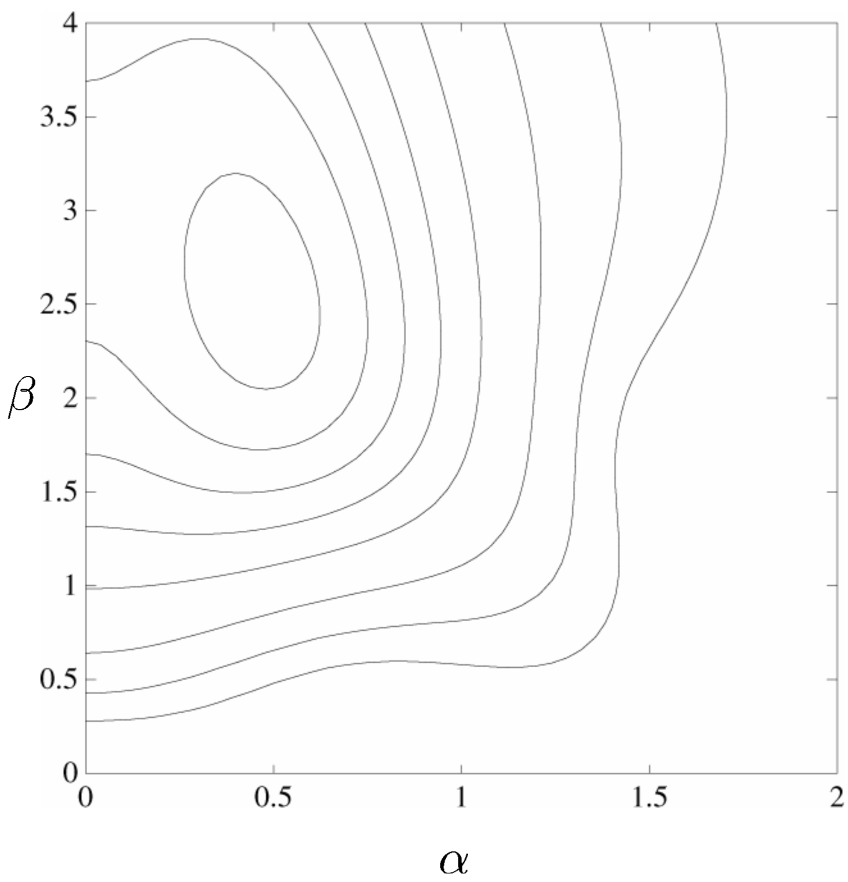
3D PPF and Blasius flow, Re=1000

$$G_{max} = \max_{t \geq 0} G(t)$$

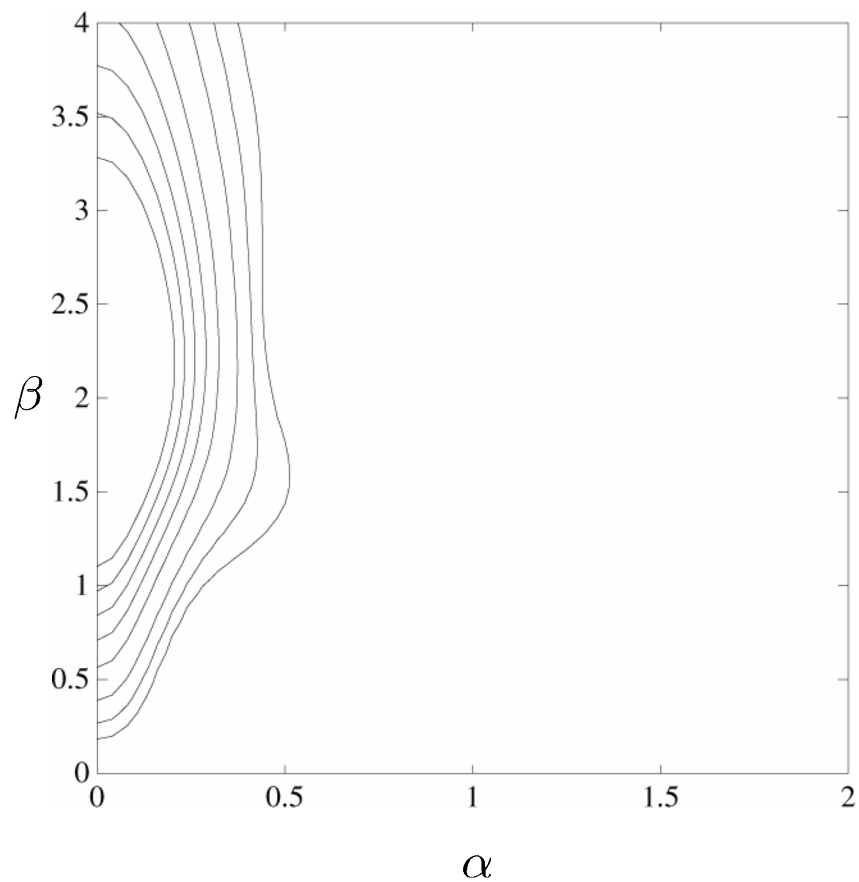




3D PPF: G(t), Re=1000



$t = 25$

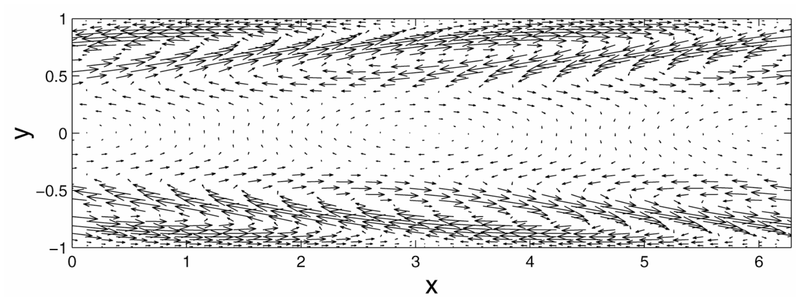


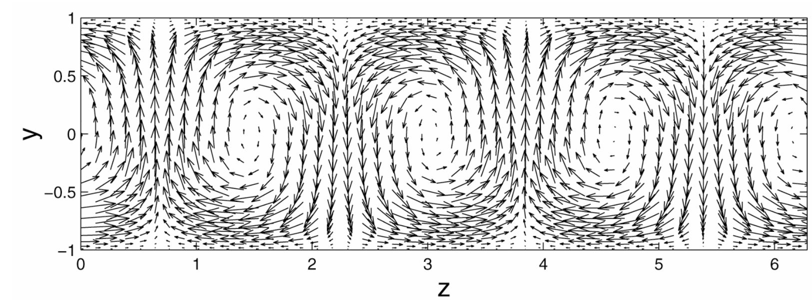
$t = 75$

Optimal disturbances PPF, Re=1000

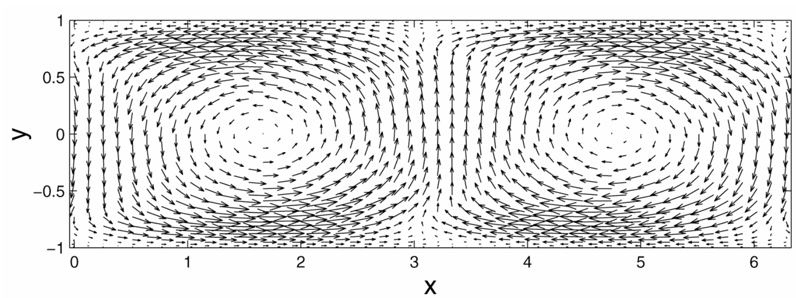
2D disturbance | 3D disturbance

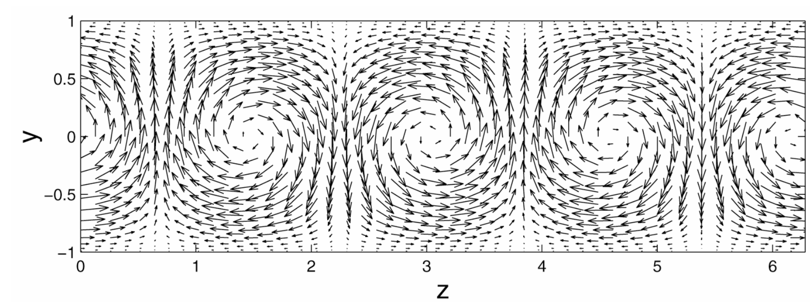
$t = 0$





$t = t_{max}$





$\alpha = 1, \beta = 0$

$\alpha = 0, \beta = 2$

The forced problem and the resolvent

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}} + \hat{\mathbf{q}}_f e^{i\omega t} \qquad \Rightarrow$$

