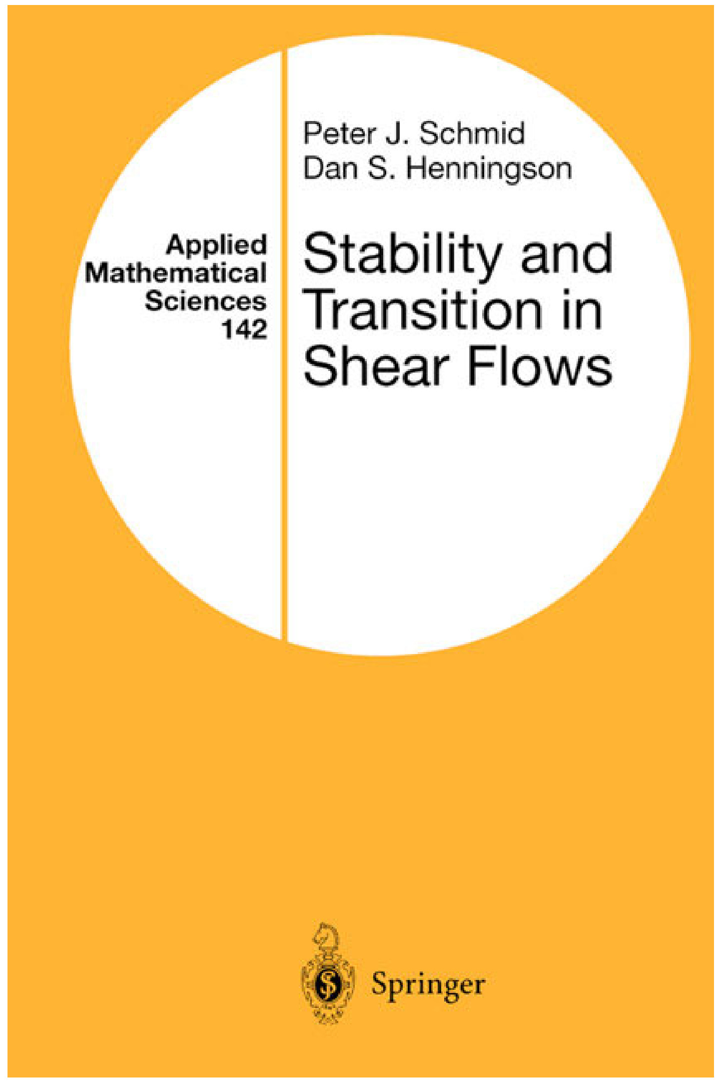
# Nonlinear Interactions and Transition



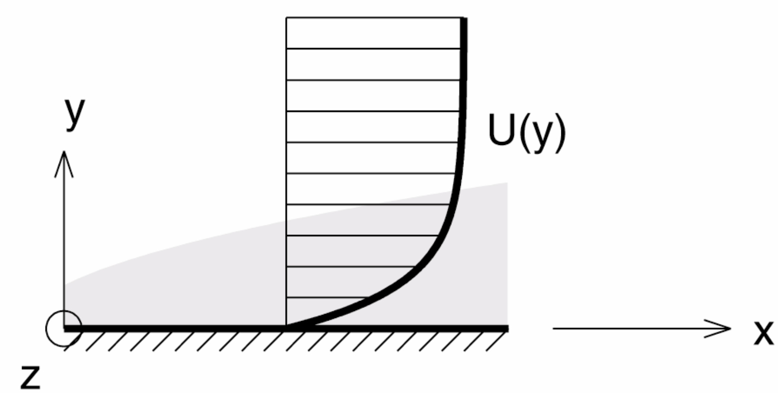
# Non-linear disturbance equations

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\text{Re}} \nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0 \quad \text{on solid boundaries}$$




$$\text{Re} = U_\infty \delta_* / \nu$$

$$u_i = U_i + u_i'$$

$$p = P + p' \quad \text{drop primes}$$

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\text{Re}} \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

# Stability definitions

$$E_V = \frac{1}{2}\int_V u_i u_i \; dV.$$

*Stable:* $\lim_{t\to\infty} \frac{E_V(t)}{E_V(0)} \to 0$

*Conditionally stable:* $\exists \;\; \delta : E(0) < \delta \;\Rightarrow$ stable

*Globally stable:* Conditionally stable with $\delta \to \infty$

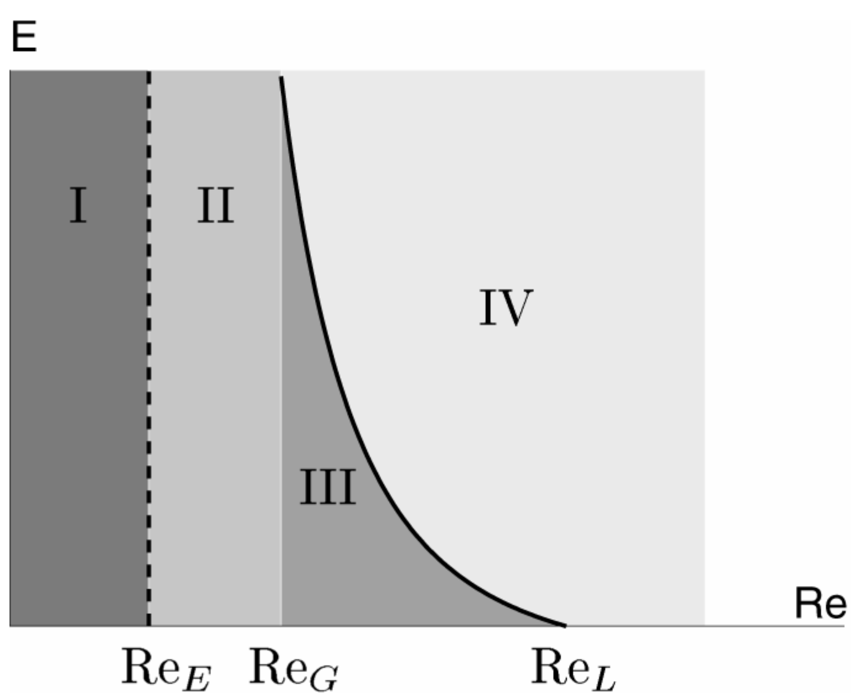
*Monotonically stable* $\frac{dE}{dt} \leq 0 \;\forall\;\; t > 0$

# Critical Reynolds numbers

$Re_E$ : $Re < Re_E$ flow monotonically stable

$Re_G$ : $Re < Re_G$ flow globally stable

$Re_L$ : $Re > Re_L$ flow linearly unstable ($\delta \to 0$)



# Reynolds-Orr equation

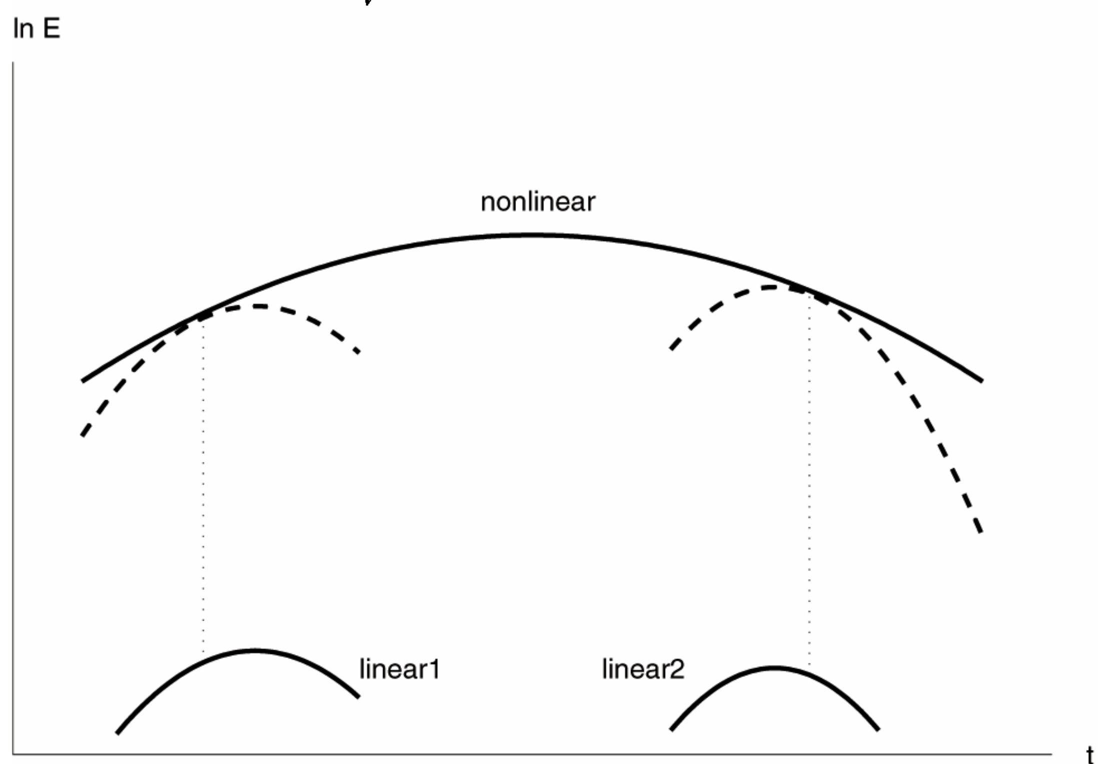
$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{\mathrm{Re}} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j} \left[ -\frac{1}{2} u_i u_i U_j - \frac{1}{2} u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{\mathrm{Re}} u_i \frac{\partial u_i}{\partial x_j} \right] \\
&\Rightarrow \\
\frac{dE_V}{dt} &= -\int_V u_i u_j \frac{\partial U_i}{\partial x_j} dV - \frac{1}{\mathrm{Re}} \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV
\end{aligned}
$$

*Theorem:* Linear mechanisms required for energy growth

*Proof:* $\frac{1}{E_V} \frac{dE_V}{dt}$ independent of disturbance amplitude

# Linear growth mechanisms

$$\frac{1}{E_V}\frac{dE_V}{dt} = \frac{d}{dt}\ln E_V$$



# Quadratic non-linear interactions

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = -u\frac{\partial u}{\partial x} \equiv -\frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

$$u = \sum_{k=-\infty}^{\infty} a_k(t)e^{ik\alpha x}$$

$$\sum_{k=-\infty}^{\infty}\left[\frac{da_k}{dt} + ik\alpha U a_k + \nu k^2\alpha^2 a_k\right] \; e^{ik\alpha x}$$

$$= \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} i(m+n)\alpha\left[a_m(t)\;a_n(t)\right]\; e^{i(m+n)\alpha x}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} i\alpha k \sum_{m+n=k}\left[a_m(t)\;a_n(t)\right]\; e^{ik\alpha x}$$

$$\frac{da_k}{dt} + ik\alpha U a_k + \nu k^2\alpha^2 a_k \; = \; \frac{1}{2}ik\alpha \sum_{m+n=k} a_m \; a_n$$

