Calculating fluid fields from the distribution function

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November 2011

General relations

The following relations hold for any distribution function $F = nf$ (e is the thermal energy per mass)

$$\begin{aligned}
\int f d^3c &= 1, \\
\mathbf{v} &= \int \mathbf{c} f d^3c, \\
\mathbf{c} &= \mathbf{v} + \mathbf{c}', \\
me &= \int \frac{m}{2} c'^2 f d^2c, \\
-\sigma_{ij} &= mn \int c'_i c'_j f d^3c, \\
q_i &= n \int \frac{m}{2} c'^2 \mathbf{c}' f d^3c.
\end{aligned}$$

T is the temperature, σ_{ij} the stress tensor and q_i the heat current density.

Maxwellian

1. The Maxwellian distribution is $F = nf$, where

$$f = A \exp(-\beta^2 c'^2).$$

Here A, β are constants.

1.1. Determine A.

$$A = \beta^3 \pi^{-3/2}$$

1.2 Show that the stress tensor only has diagonal components which are all the same

$$\sigma_{ij} = -p\delta_{ij},$$

and that

$$p = \frac{nm}{2\beta^2}.$$

As the gas is ideal, we know that it should satisfy the gas law $p = nkT$. Hence that

$$\beta^2 = \frac{m}{2kT}.$$

The Maxwellian is thus

$$f = (\frac{m}{2\pi kT})^{3/2} \exp(-\frac{mc'^2}{2kT}).$$

1.3 Calulate the average thermal kinetic energy per particle. Show that

$$me = \frac{3kT}{2},$$

which means $kT/2$ per degree of freedom.

Small Knudsen number

2. We now consider a simple shear flow where the velocity is in the x-direction and is a function of y only, $v(y)$. In the continuum limit $Kn << 1$ the simple relaxation model gives the distribution function as

$$\begin{aligned} F &= n(\frac{\beta}{\sqrt{\pi}})^3 \exp(-\beta^2 c'^2)[1 - \frac{\tau}{2}\beta^2 \frac{dv}{dy} c'_x c'_y], \\ \beta^2 &= \frac{m}{2kT} \end{aligned}$$

A strict calculation by the Chapman-Enskog method will give the same result except that τ is a function of $\beta^2 c'^2$ that is given by the forces between the molecules.

2.1. Show that n is the number density and that the term in F containing dv/dy does not contribute to the fluid velocity. Show that the heat current q_i vanishes. Calculate the shear stress

$$\sigma_{xy} = \frac{n\tau kT}{4}\frac{dv}{dy} = \frac{1}{4}\tau\frac{dv}{dy}p.$$

Large Knudsen number

3. Suppose that a solid wall has diffuse reflection so that the outgoing particles have a Maxwellian distribution ($\mathbf{n}$ is the normal pointing into the gas)

$$F = n_w \beta_w^3 \pi^{-3/2} \exp(-\beta_w^2 c'^2),\ \mathbf{c}' \cdot \mathbf{n} \geq 0.$$

Let us now consider the Couette problem with gas flowing between two parallel planes in $y = 0$ and $y = d$. The latter plane has the speed V in the positive x-direction. For $Kn >> 1$ all the collisions are with the walls. As a consequence the distribution is the sum of two half Maxwellians

$$F = \{ \begin{array}{c} n_w \beta_w^3 \pi^{-3/2} \exp(-\beta_w^2 c^2),\ c_y > 0. \\ n_w \beta_w^3 \pi^{-3/2} \exp(-\beta_w^2 (\mathbf{c} - \mathbf{V})^2),\ c_y < 0. \end{array}$$

Here

$$\beta_w^2 = \frac{m}{2kT_w},$$

The two walls have the same temperature T_w.

3.1. Find the number density in the gas. Show that

$$n = n_w.$$

3.2. Then show that $v_x = V/2$ and that v_y vanishes and hence no particles are entering or leaving the walls. Now it is time to introduce the thermal velocity through

$$\mathbf{c} = \mathbf{v} + \mathbf{c}'.$$

3.3. Find the mean thermal kinetic energy and from the formula

$$\overline{\frac{m}{2} c'^2} = \frac{3}{2} kT$$

the temperature in the gas. Show that

$$kT = kT_w + \frac{mV^2}{12}.$$

3.4. Also find the shear stress

$$\sigma_{xy} = \frac{mnV}{2\sqrt{\pi}\beta_w} = \sqrt{\frac{kT_w m}{2\pi}} nV.$$

Hint

Partial integration gives (n is a nonnegative integer)

$$\begin{aligned} & \int_0^\infty \exp(-\xi^2) \xi^{2n+1} d\xi \\ = \ & \frac{1}{2} n! \end{aligned}$$

Further,

$$\int_0^\infty \exp(-\xi^2) d\xi = \frac{\sqrt{\pi}}{2}.$$

and by partial integration

$$\begin{aligned} & \int_0^\infty \exp(-\xi^2) \xi^{2n} d\xi \\ = \ & (n - \frac{1}{2})(n - \frac{3}{2}) ... \frac{1}{2} \frac{\sqrt{\pi}}{2} \end{aligned}$$