$$\hat{\mathbf{q}} = e^{\mathbf{L}_1 t}\hat{\mathbf{q}}_0 + (\mathbf{L}_1 - i\omega\mathbf{I})^{-1}\hat{\mathbf{q}}_f e^{i\omega t}$$

$$\frac{\partial}{\partial t}\hat{\mathbf{q}} = \mathbf{L}_1\hat{\mathbf{q}} \qquad \tilde{\mathbf{q}} = \int_0^\infty e^{-st}\hat{\mathbf{q}}(t)dt$$

$$s\tilde{\mathbf{q}} - \mathbf{L}_1\tilde{\mathbf{q}} = \hat{\mathbf{q}}_0$$

$$\tilde{\mathbf{q}} = (s\mathbf{I} - \mathbf{L}_1)^{-1}\hat{\mathbf{q}}_0$$

Discrete formulation

$$\begin{aligned}
\tilde{\mathbf{q}} &= \int_0^\infty e^{-st}\hat{\mathbf{q}}(t)dt \\
&= \int_0^\infty e^{-st}\sum_{n=1}^{N}\kappa_n\tilde{\mathbf{q}}_n e^{\lambda_n t}dt \\
&= \sum_{n=1}^{N}\kappa_n\tilde{\mathbf{q}}_n\int_0^\infty e^{-(s-\lambda_n)t}dt \\
