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Semi-Relativistic Two-Body States of Spinless Particles with a Scalar-Type Interaction Potential

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Abstract *A semi-relativistic quantum approximation for mutual scalar interaction potentials is outlined and discussed. Equations are consistent with two-body Dirac equations for bound states of zero total angular momentum. Two-body effects near the non-relativistic limit for a linear scalar potential is studied in some detail.*

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1 Introduction

Relativistic two-body effects may be studied using quantum field equations,[1] two-body Dirac equations,[2−6] and/or by equations resulting from direct quantum-operator substitutions of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$.[7−10] The latter approach explored in this study ignores spins other than orbital angular momenta. Many relevant references can be found in those cited.

Semay *et al.*[2] and Ferreira[3] showed explicit results from a 16-components Dirac approach for scalar potentials of the confining type and bound states with vanishing total spin. The main interest of these authors is related to quark spectra. The relevant second-order differential equations obtained are simple and provide some understanding of important two-body effects.

Duviryak (2008),[4] also applying a Dirac-type method, presented solvable two-body models in connection with light mesons and Regge trajectories. No explicit results for scalar potentials are given. However, the general results seem to be relevant in the present context.

Moshinsky and Requer (2003)[6] studied two equal fermionic masses in the context of positronium formations. It seems close to other procedures related to sub-atomic interactions. No explicit results for scalar potentials are given.

In the present study the semi-relativistic approach[8−10] is applied with scalar (mass-type) potentials. In addition a "local-momentum" approximation is suggested to find the Dirac-type equations of Refs. [2–4] for two-body spectra with vanishing total spin.

The basic equations for calculating bound state energies are presented in Sec. 2. Section 3 is devoted to a linear quark-type potential model. Two-body effects on selected bound state energies near the non-relativistic limit are illustrated. Conclusions are in Sec. 4.

2 Semi-relativistic Local-Momentum Equations

In this section the "local-momentum" approximation used to simplify the semi-relativistic equation is outlined. This approach appears to be closely related to the one of Krolikowski.[7] The semi-relativistic quantum (SRQ) approximation of two interacting spinless particles starts from a Hamiltonian of classical special relativity. For an instantaneous scalar potential $S(r)$ in the center-of-mass frame of two massive particles, the stationary SRQ quantal wave function ψ satisfies the equation

$$\Bigl(\sqrt{\tilde{m}_1^2c^4+\hat{p}^2c^2}+\sqrt{\tilde{m}_2^2c^4+\hat{p}^2c^2}\Bigr)\psi=E\psi\,. \tag{1}$$

Here, E the relativistic energy, c is the speed of light and $\pm\hat{p}$ the two momentum operators. The mutual scalar potential $S(r)$ and masses are combined:

$$\tilde{m}_{1,2}=m_{1,2}+S(r)/c^2, \tag{2}$$

with $m_{1,2}$ being the rest masses.

The momentum operator $\hat{p}^2$ is the same for both particles in a centre-of-mass frame (although moving in opposite directions). The momentum operator is given by the cartesian and the radial expressions as

$$\hat{p}^2=-\hbar^2\Bigl(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Bigr)\,, \tag{3}$$

$$\hat{p}^2=\hat{p}_r^2+\frac{\hat{L}^2}{r^2}=\Bigl(-\hbar^2\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r+\frac{\hat{L}^2}{r^2}\Bigr)\,, \tag{4}$$

where $\hat{L}$ is the orbital angular momentum operator and $\hbar$ the reduced Planck's constant.

The equations of the local-momentum approximation can be derived by imagining two particles entering from free space towards a finite interaction region. In free space a plane wave $\mathrm{e}^{\,ikz}$, with a given wave number k, is repre-

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sented by the partial wave series[11]

$$\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}kr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)\mathrm{i}^l j_l(kr)P_l(\cos\theta)\,, \tag{5}$$

where $l = 0, 1, \ldots$ are the orbital angular momentum quantum numbers. The Legendre polynomials $P_l(\cos\theta)$ are expressed in terms of the angle θ between the initial z-direction and the relative position vector.