# Non-linear v-eta formulation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4 v = \underbrace{-\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)S_2 - \frac{\partial^2 S_1}{\partial x\,\partial y} - \frac{\partial^2 S_3}{\partial y\,\partial z}\right]}_{N_v}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta + U'\frac{\partial v}{\partial z} - \frac{1}{\mathrm{Re}}\nabla^2 \eta = \underbrace{-\left(\frac{\partial S_1}{\partial z} - \frac{\partial S_3}{\partial x}\right)}_{N_\eta}$$

$$S_1 = \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}$$

$$S_2 = \frac{\partial(uv)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(vw)}{\partial z}$$

$$S_3 = \frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y} + \frac{\partial(w^2)}{\partial z}$$

# Fourier-transformed equations

$$v = \sum_m \sum_n \hat{v}_{mn}(y,t) e^{i\alpha_m x + i\beta_n z}$$

$$\frac{\partial}{\partial t} \underbrace{\begin{pmatrix} -D_{mn}^2 + k_{mn}^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}_{mn}} \begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix} - \underbrace{\begin{pmatrix} \mathcal{L}_{OS}^{mn} & 0 \\ -i\beta_n U' & \mathcal{L}_{SQ}^{mn} \end{pmatrix}}_{\mathbf{L}_{mn}} \underbrace{\begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix}}_{\hat{\mathbf{q}}_{mn}} = \sum_{k+p=m} \sum_{l+q=n} \underbrace{\begin{pmatrix} \hat{N}_v^{mn} \\ \hat{N}_\eta^{mn} \end{pmatrix}}_{\mathbf{n}_{mn}}$$

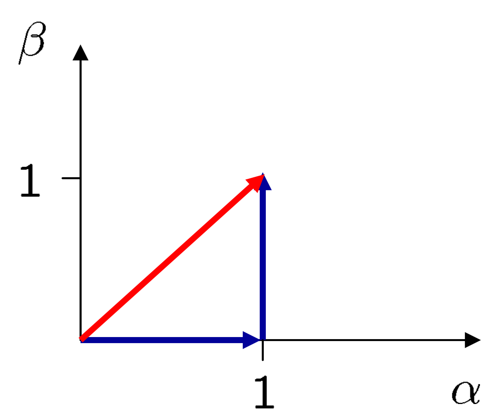
$$\left( \frac{\partial}{\partial t} \mathbf{M}_{mn} - \mathbf{L}_{mn} \right) \hat{\mathbf{q}}_{mn} = \sum_{k+p=m} \sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})$$

# Convolution sums and triad interactions

$$\left(\frac{\partial}{\partial t}\mathbf{M}_{mn} - \mathbf{L}_{mn}\right)\hat{\mathbf{q}}_{mn} = \sum_{k+p=m}\sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})$$

non-linear interactions by $\hat{\mathbf{q}}_{kl}$ and $\hat{\mathbf{q}}_{pq}$ contributes to $\hat{\mathbf{q}}_{mn}$ if

$$(\alpha_m, \beta_n) = (\alpha_k, \beta_l) + (\alpha_p, \beta_q)$$




$$(1,1) = (1,0) + (0,1)$$

# Example

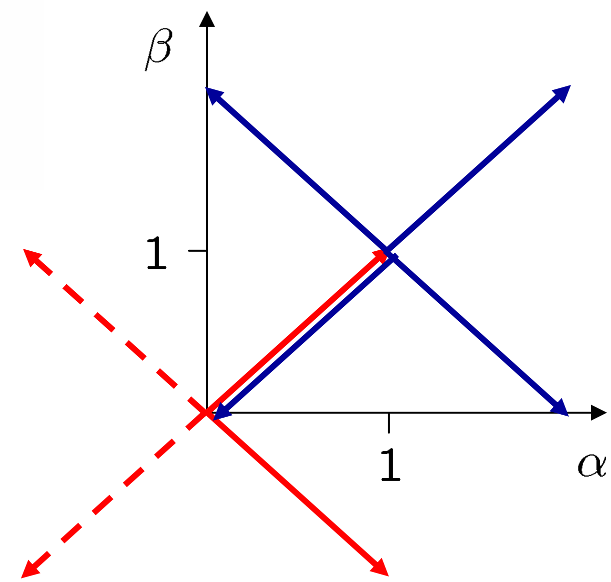
- interaction possible with $(1,1)$ and $(1,-1)$
- real solution vector $\widehat{v}_{mn}^{*} = \widehat{v}_{-m,-n}$
- spanwise symmetry $\widehat{v}_{mn} = \widehat{v}_{m,-n}$

$$(2,2) = (1,1) + (1,1)$$

$$(2,0) = (1,1) + (1,-1)$$

$$(0,2) = (1,1) + (-1,1)$$

$$(0,0) = (1,1) + (-1,-1)$$

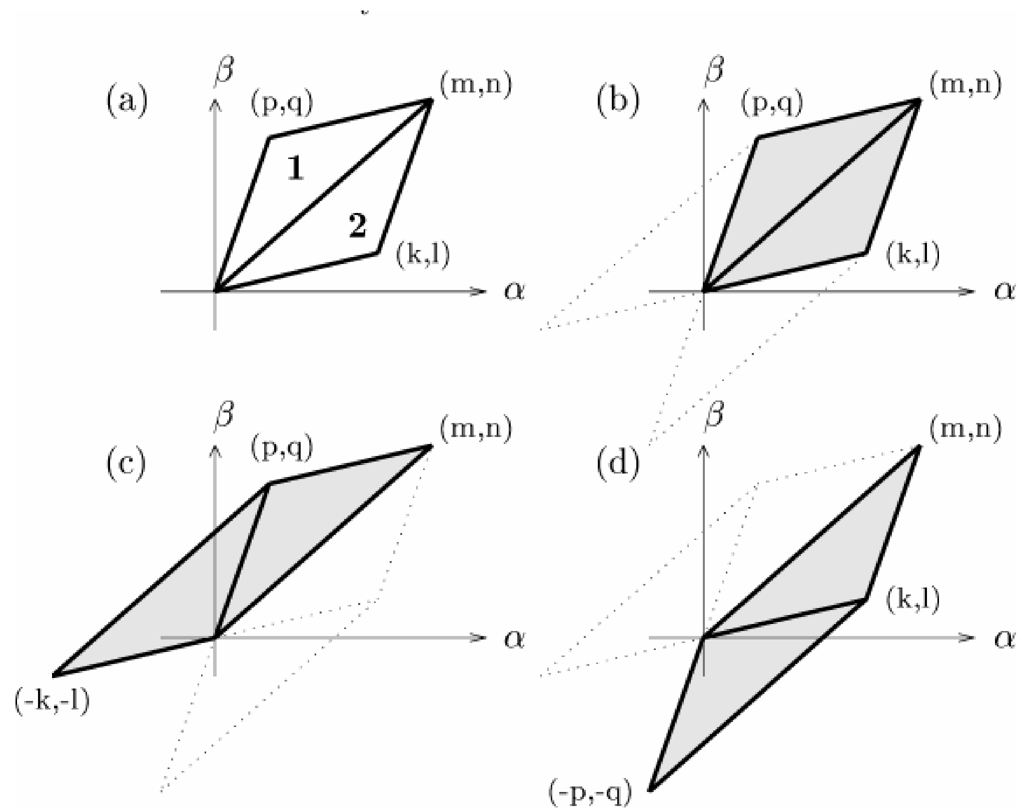


# Rate of change of energy

$$E_{mn} = \frac{1}{2k_{mn}^2}\int_y \mathbf{q}_{mn}^* \mathbf{M}_{mn}\mathbf{q}_{mn}\,dy$$

$$\begin{aligned}\frac{d}{dt}E_{mn} &= \frac{1}{k_{mn}^2}\Re\left\{\int_y \frac{\partial \mathbf{q}_{mn}^*}{\partial t}\mathbf{M}_{mn}\mathbf{q}_{mn}\,dy\right\} \\ &= \frac{1}{k_{mn}^2}\Re\left\{-\int_y \mathbf{q}_{mn}^*\mathbf{L}_{mn}\mathbf{q}_{mn}\,dy\right\} \\ &+ \frac{1}{k_{mn}^2}\Re\left\{\underbrace{\sum_{k+p=m}\sum_{l+q=n}\int_y \mathbf{q}_{mn}^*\mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl},\hat{\mathbf{q}}_{pq})\,dy}_{\dot{E}([m,n],[p,q],[k,l])}\right\}\end{aligned}$$

# Conservation of energy in triads




$$T([m,n],[p,q],[k,l]) \equiv \dot{E}([m,n],[p,q],[k,l]) + \dot{E}([m,n],[k,l],[p,q])$$

$$T([m,n],[p,q],[k,l]) + T([p,q],[-k,-l],[m,n]) + T([k,l],[-p,-q],[m,n]) = 0$$

# Non-linear equilibrium states

$$\left(\frac{\partial}{\partial t}\mathbf{M}_{mn} - \mathbf{L}_{mn}\right)\hat{\mathbf{q}}_{mn} = \sum_{k+p=m}\sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{q}}_{kl}, \hat{\mathbf{q}}_{pq})$$

find stationary states moving with speed $C$

$$\mathbf{q}(x,y,z,t) = \epsilon \cdot \mathbf{r}(x',y,z) \qquad x' = x - Ct \qquad \Rightarrow$$