&= \sum_{n=1}^{N}\underbrace{\frac{\kappa_n}{s-\lambda_n}}_{\kappa_n(s)}\tilde{\mathbf{q}}_n
\end{aligned}$$

$$\kappa(s) = \begin{pmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_N} \end{pmatrix}\begin{pmatrix} \kappa_1^0 \\ \vdots \\ \kappa_N^0 \end{pmatrix}$$

Maximum response to forcing

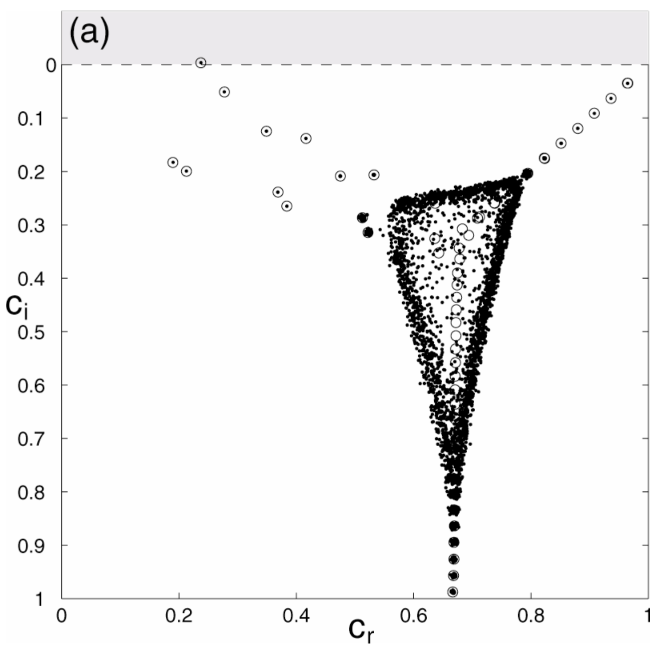
$$
\begin{aligned}
R(s) &= \max_{\hat{\mathbf{q}}_0 \neq 0} \frac{\|(s\mathbf{I} - \mathbf{L}_1)^{-1}\hat{\mathbf{q}}_0\|}{\|\hat{\mathbf{q}}_0\|} \\
&= \max_{\kappa_0 \neq 0} \frac{\|\kappa(s)\|_E}{\|\kappa_0\|_E} \\
&= \|F \ \text{diag}\{\frac{1}{s-\lambda_1}, \ldots, \frac{1}{s-\lambda_N}\} F^{-1}\|_2 \\
&\leq \|F\|_2 \ \|F^{-1}\| \ \frac{1}{\text{min.dist}(\lambda - s)}
\end{aligned}
$$

