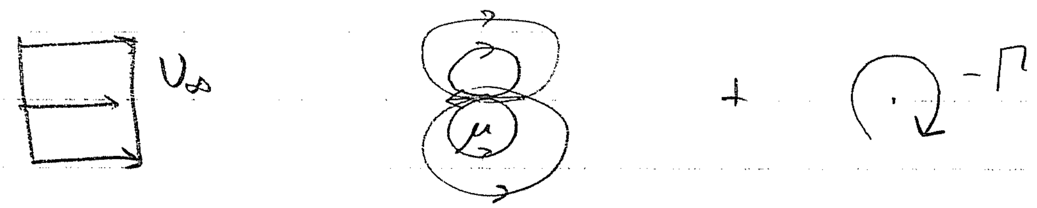
How to fly?

Flow past a circular cylinder with circulation.



uniform stream | doublet $\mu = 2\pi U_\infty a^2$ | vortex

$$F(z) = U_\infty\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln z \quad , \quad z = re^{i\theta}$$

$$\Rightarrow \quad \Psi = U_\infty \sin\theta\left(r - \frac{a^2}{r}\right) + \frac{\Gamma}{2\pi}\ln(r/a)$$

$r = a \Rightarrow \Psi = 0$ ∵ streamline

Complex velocity

$$W = F'(z) = U_\infty\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi z} = \ldots =$$

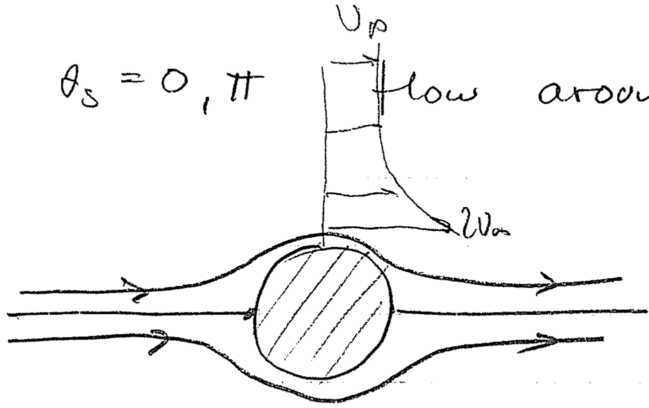
$$= \Bigg\{\underbrace{U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta}_{= u_r} + i\underbrace{\left[U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right]}_{-u_\theta}\Bigg\} e^{-i\theta}$$

Stagnation points: $u_r = 0 \Rightarrow r = a$

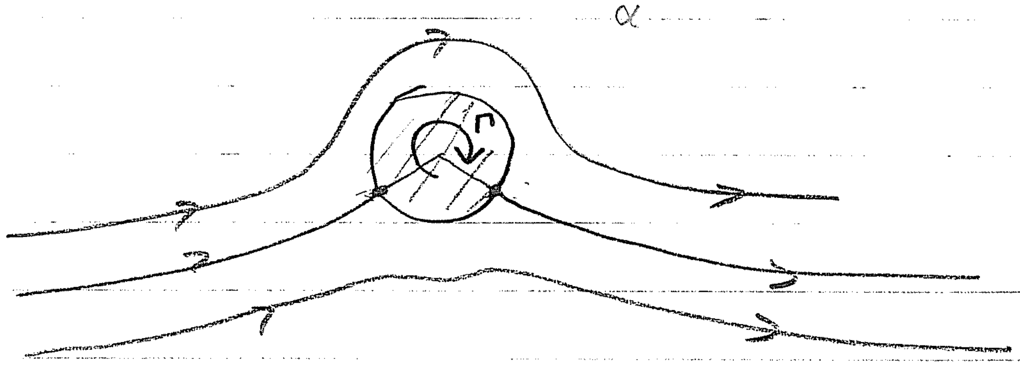
$$u_\theta = -2U_\infty \sin\theta_s - \frac{\Gamma}{2\pi a} = 0$$

$$\Rightarrow \quad \sin\theta_s = -\frac{\Gamma}{4\pi a U_\infty} \qquad \text{ok if } \Gamma < 4\pi a U_\infty$$

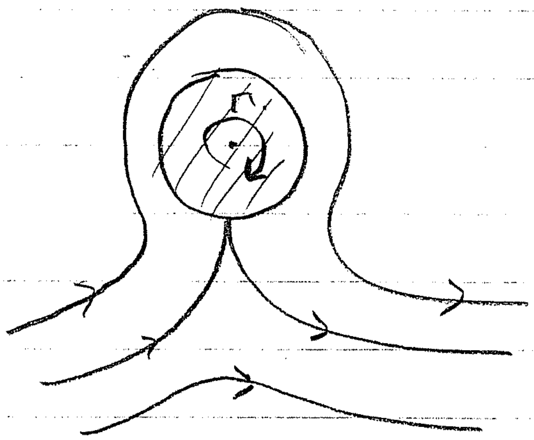
$\underline{\Gamma = 0}$ $\theta_s = 0, \pi$ flow around cylinder