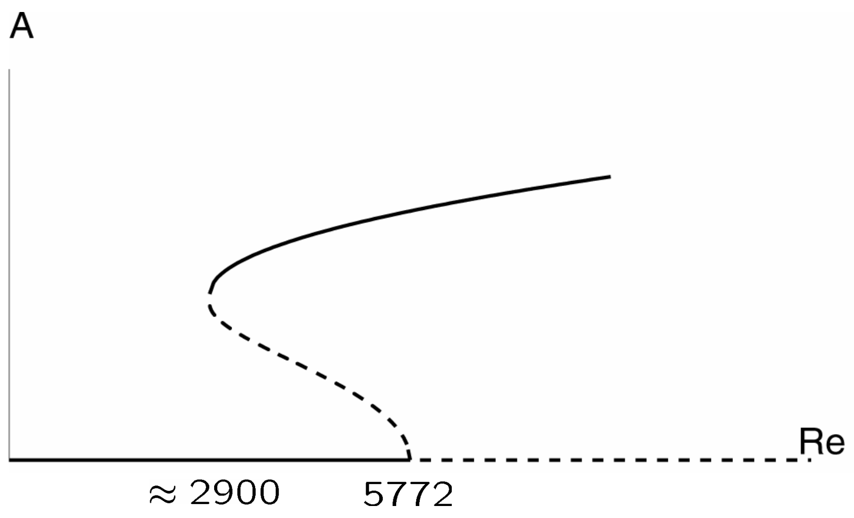
$$(i\alpha_m C\mathbf{M}_{mn} + \mathbf{L}_{mn})\,\hat{\mathbf{r}}_{mn} = -\epsilon \cdot \sum_{k+p=m}\sum_{l+q=n} \mathbf{n}_{mn}(\hat{\mathbf{r}}_{kl}, \hat{\mathbf{r}}_{pq})$$

non-linear eigenvalue problem $F(\mathbf{u}, \lambda) = 0$, with

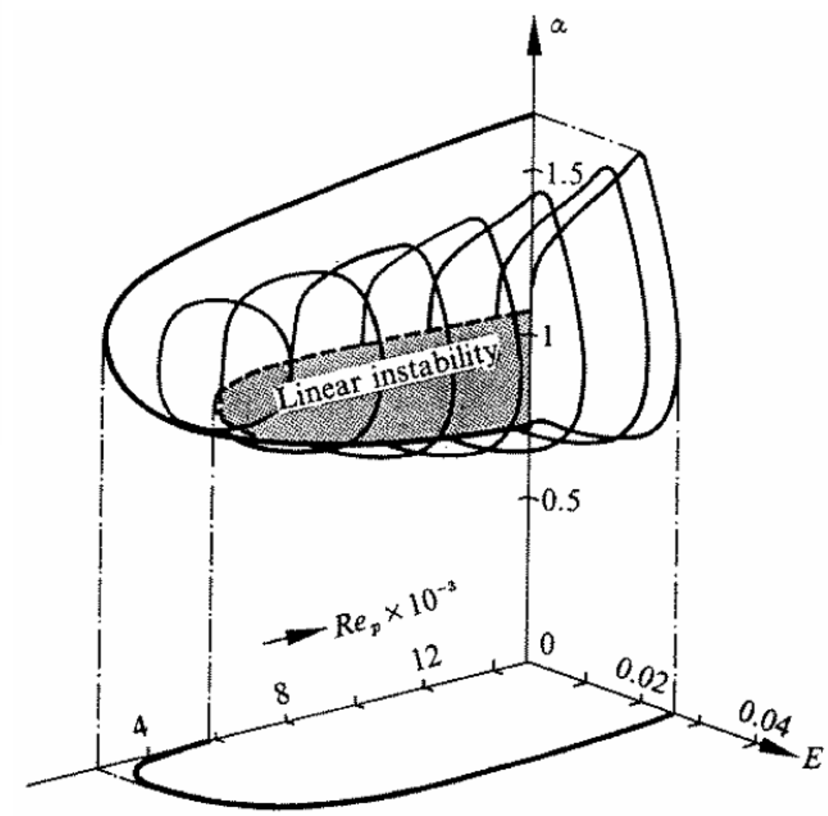
$$\mathbf{u} = \{\epsilon, C, [\hat{\mathbf{r}}_{mn}]|_{m=0,M;n=0,N}\}$$

$$\lambda = \{\alpha, \beta, \mathrm{Re}\}$$

# 2D finite amplitude states in PPF




subcritical bifurcation



# Secondary instability of 2D waves

3D bifurcation for 2D finite amplitude states

$$U_i = [U(y) - C]\delta_{i1} + A[u^{2D}(x', y)\delta_{i1} + v^{2D}(x', y)\delta_{i2}]$$

$$\frac{\partial u}{\partial t} + (U - C)\frac{\partial u}{\partial x'} + U'v + \frac{\partial p}{\partial x'} - \frac{1}{\mathrm{Re}}\nabla^2 u$$

$$= -A[\frac{\partial}{\partial x'}(uu^{2D}) + \frac{\partial}{\partial y}(vu^{2D} + uv^{2D}) - u^{2D}\frac{\partial w}{\partial z}]$$

$$\frac{\partial v}{\partial t} + (U - C)\frac{\partial v}{\partial x'} + \frac{\partial p}{\partial y} - \frac{1}{\mathrm{Re}}\nabla^2 v$$

$$= -A[\frac{\partial}{\partial x'}(uv^{2D} + vu^{2D}) + \frac{\partial}{\partial y}(vv^{2D}) - v^{2D}\frac{\partial w}{\partial z}]$$

$$\frac{\partial w}{\partial t} + (U - C)\frac{\partial w}{\partial x'} + \frac{\partial p}{\partial z} - \frac{1}{\mathrm{Re}}\nabla^2 w = 0$$

# Form of the solution

- Shape assumption $u^{2D}(x', y) = \tilde{u}_{TS}(y)e^{i\alpha x'} + \tilde{u}^*_{TS}(y)e^{-i\alpha x'}$

- Floquet theory for PDEs with periodic coefficients

$$u(x', y, z, t) = \hat{u}(x', y)\ e^{\gamma x'}\ e^{\sigma t}\ e^{i\beta z}$$

- Temporal instability $\gamma_r = 0, \quad \sigma_r \neq 0$

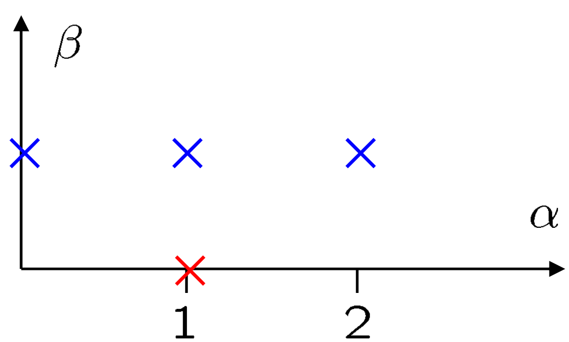
- Expand in Fourier series

$$u(x', y, z, t) = e^{\sigma_r t} e^{i\beta z} \sum_m \tilde{u}_m(y) e^{i(m\alpha + \gamma_i)x'}$$

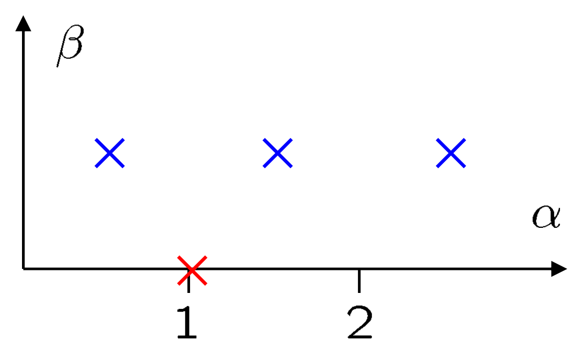
# Classification of modes

$$u(x', y, z, t) = e^{\sigma_r t} e^{i\beta z} \sum_m \tilde{u}_m(y) e^{i(m\alpha + \gamma_i)x'}$$

$\gamma_i = 0$ fundamental instability




$\gamma_i = \alpha/2$ subharmonic instability



# Secondary instability equations

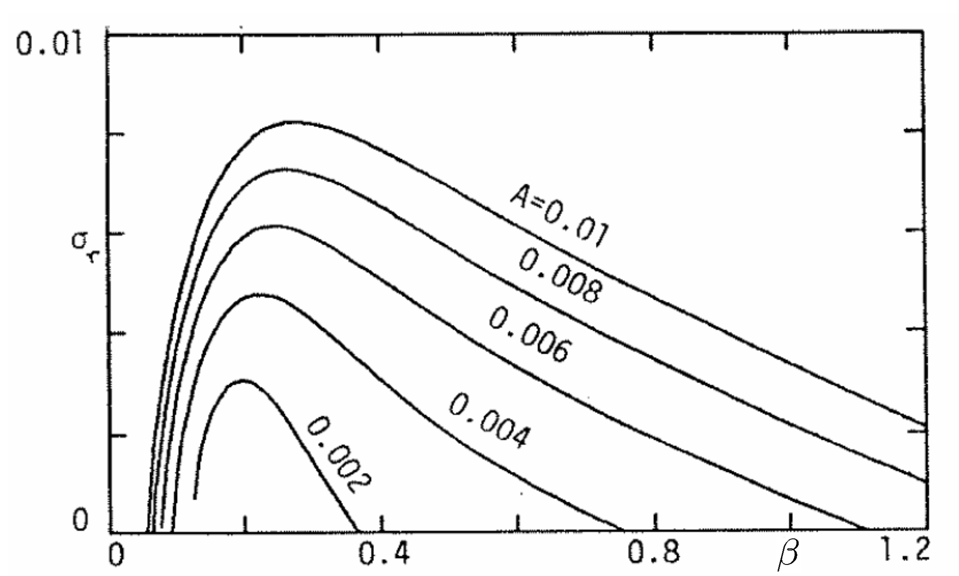
$$
\begin{aligned}
\sigma\tilde{u}_m + i\alpha_m(U-C)\tilde{u}_m + \tilde{v}_m U' + i\alpha_m\tilde{p} - \frac{1}{\mathrm{Re}}(D^2-k_m^2)\tilde{u}_m &= N_v \\
\sigma\tilde{v}_m + i\alpha_m(U-C)\tilde{v}_m + D\tilde{p} - \frac{1}{\mathrm{Re}}(D^2-k_m^2)\tilde{v}_m &= N_v \\
\sigma\tilde{w}_m + i\alpha_m(U-C)\tilde{w}_m + i\beta\tilde{p} - \frac{1}{\mathrm{Re}}(D^2-k_m^2)\tilde{w}_m &= 0 \\
i\alpha_m\tilde{u}_m + D\tilde{v}_m + i\beta\tilde{w}_m &= 0
\end{aligned}
$$

$$
\begin{aligned}
N_u &= -A\left[i\alpha_m\tilde{u}_{m\pm1}\tilde{u}^{TS} + D(\tilde{v}_{m\pm1}\tilde{u}^{TS} + \tilde{u}_{m\pm1}\tilde{v}^{TS}) - i\beta\tilde{w}_{m\pm1}\tilde{u}^{TS}\right] \\
N_v &= -A\left[i\alpha_m(\tilde{u}_{m\pm1}\tilde{v}^{TS} + \tilde{v}_{m\pm1}\tilde{u}^{TS}) + D(v_{m\pm1}\tilde{v}^{TS}) - i\beta\tilde{w}_{m\pm1}\tilde{v}^{TS}\right]
\end{aligned}
$$