By expanding the square roots in the basic SRQ equation (1), and using the cartesian representation (3) of $\hat{p}^2$, this equation (1) provides the exact asymptotic wave number k for freely propagating plane waves $\mathrm{e}^{\pm\mathrm{i}kz}$:

$$k = \frac{1}{2\hbar c}\sqrt{\frac{(E^2 - m^2c^4)(E^2 - m_-^2c^4)}{E^2}}\,, \tag{6}$$

where

$$m = m_1 + m_2, \quad m_- = m_1 - m_2\,. \tag{7}$$

The radial component $\mathrm{j}_l(kr)$ of the plane wave is the spherical Bessel functions behaving as

$$\mathrm{j}_l(kr) \sim \frac{1}{kr}\sin(kr - \pi l/2), \quad r \to +\infty\,. \tag{8}$$

The constant k-eigenvalue of the partial wave components $\mathrm{j}_l(kr)$ is related to the equation:

$$\begin{aligned}\hat{p}^2\mathrm{j}_l(kr) &= \Big(-\hbar^2\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{\hbar^2 l(l+1)}{r^2}\Big)\mathrm{j}_l(kr)\\ &= \hbar^2k^2\mathrm{j}_l(kr), \quad l = 0, 1, \ldots\end{aligned} \tag{9}$$

A formal expansion of the operator in terms of $\hat{p}^2$ in the l-term of Eq. (1), i.e. in

$$\Big[\sqrt{m_1^2c^4 + \hat{p}^2c^2} + \sqrt{m_2^2c^4 + \hat{p}^2c^2}\Big]\mathrm{j}_l(kr) = E\mathrm{j}_l(kr)\,, \tag{10}$$

leads after some algebra to Eq. (6).

Hence, the plane wave satisfies Eq. (1) and its partial-wave component satisfies (9) as well as (10).

A generalization of the above semi-relativistic observations for plane waves leads to the *local-momentum approximation.* To this end, let a general wave be expanded as

$$\psi = \sum_{l=0}^{\infty}\psi_l P_l(\cos\theta)\,. \tag{11}$$

Assume ψ_l satisfies the second-order partial wave equation

$$\begin{aligned}\hat{p}^2\psi_l &= \Big(-\hbar^2\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{\hbar^2 l(l+1)}{r^2}\Big)\psi_l\\ &= \hbar^2\tilde{K}_{\mathrm{LM}}^2(r)\psi_l, \quad l = 0, 1, \ldots\,,\end{aligned} \tag{12}$$

where $\tilde{K}_{\mathrm{LM}}^2(r)$ in Eq. (12) is an *unspecified* scalar function of r. If $\tilde{K}_{\mathrm{LM}}^2(r)$ is not constant, higher powers of $\hat{p}^2$ are now assumed to satisfy the "*approximate*" relations

$$\hat{p}^{2n}\psi_l \approx \hbar^{2n}\tilde{K}_{\mathrm{LM}}^{2n}(r)\psi_l, \quad n = 2, 3, \ldots\,, \tag{13}$$

being accurate for sufficiently slowly varying functions $\tilde{K}_{\mathrm{LM}}^2(r)$. It follows that the wave function ψ_l in Eq. (12) solves Eq. (1) approximately. The left hand member in Eq. (1) is approximated, yielding

$$\Big(\sqrt{\tilde{m}_1^2(r)c^4 + \hbar^2c^2\tilde{K}_{\mathrm{LM}}^2(r)} + \sqrt{\tilde{m}_2^2(r)c^4 + \hbar^2c^2\tilde{K}_{\mathrm{LM}}^2(r)}\Big)\psi_l = E\psi_l\,. \tag{14}$$

The scalar functions on both sides of this equation have to be equal. Algebraic manipulations determine the local-momentum coefficient, yielding

$$\tilde{K}_{\mathrm{LM}}^2(r) = \frac{1}{4\hbar^2c^2}(E^2 - m_-^2c^4)(1 - \tilde{m}^2(r)c^4/E^2)\,, \tag{15}$$

where $\tilde{m}(r) = \tilde{m}_1(r) + \tilde{m}_2(r)$ and $m_- = m_1 - m_2$. This coefficient is consistent with the 16-component two-body Dirac approach;[2–3] see also Ref. [7].