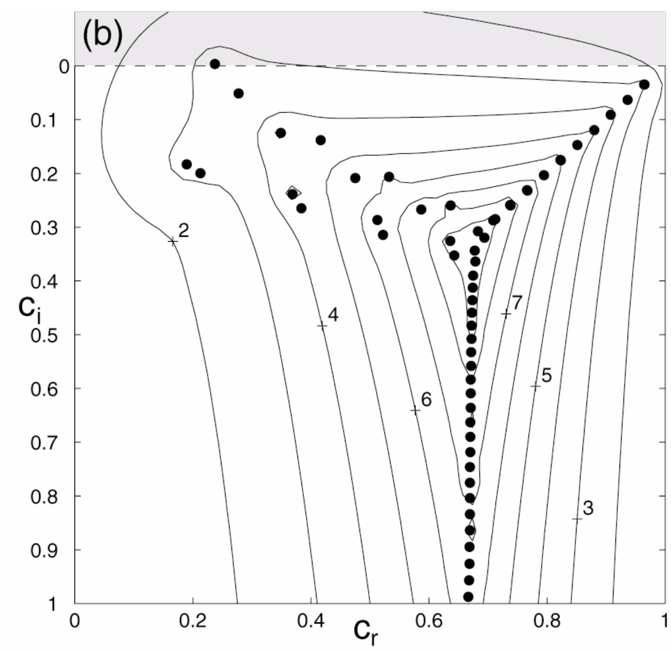
Pseudospectra, resolvents and sensitivity

Definition: for $\epsilon \geq 0$, s is in the ϵ-pseudospectra of $\mathbf{L}$ if any of the following equivalent conditions hold