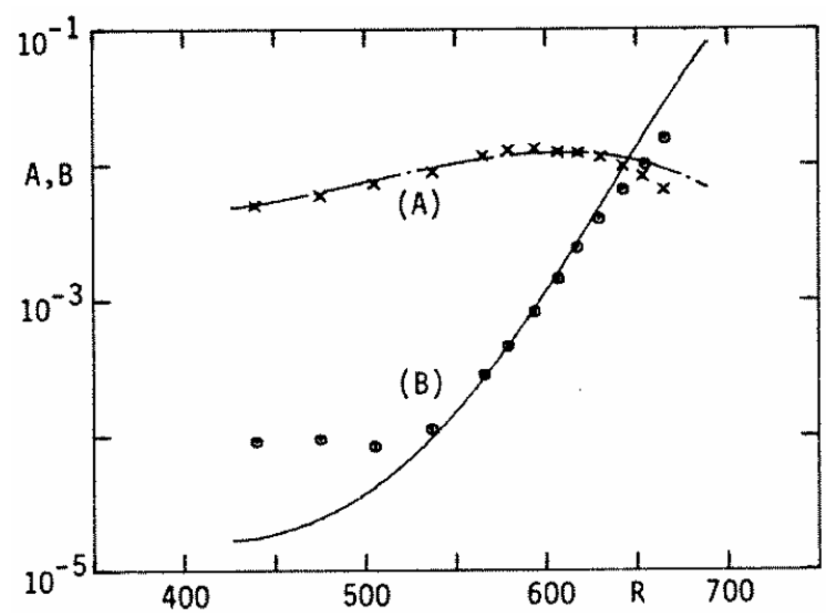
# Secondary instability of 2D TS waves

Subharmonic secondary instability most unstable:

dependence on amplitude

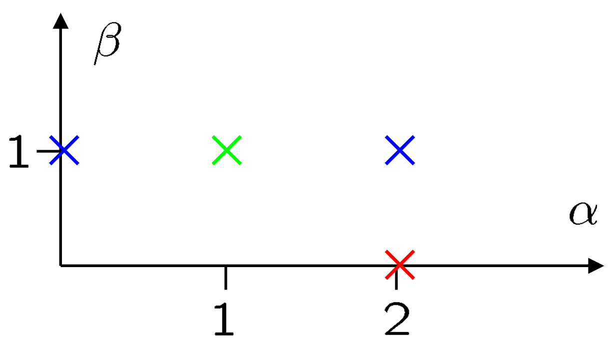



comparisons with experiments



# Transition to turbulence: 3 scenarios

2nd instability of TS-waves

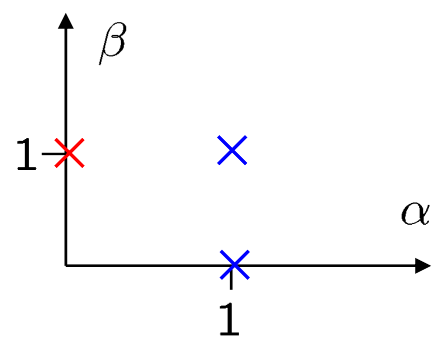



× TS-wave

× subharmonic mode

× fundamental mode

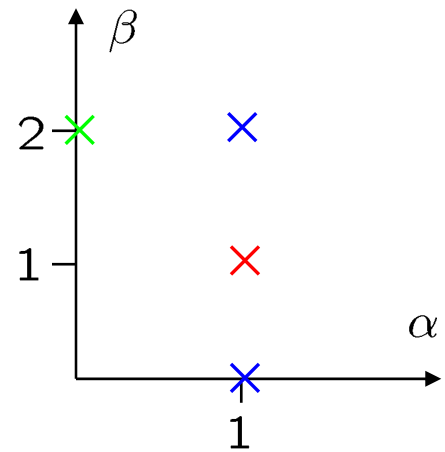
Streak breakdown




× streak

× fundamental mode

Oblique transition

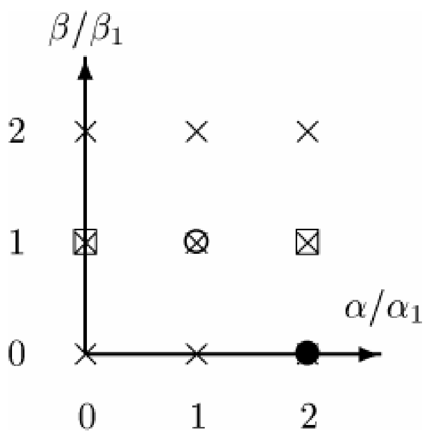


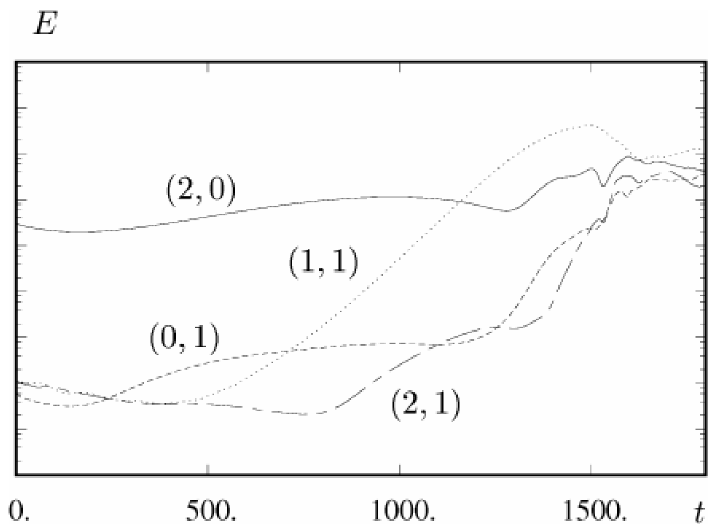

× oblique mode

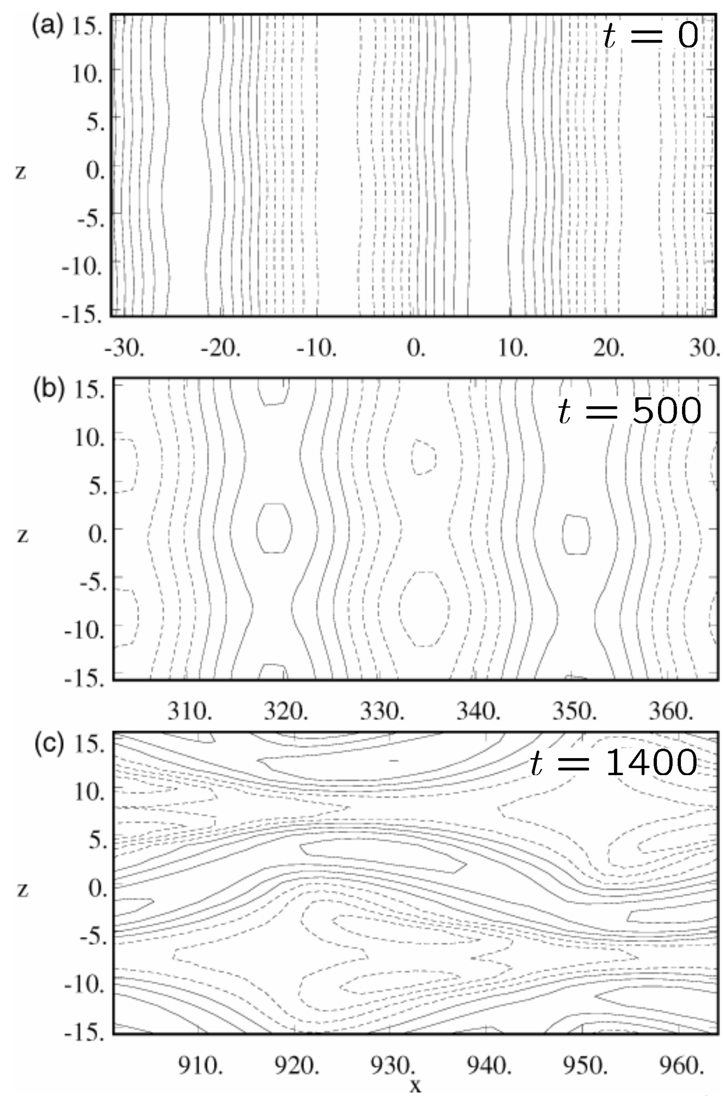