An alternative mass notation is given in terms of the reduced mass μ and the total mass m

$$\mu = \frac{m_1m_2}{m}, \quad m = m_1 + m_2\,, \tag{16}$$

leading to the explicit expression

$$\tilde{K}_{\mathrm{LM}}^2(r) = \frac{1}{4\hbar^2c^2}(E^2 - m(m - 4\mu)c^4)(1 - (mc^2 + 2S(r))^2/E^2)\,. \tag{17}$$

Hence, the leading-order local-momentum approximation is based on the second-order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\psi_l) + \Big(\tilde{K}_{LM}^2(r) - \frac{l(l+1)}{r^2}\Big)(r\psi_l) = 0, \quad l = 0, 1, \ldots \tag{18}$$

Once Eq. (18) is derived it may be applied to both scattering and bound states.

Equation (18) for mutual scalar interaction potentials is identical to that of Semay *et al.*[2] and Ferreira.[3] These authors used a two-body Dirac approach for $J = 0$ with $l = 0$ and $l = 1$ particle-antiparticle bound states. The present approach treats $l = 0, 1, 2, 3, \ldots$ as a "good" quantum number.

2.1 *Non-relativistic limit*

By letting the relevant non-relativistic energy ϵ be defined by

$$\epsilon = E - mc^2, \quad m = m_1 + m_2\,, \tag{19}$$

the coefficient $\tilde{K}^2_{\rm LM}(r)$ is expanded in powers of c^{-2}, yielding

$$\tilde{K}^2_{\rm LM}(r) = \frac{2\mu}{\hbar^2}(\epsilon - 2S(r))\Big\{1 + \frac{1}{2\mu c^2}\Big(\Big[1 - 3\frac{\mu}{m}\Big]\epsilon + \frac{\mu}{m}(2S)\Big) + O(c^{-4})\Big\}. \tag{20}$$

For equal masses $m = 4\mu$ and for extreme light-heavy mass systems $\mu/m \to 0$, which can be considered also the single-mass limit in an external scalar potential $2S$.

In the single-mass limit the coefficient in Eq. (20) simplifies to

$$\tilde{K}^2_{\rm LM}(r) = \frac{2\mu}{\hbar^2}(\epsilon - 2S(r))\Big\{1 + \frac{1}{2\mu c^2}\epsilon + O(c^{-4})\Big\}, \quad m \to +\infty. \tag{21}$$

Table 1 Local-momentum energy levels for the linear potential in Eq. (28) with selected values of $\alpha^2/4$ and μ/m.

l, n	μ/m	$W(\alpha^2/4 = 0)$	$W(\alpha^2/4 = 1/8)$	$W(\alpha^2/4 = 1/4)$
0, 0	0	2.338 107 41	2.159 115 25	2.038 207 71
0, 1	0	4.087 949 44	3.610 634 50	3.339 201 16
0, 2	0	5.520 559 83	4.728 796 92	4.324 057 22
0, 0	1/4	2.338 107 41	2.265 472 95	2.204 232 08
0, 1	1/4	4.087 949 44	3.879 881 78	3.721 580 36
0, 2	1/4	5.520 559 83	5.158 860 36	4.902 267 20
1, 0	0	3.361 254 52	3.020 832 18	2.814 360 32
1, 1	0	4.884 451 84	4.239 003 78	3.894 057 81
1, 2	0	6.207 623 29	5.247 054 50	4.777 075 20
1, 0	1/4	3.361 254 52	3.206 989 76	3.086 488 46
1, 1	1/4	4.884 451 84	4.588 068 12	4.372 802 18
1, 2	1/4	6.207 623 29	5.754 298 07	5.443 479 99

This agrees with an *exact spin symmetry* model of the light-heavy quark-mass system in Ref. [12], provided the light mass component is represented by μ. Also, $2S(r)$ in Eq. (21) represents the sum of the equal "external" scalar and (time-component) vector potentials in Ref. [12]. As realized from Eq. (20), two-body effects (relative to non-relativistic results) relate to the total mass m being finite rather than infinite.