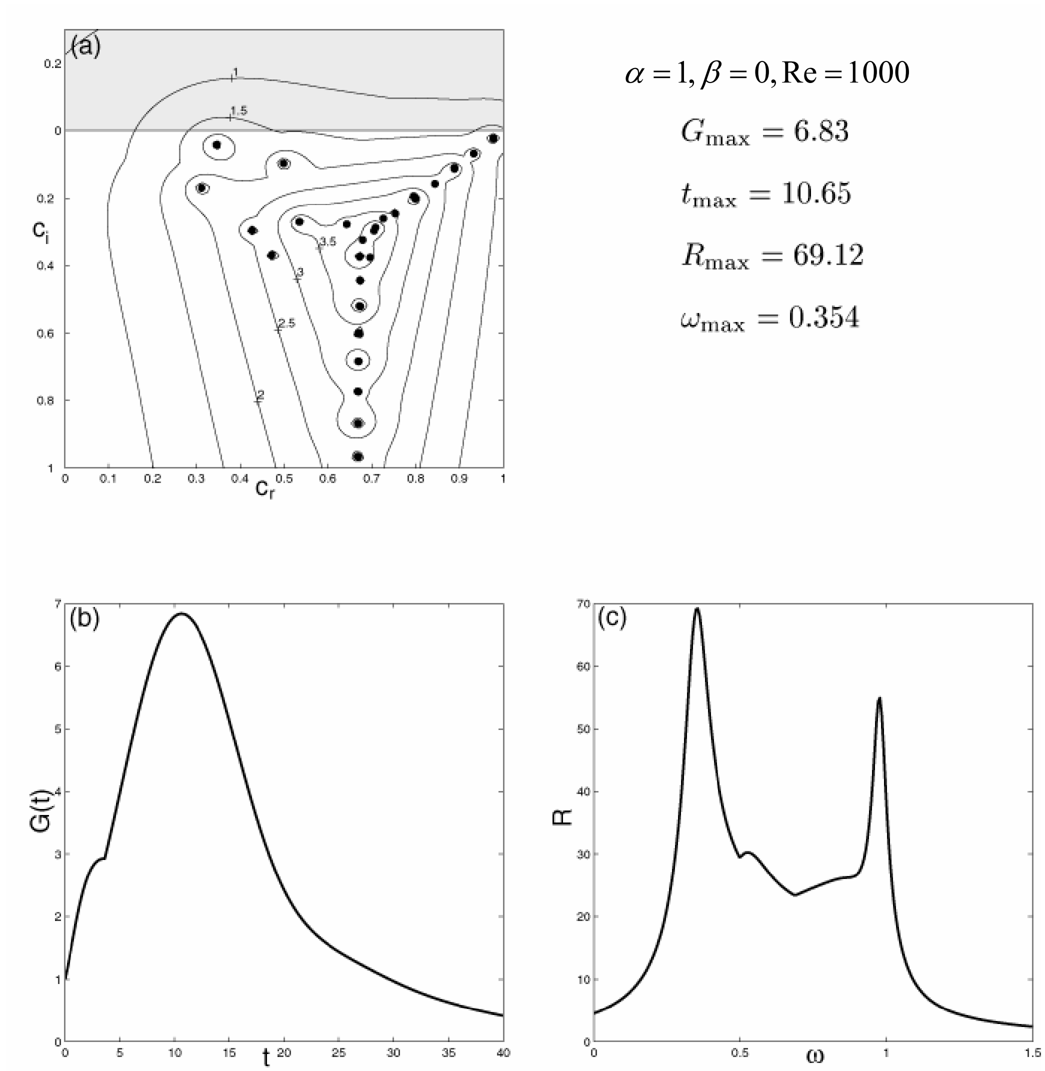
(i) s is an eigenvalue of $\mathbf{L}+\mathbf{E}$, where $\|\mathbf{E}\| \leq \epsilon$