× induced streak

× fundamental mode

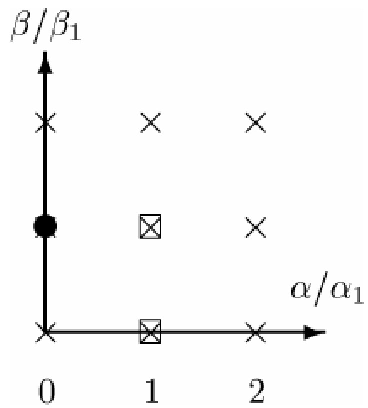
# Secondary instability of TS-waves

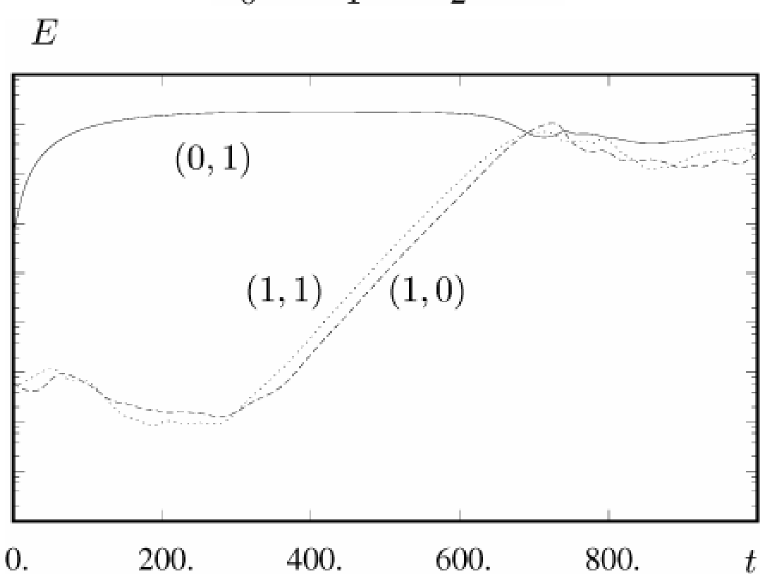


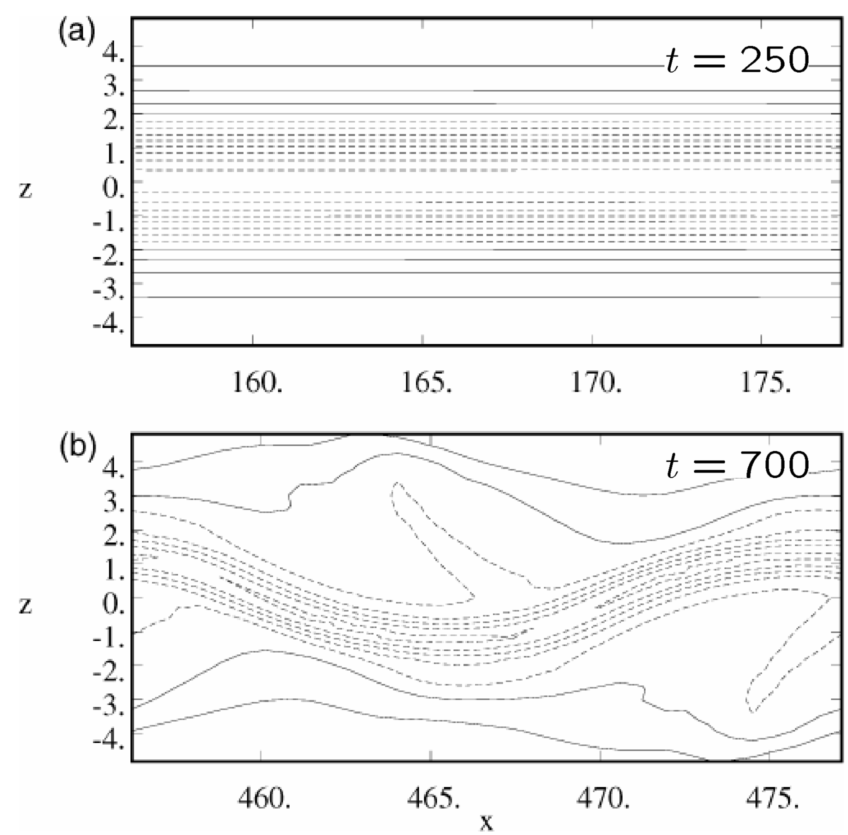



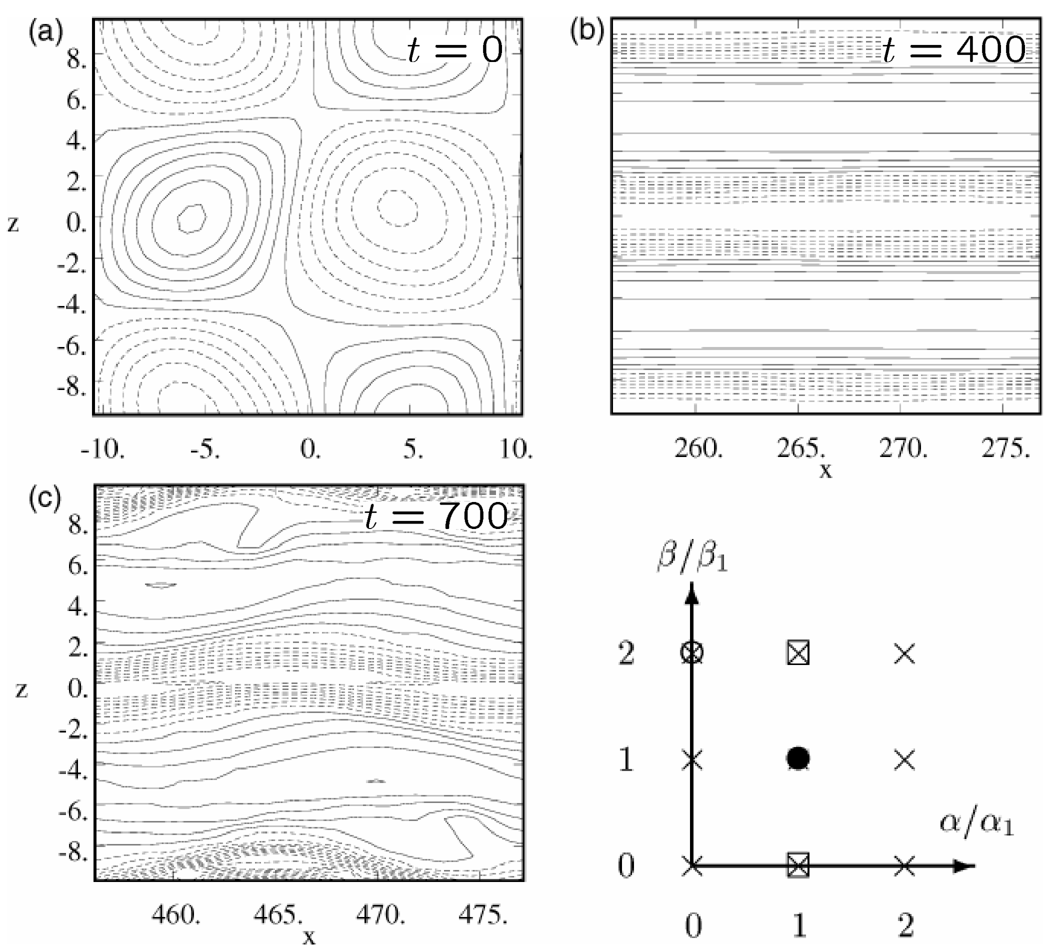
# Streak breakdown

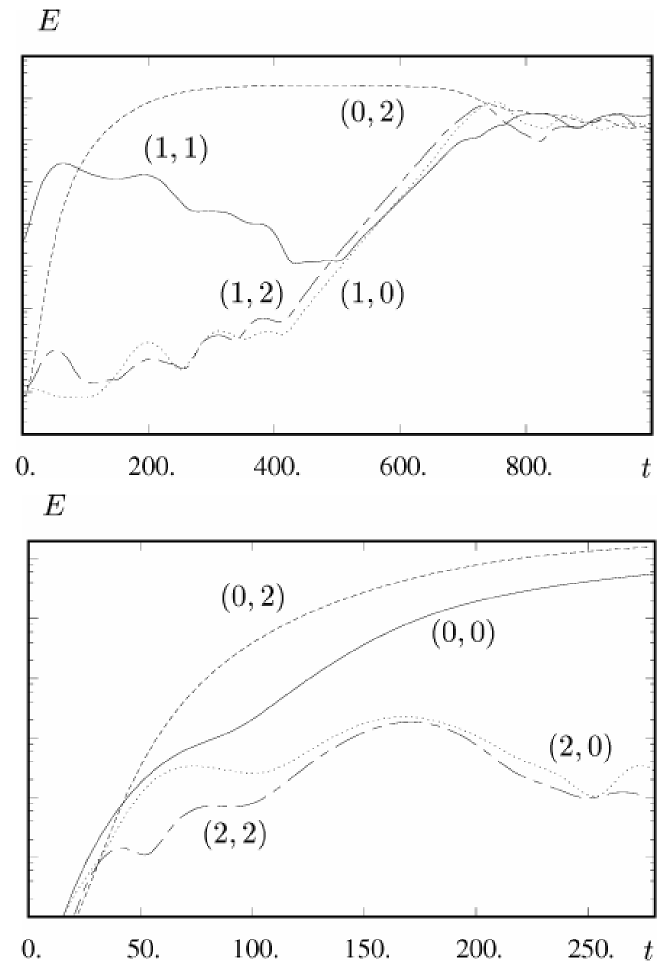


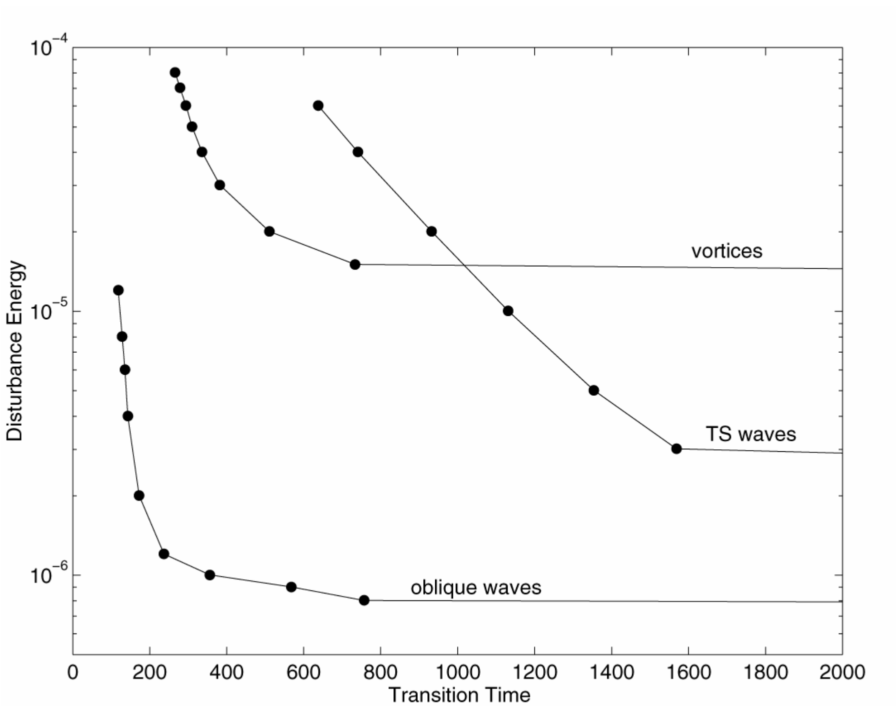
# Oblique transition



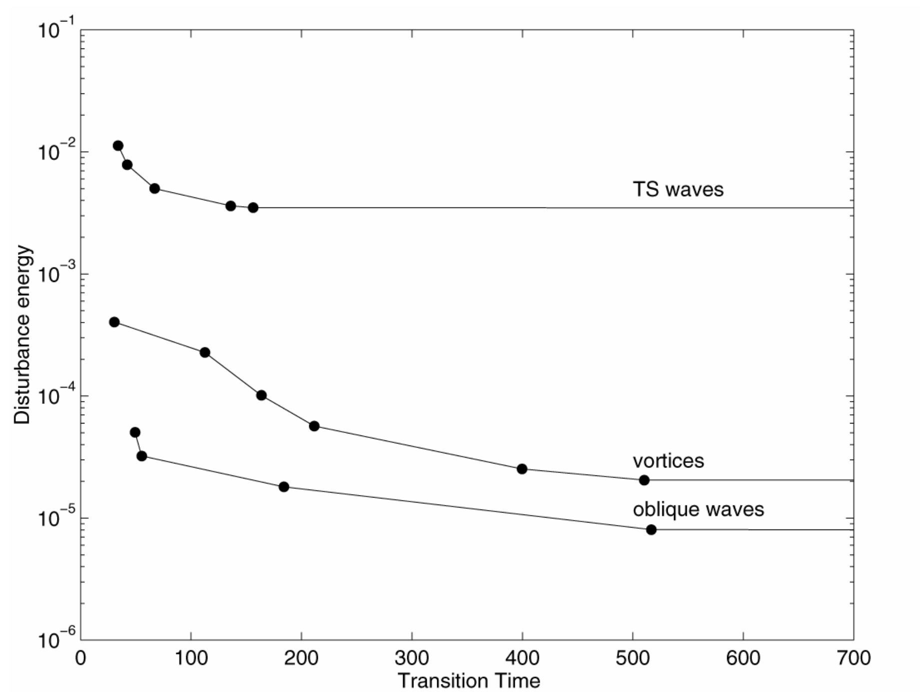



# Transition times vs disturbance energy

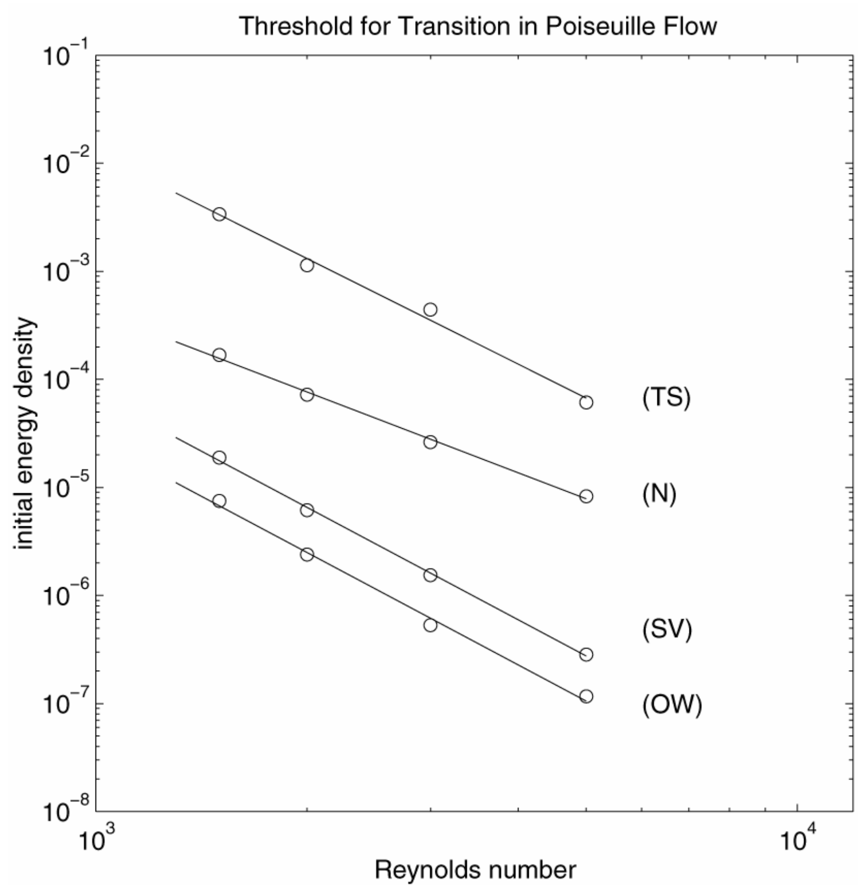