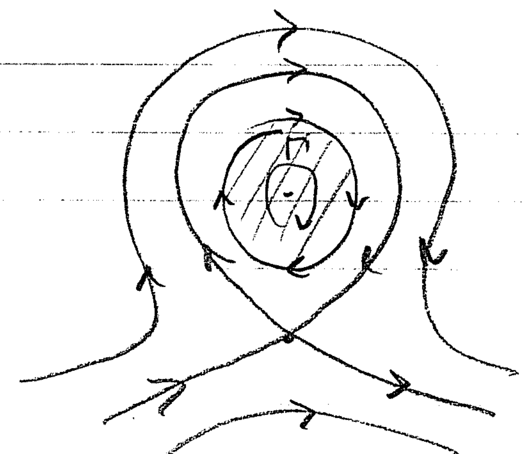
$\Gamma < 4\pi a U_\infty$ $\theta_s = \underbrace{-\arcsin \Gamma/4\pi a U_\infty}_{\alpha}, \ \pi + \arcsin \Gamma/4\pi a U_\infty$



$\Gamma = 4\pi a U_\infty$



$\Gamma > 4\pi a U_\infty$



Pressure on cylinder surface

Bernoulli's eq. $\Rightarrow$ $p(\theta) + \frac{1}{2}\rho U_\theta^2(r=a,\theta) = p_\infty + \frac{1}{2}\rho U_\infty^2$

$$U_\theta^2(r=a,\theta) = \left(-2U_\infty \sin\theta - \frac{\Gamma}{2\pi a}\right)^2 = 4U_\infty^2\left(\sin\theta + \sin\alpha\right)^2$$

$$= 4U_\infty^2 \sin^2\theta + 4U_\infty^2 \sin^2\alpha + 8U_\infty^2 \sin\theta \sin\alpha$$

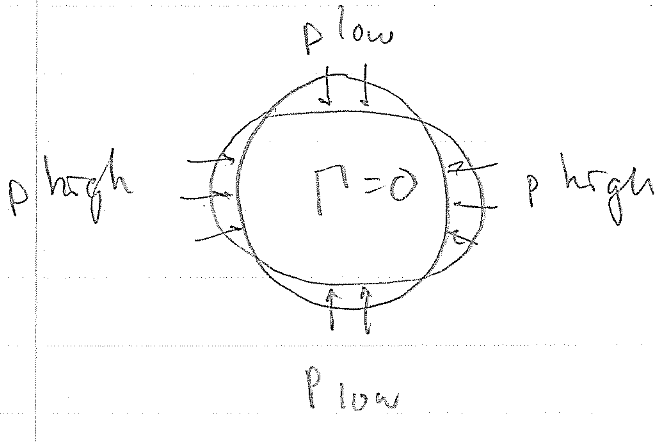
Pressure coefficient $c_p \hat{=} \dfrac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} =$

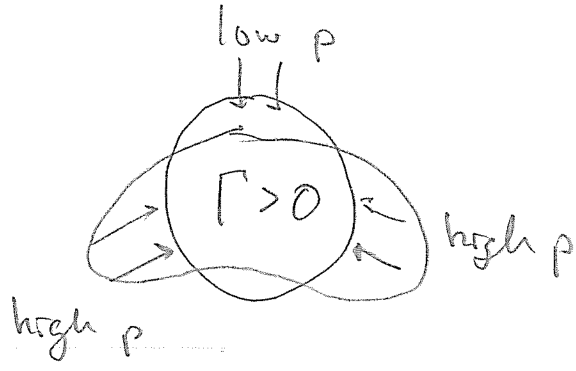
$$= 1 - 4\sin^2\theta - 4\sin^2\alpha - 8\sin\theta\sin\alpha$$ {see graph}