(ii) $\|(s\mathbf{I}-\mathbf{L})^{-1}\| \geq \dfrac{1}{\epsilon}$





PPF: resolvents, growth and forcing



$\alpha = 1, \beta = 0, \mathrm{Re} = 1000$

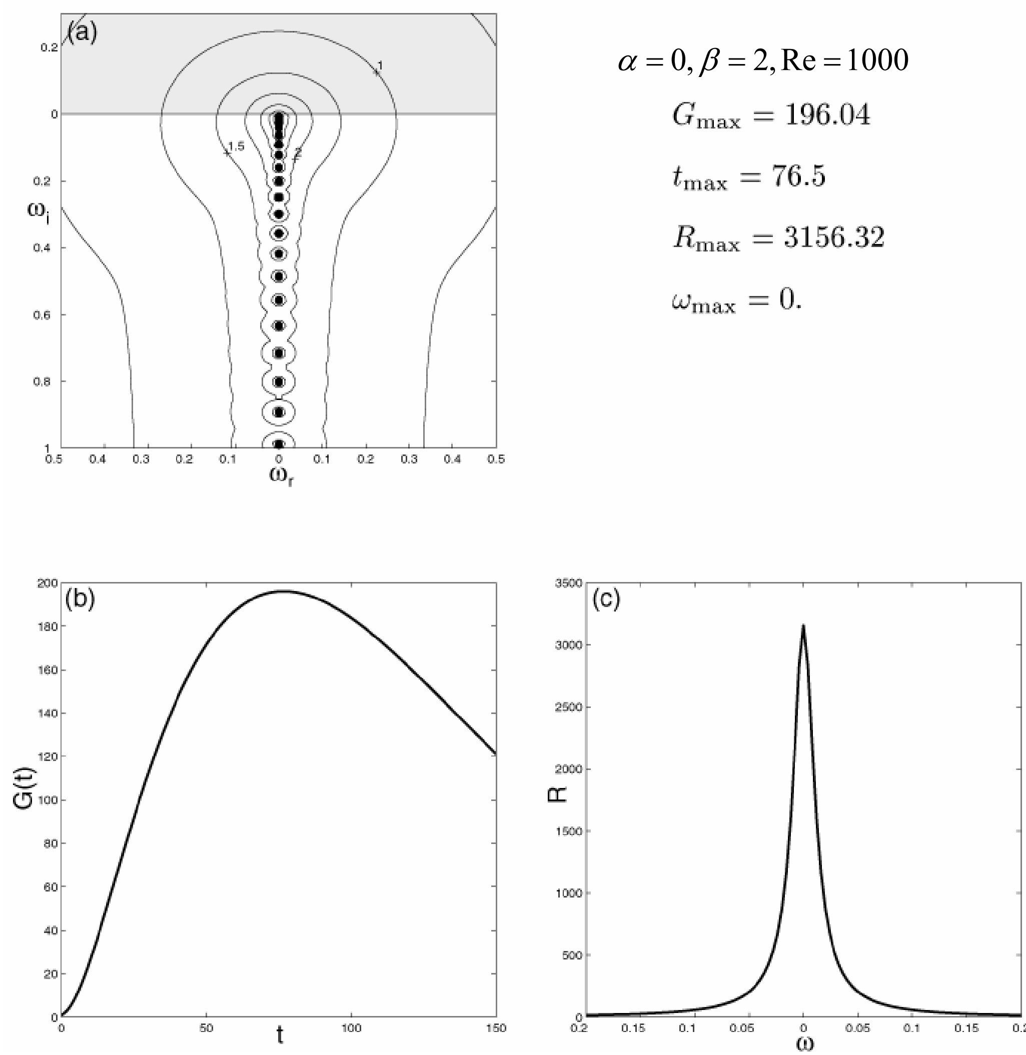
$G_{\max} = 6.83$

$t_{\max} = 10.65$

$R_{\max} = 69.12$

$\omega_{\max} = 0.354$

PPF: resolvents, growth and forcing



$\alpha = 0, \beta = 2, \mathrm{Re} = 1000$

$G_{\max} = 196.04$

$t_{\max} = 76.5$

$R_{\max} = 3156.32$

$\omega_{\max} = 0.$

Re-dependence of growth and response

scaling:

$$\bar{t} = t/\text{Re}, \quad \bar{\eta}(y,\bar{t}) = \hat{\eta}(y,t/\text{Re})/\beta\text{Re}, \quad \bar{v}(y,\bar{t}) = \hat{v}(y,t/\text{Re}) \quad \Rightarrow$$

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix} -D^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}}\begin{pmatrix} \bar{v} \\ \bar{\eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{\mathcal{L}}_{OS} & 0 \\ -iU' & \bar{\mathcal{L}}_{SQ} \end{pmatrix}}_{\bar{\mathbf{L}}}\underbrace{\begin{pmatrix} \bar{v} \\ \bar{\eta} \end{pmatrix}}_{\bar{\mathbf{q}}}$$

$$\bar{\mathcal{L}}_{OS} = -i\alpha\text{Re}U(k^2-D^2) - i\alpha\text{Re}U'' - (k^2-D^2)^2$$

$$\bar{\mathcal{L}}_{SQ} = -i\alpha\text{Re}U - k^2 + D^2$$

Re-dependence, cont.