## Blasius boundary layer



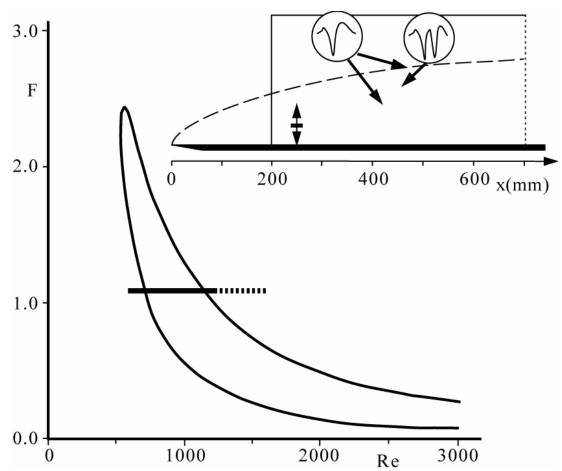

## Plane Poiseuille flow

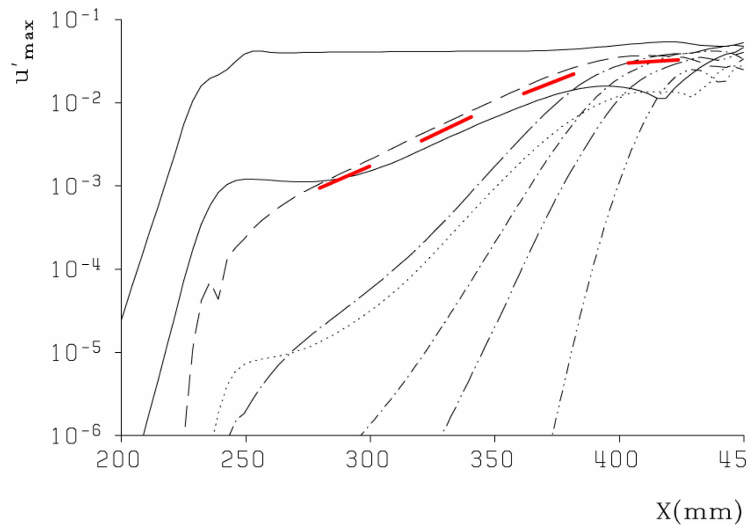


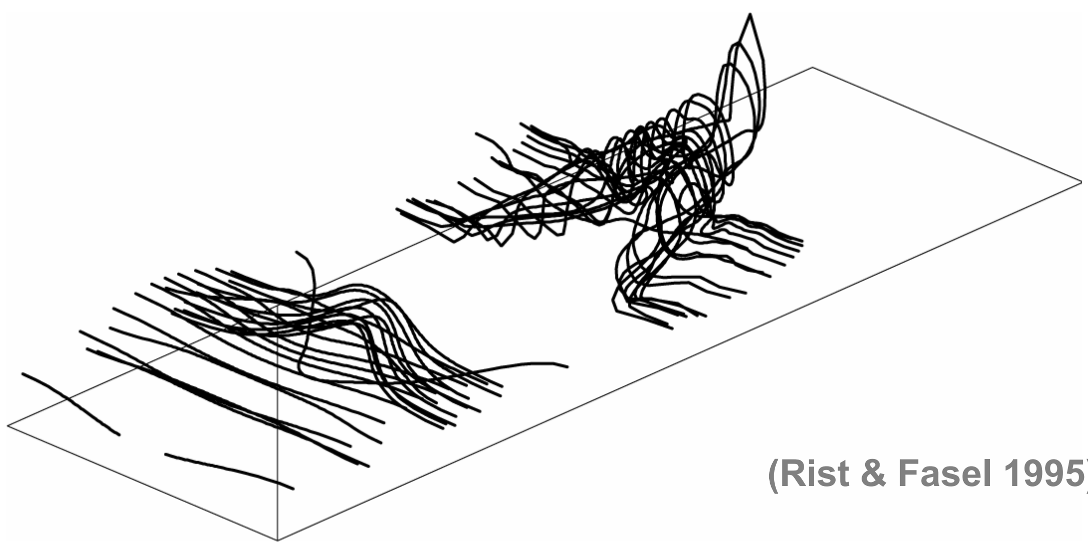
# Threshold for transition in PPF



# DNS of  becondary breakdown of TS-waves

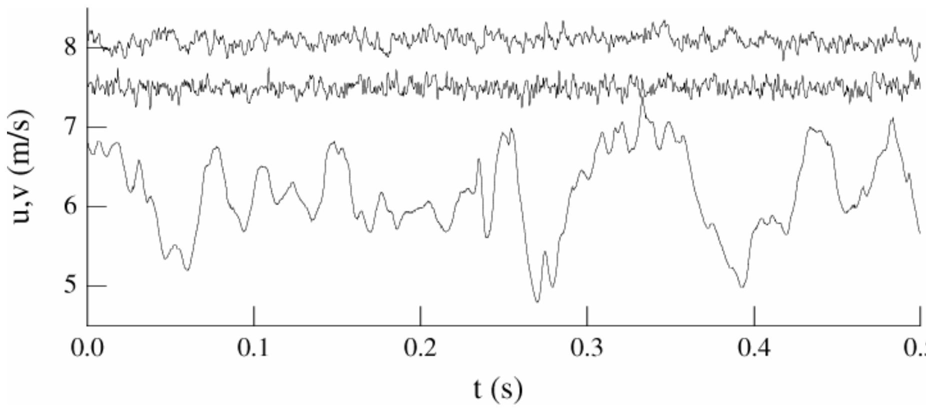


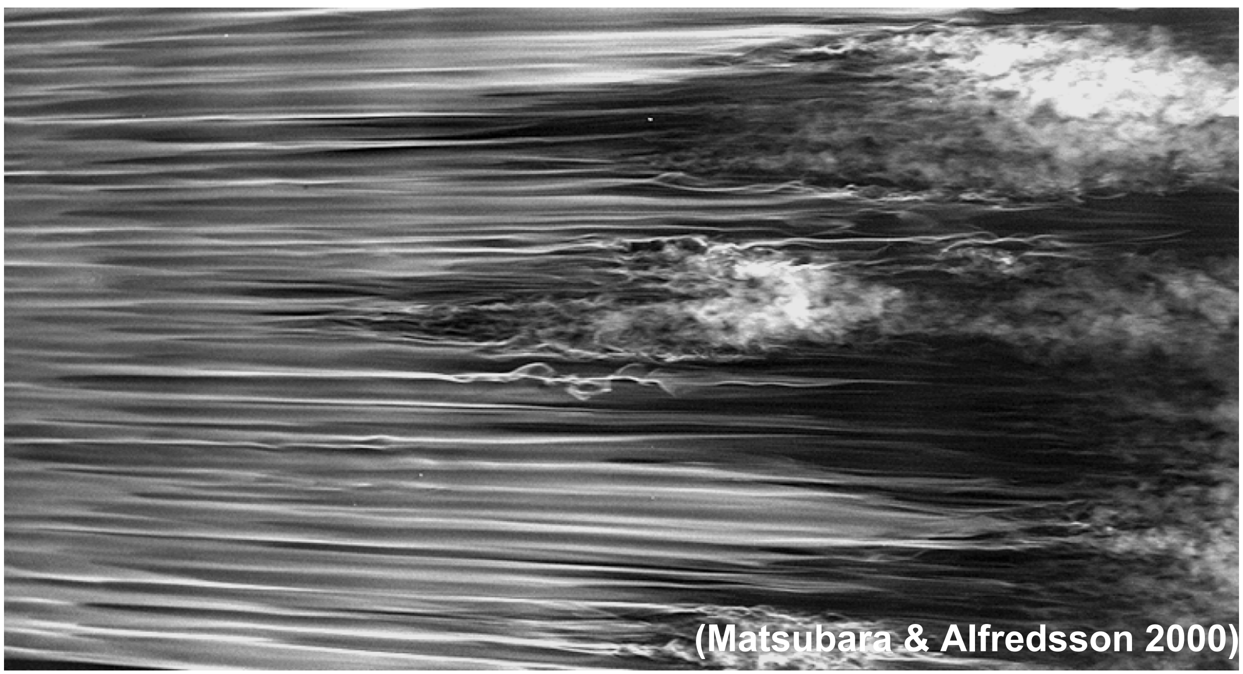



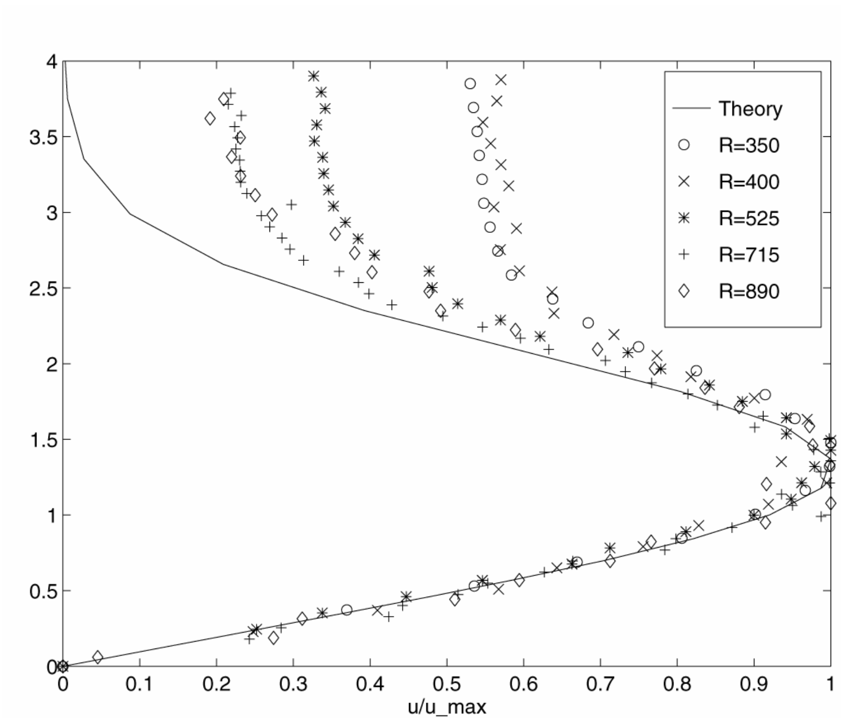
(Rist & Fasel 1995)

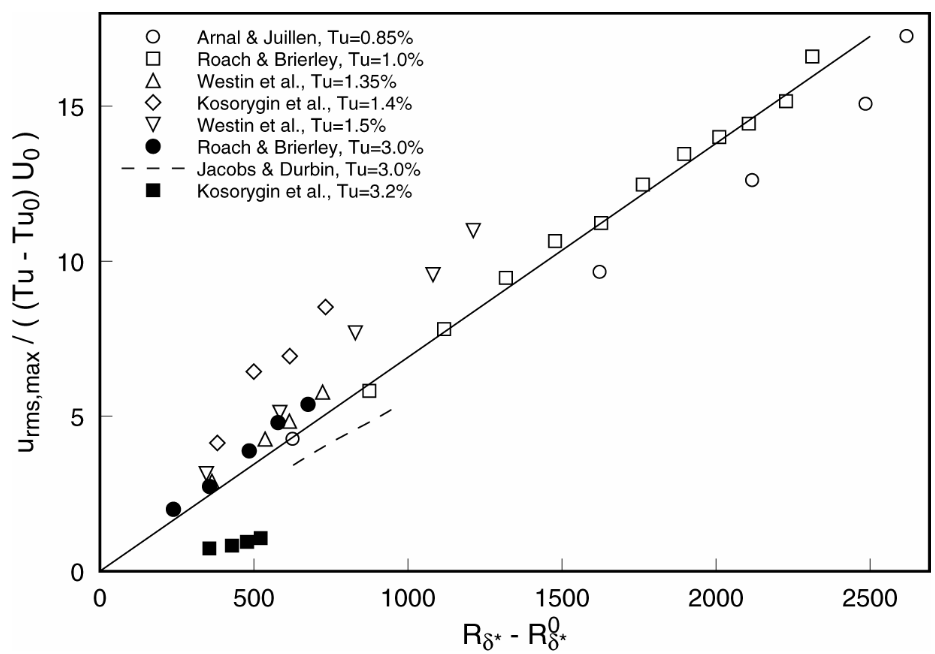
# Free-stream turbulence and streak breakdown