3 Linear Scalar Potential

Equation (18) is transformed into non-dimensional form for a linear scalar potential defined by

$$2S(r) = qr, \quad q > 0. \tag{22}$$

A unit length scale is chosen as in Ref. [12]:

$$r_* = \Big(\frac{\hbar^2}{2\mu q}\Big)^{1/3}, \tag{23}$$

and a dimensionless length x is introduced by

$$x = r/r_*. \tag{24}$$

The parameter responsible for relativistic effects in general is

$$\alpha = \frac{\hbar}{\mu c r_*}, \tag{25}$$

so that energy eigenvalues are scaled and represented by

$$W = \frac{2\mu r_*^2}{\hbar^2}\epsilon = \frac{\mu^2 c^2 r_*^2}{\hbar^2}\frac{2\epsilon}{\mu c^2} = \frac{2\epsilon}{\alpha^2 \mu c^2}, \tag{26}$$

where r_* and $\alpha^2\mu c^2$ are independent of the speed of light. The potential $S(r)$ is likewise reduced to

$$\Sigma(x) = \frac{2S(r_* x)}{\alpha^2 \mu c^2} = \frac{q r_*}{\alpha^2 \mu c^2}x = \frac{mq}{\hbar^2}r_*^3 x = x. \tag{27}$$

The reduced differential equation becomes

$$\frac{{\rm d}^2 F_l}{{\rm d}x^2} + \Big(\kappa^2_{\rm LM}(x) - \frac{l(l+1)}{x^2}\Big)F_l = 0, \quad l = 0, 1, \ldots, \tag{28}$$

with

$$\kappa^2_{\rm LM}(x) = (W - x)\Big\{1 + \frac{\alpha^2}{4}\Big(\Big[1 - 3\frac{\mu}{m}\Big]W + \frac{\mu}{m}x\Big)\Big\}. \tag{29}$$

In Eq. (29) μ/m represents the two-body parameter being magnified by the relativistic parameter α^2 as this becomes large. The two extreme cases are $\mu/m = 0$ and $\mu/m = 1/4$. A semiclassical analysis of the coefficient $\kappa^2_{\rm LM}(x)$ indicates that turning points are not affected by the relativistic terms. Also $\kappa^2_{\rm LM}(x) \geq \kappa^2_{\rm LM}\ (x, \alpha = 0)$ in the oscillating region of the effective potential, implying that energy levels are expected to appear shifted to lower values as α increases.

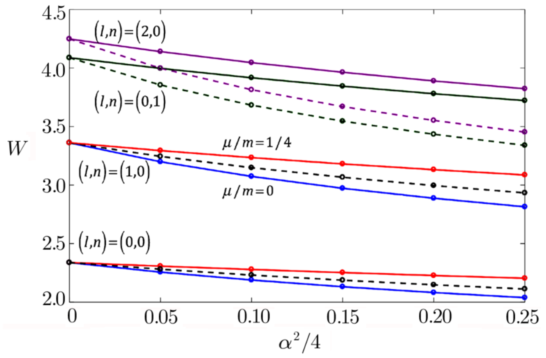

Fig. 1 (Color online) Energy levels (W) as function of relativity ($\alpha^2/4$) and the two-body parameter μ/m. The quantum numbers are l (orbital angular momentum) and n (radial nodes). From top: Purple lines: $l = 2$, $n = 0$. Solid line corresponds to equal masses, dashed line to the single mass limit. Green lines: $l = 0$, $n = 1$. Solid line corresponds to equal masses, dashed line to the single mass limit. Red, black respectively blue lines: $l = 1$, $n = 0$: $\mu/m = 0.25$ (equal masses), $= 0.125$ (in between), respectively $= 0$ (single mass). Red, black respectively blue lines: $l = 0$, $n = 0$: $\mu/m = 0.25$ (equal masses), $= 0.125$ (in between), respectively $= 0$ (single mass).

Numerical computations based on Eq. (28) are performed using an amplitude-phase method.[13] Figure 1 shows how energy levels are shifted as function of $\alpha^2/4$ with different values of the two-body parameter μ/m. The reduced mass μ is considered fixed and the quantum numbers are l (orbital angular momentum) and n (radial nodes). Levels corresponding to the single-mass limit are the ones most sensitive to relativistic corrections. A possible explanation is that in this limit one of the masses is as small as possible for a given reduced mass μ. Note that all levels investigated are shifted to lower values as α increases.

The level spacing with respect to n is wider than that with respect to l (see Fig. 1), and only the lowest energy levels are considered in Fig. 1.

4 Summary

An approximation of the semi-relativistic approach, the "local-momentum approximation", is outlined. Bound-state conditions appear similar to those of two-body approaches based on the Dirac theory for fermions.

The two-body effect found is that single-mass conditions are more sensitive to relativistic corrections. In the single-mass limit a spectrum corresponding to the spin symmetry of the single-particle Dirac equation is obtained.

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