for $\alpha\text{Re} = 0$ all eigensolutions are damped and

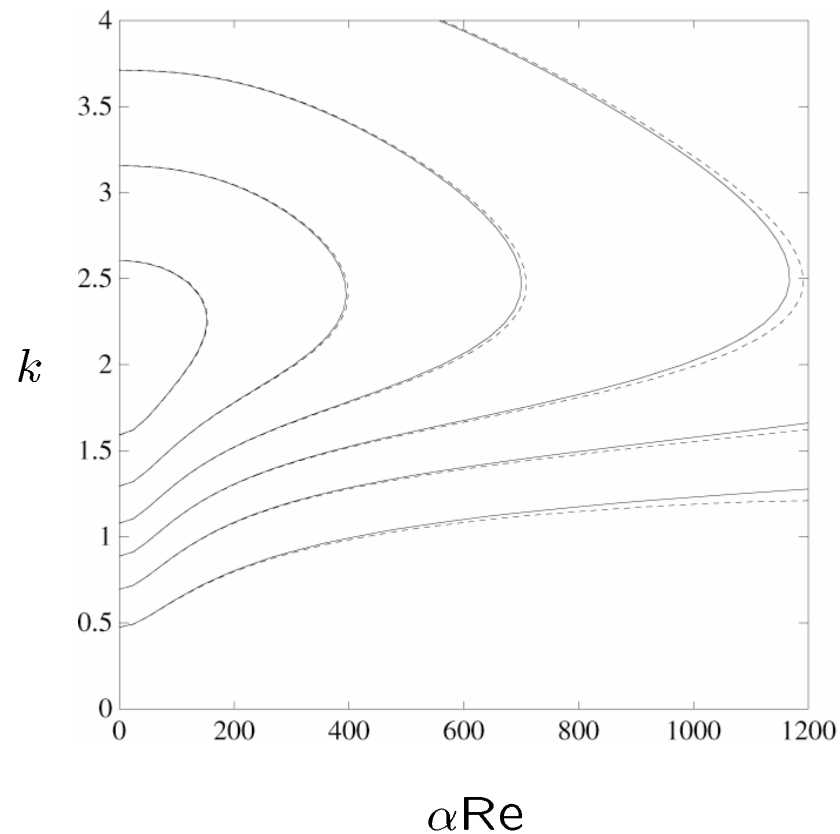
$$\mathcal{L}_{OS} \text{ normal} \quad \Rightarrow \quad E_V(\bar{t}) \le E_V(0)$$
$$\mathcal{L}_{SQ} \text{ normal} \quad \Rightarrow \quad E_\eta(\bar{t}) \le E_\eta(0) \text{ if } \bar{v}_0 = 0$$

$$\begin{aligned} 2k^2 E(\bar{t}) &= \|\bar{\mathbf{q}}\|^2 = \int_{-1}^{1} \left(k^2|\bar{v}|^2 + |D\bar{v}|^2 + \beta^2\text{Re}^2|\bar{\eta}|^2\right) dy \\ &= E_V(\bar{t}) + \beta^2\text{Re}^2 E_\eta(\bar{t}) \end{aligned}$$

$$\begin{aligned} G(\bar{t}_{max}) &= \max \frac{E_V(\bar{t}_{max}) + \beta^2\text{Re}^2 E_\eta(\bar{t}_{max})}{E_V(0) + \beta^2\text{Re}^2 E_\eta(0)} \\ &\approx \beta^2\text{Re}^2 \max \frac{E_\eta(\bar{t}_{max})}{E_V(0)} \end{aligned}$$