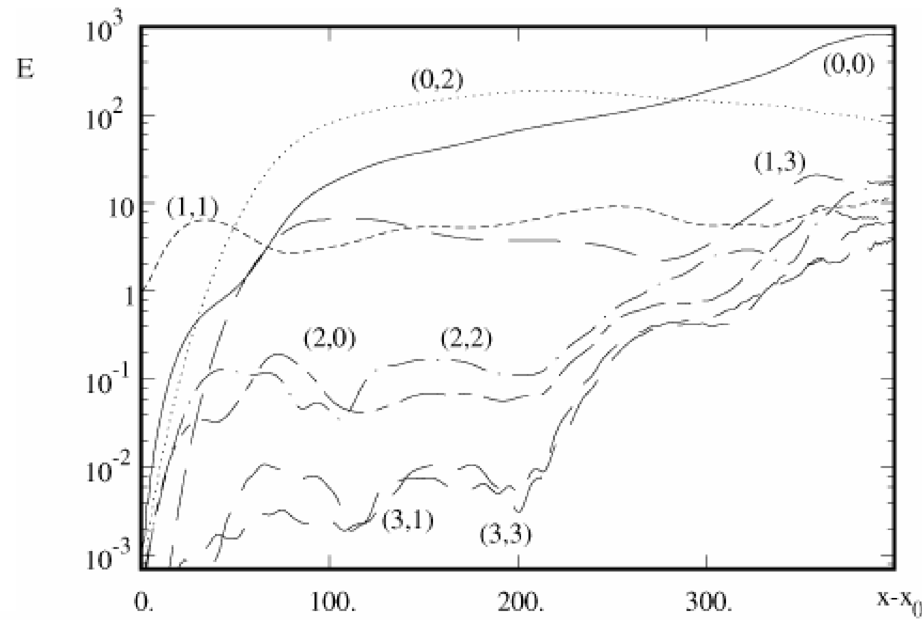



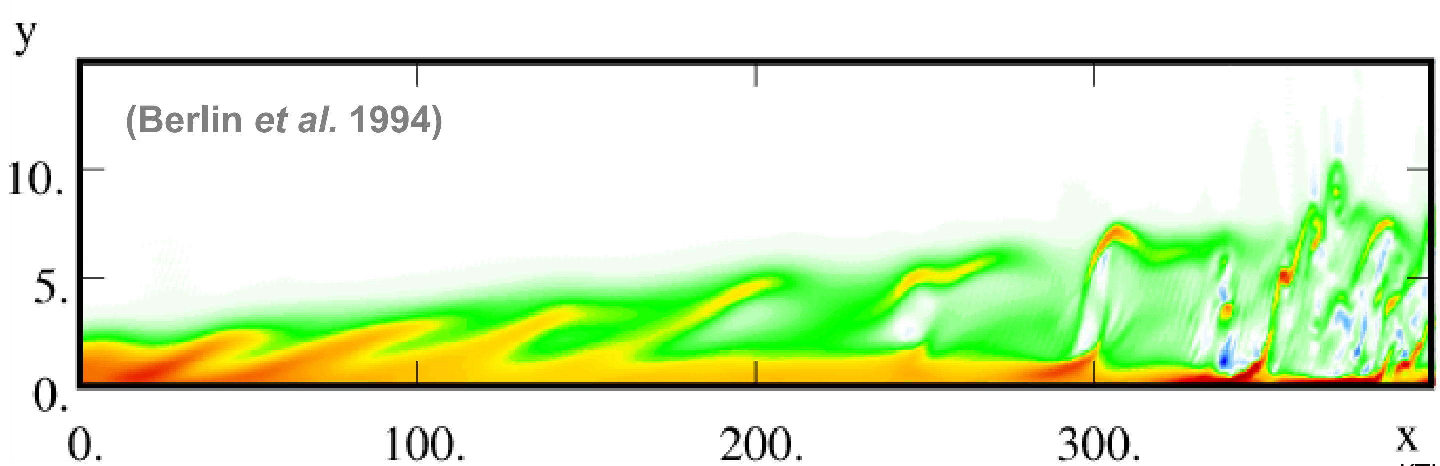
# Klebanoff mode and optimal growth



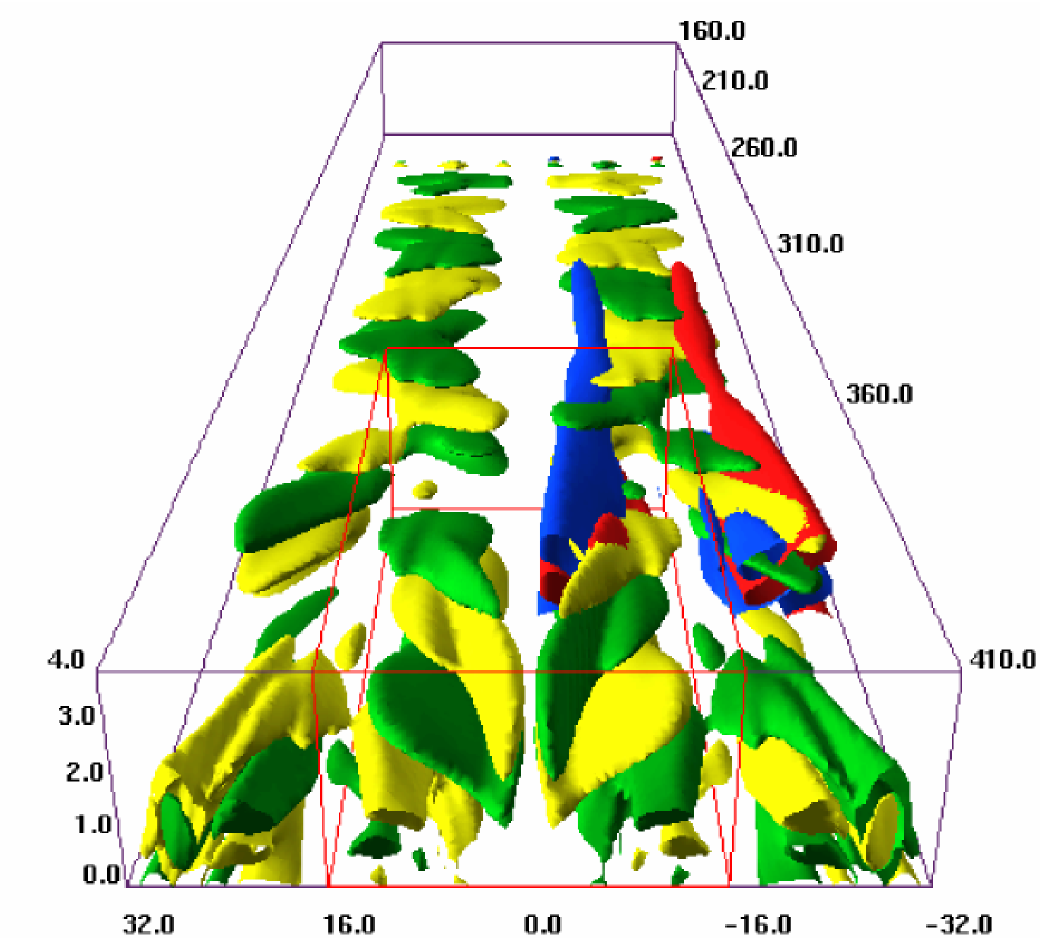



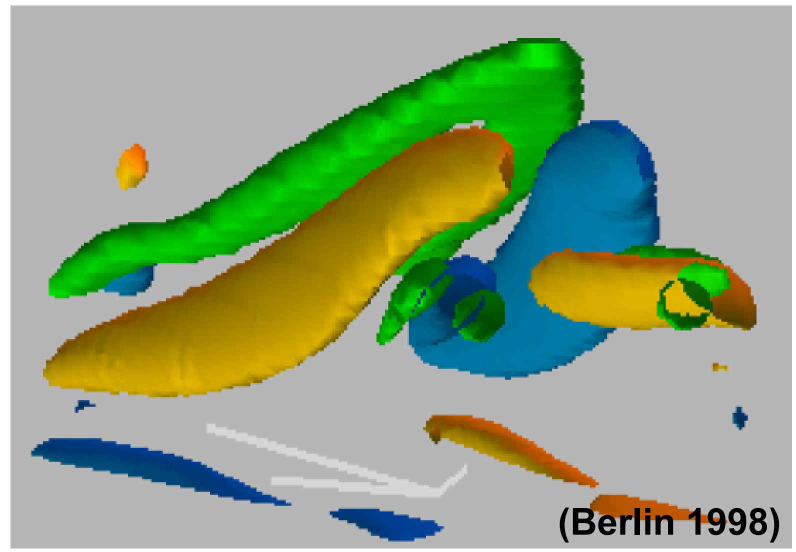
# DNS of oblique transition



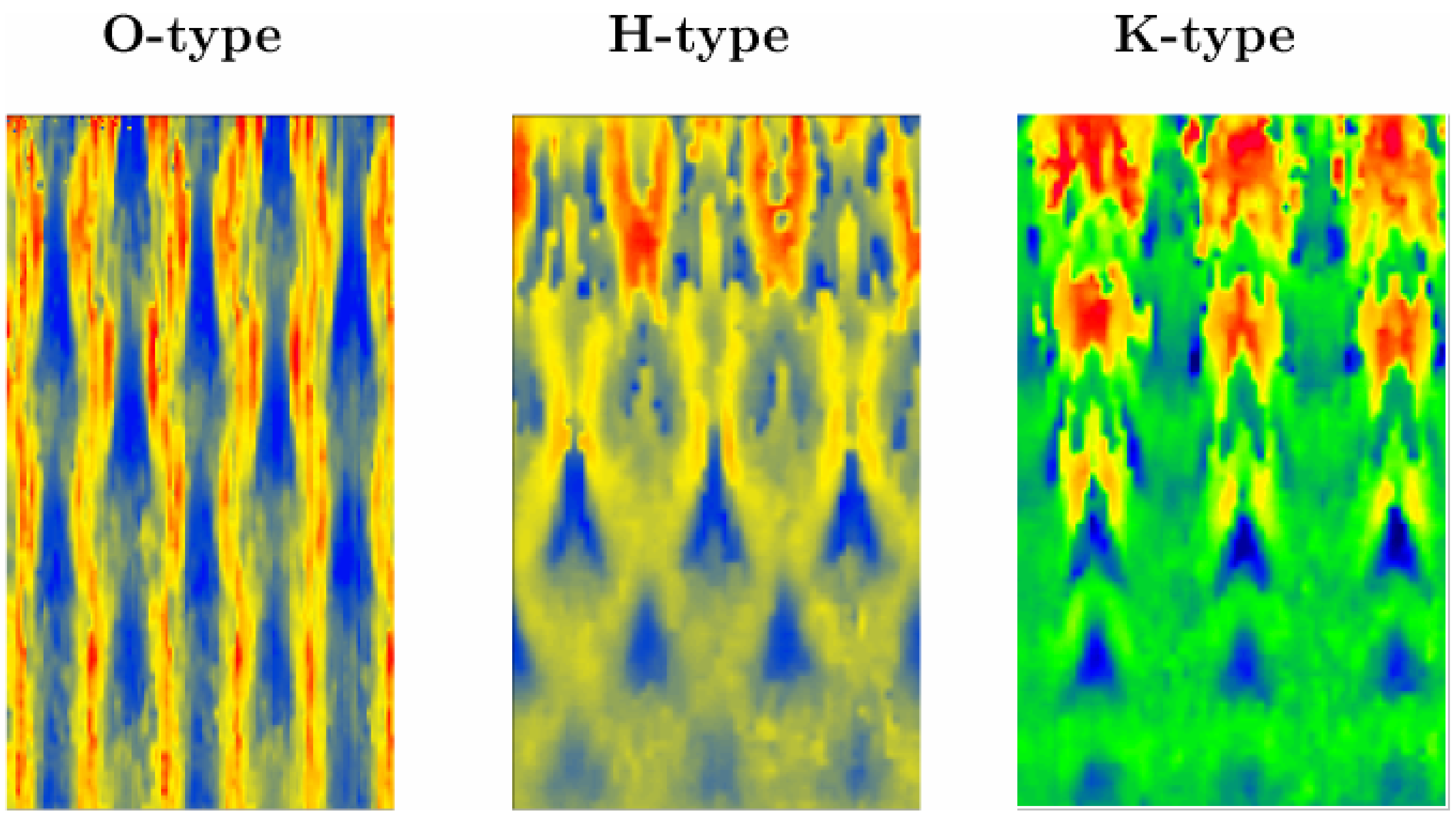



# DNS of Wiegel experiment

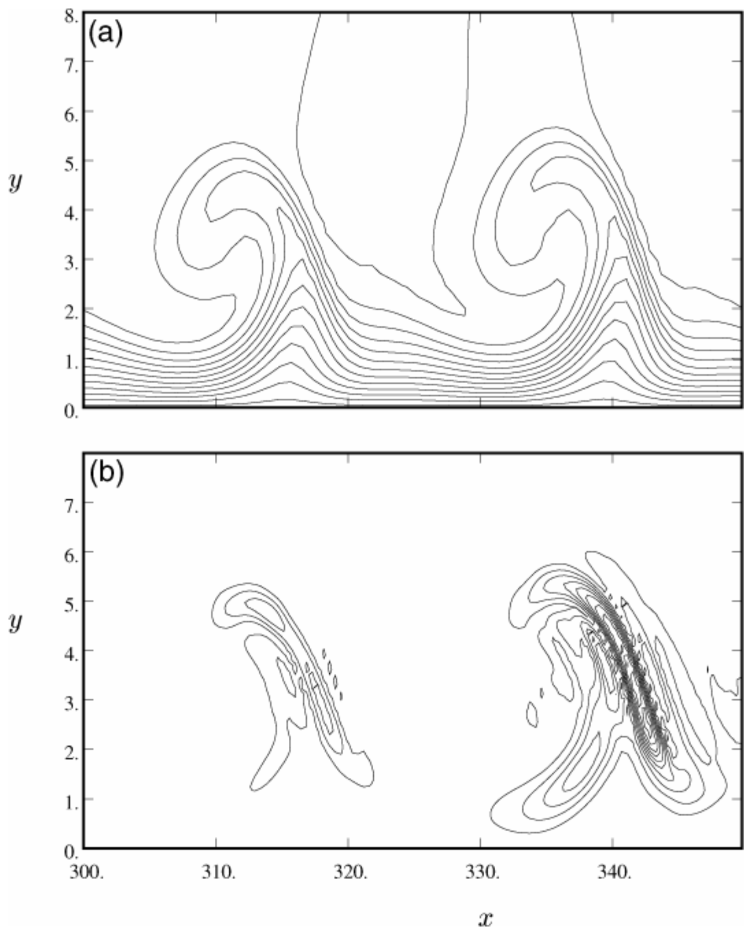


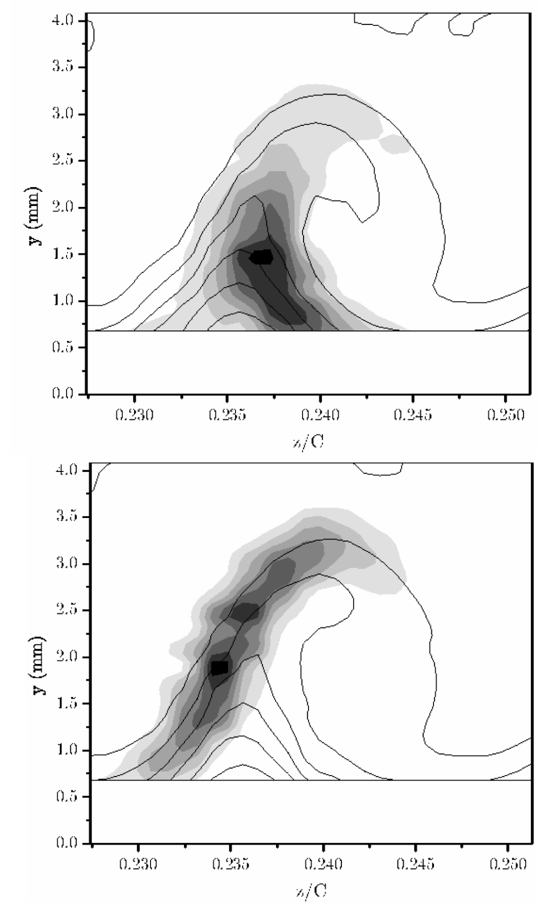
# Transition types in Wiegel experiment



# Secondary instability of cross-flow vortices




(Högberg & Henningson 1998)




(Kawakami *et al.* 1999)

# Lerche experiment of CF-vortices