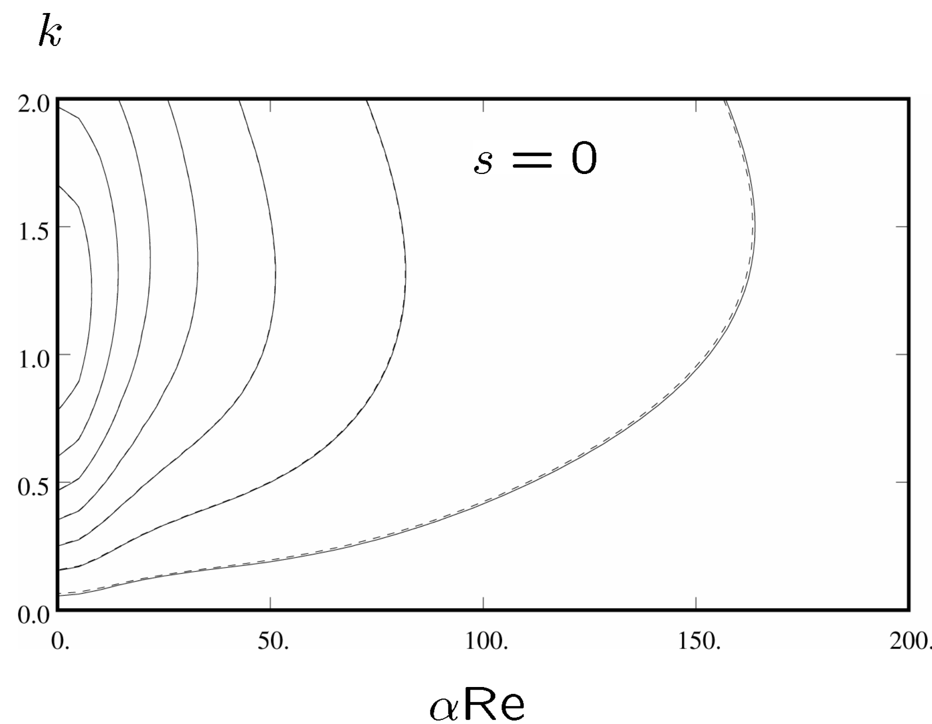
Re-dependence of Gmax for PPF

$$\frac{k^2 G(\bar{t}_{max})}{\beta^2 \mathrm{Re}^2}$$ depends only on k and αRe



Re-dependence of Rmax for PCF

$$\begin{aligned}(s\mathbf{I}-\mathbf{L})^{-1} &= \int_0^\infty e^{-st}\,\hat{\mathbf{q}}(t)dt \\ &= \int_0^\infty e^{-\bar{s}\bar{t}}\,\bar{\mathbf{q}}(\bar{t})\mathrm{Re}\,d\bar{t} \\ &= \mathrm{Re}(\bar{s}\mathbf{I}-\bar{\mathbf{L}})^{-1}\end{aligned}$$



$$\frac{k^2\|(s\mathbf{I}-\mathbf{L})^{-1}\|^2}{\beta^2\mathrm{Re}^4}$$ depends only on k and $\alpha\mathrm{Re}$

Re-dependence of Gmax and Rmax

Flow	$G_{\text{max}} \quad (10^{-3})$	t_{max}	α	β
plane Poiseuille	$0.20\,\text{Re}^2$	$0.076\,\text{Re}$	0	2.04
plane Couette	$1.18\,\text{Re}^2$	$0.117\,\text{Re}$	$35/\text{Re}$	1.6
circular pipe	$0.07\,\text{Re}^2$	$0.048\,\text{Re}$	0	1
Blasius boundary layer	$1.50\,\text{Re}^2$	$0.778\,\text{Re}$	0	0.65

Flow	$\max_{\omega \in \mathcal{R}} R(\omega)$	α	β
plane Poiseuille	$(\text{Re}/17.4)^2$	0	1.62
plane Couette	$(\text{Re}/8.12)^2$	0	1.18
Blasius boundary layer	$(\text{Re}/1.83)^2$	0	0.21