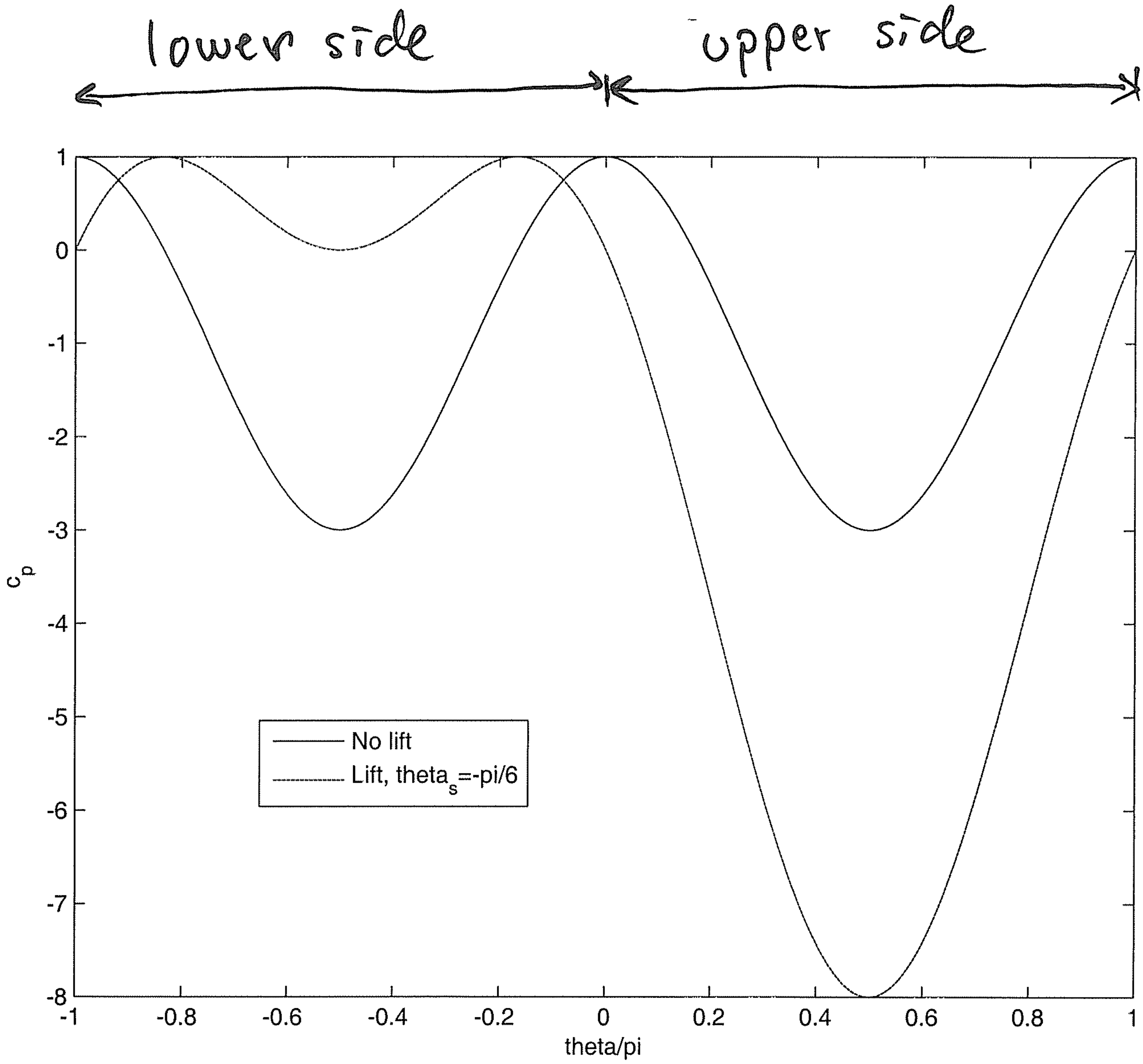
$$\sin\alpha = \frac{\Gamma}{4\pi U_\infty a}$$

$\Gamma = 0 \Rightarrow \sin\alpha = 0$ symmetric pressure distribution
no lift force, no drag force

$\Gamma > 0 \Rightarrow \sin\alpha > 0$ lift force > 0, drag force $= 0$

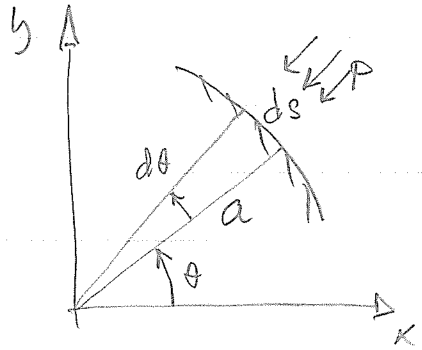






lower side
upper side
1
0
-1
-2
-3
-4
-5
-6
-7
-8
c_p
No lift
Lift, theta_s=-pi/6
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
theta/pi

Calculation of lift



$$dL_y' = -p(r=a)\,\underbrace{a\,d\theta}_{ds}\,\sin\theta$$

$$L_y' = \int_0^{2\pi} dL_y' = \dots = \frac{1}{2}\varrho U_\infty^2 a \int_0^{2\pi} (-c_p)\sin\theta\, d\theta =$$

$$= \left\{ \int_0^{2\pi} \sin^2\theta\, d\theta = \pi,\ \text{others} = 0 \right\} =$$

$$= \frac{1}{2}\varrho U_\infty^2 a \; 8\pi \sin\alpha$$

$$C_L = \frac{L_y'}{\frac{1}{2}\varrho U_\infty^2 \cdot 2a} = 4\pi\sin\alpha$$

"angle of attack"

$$L_y' = \frac{1}{2}\varrho U_\infty^2 a \; 8\pi \frac{\Gamma}{4\pi U_\infty a} = \underline{\underline{\varrho \Gamma U_\infty}}$$

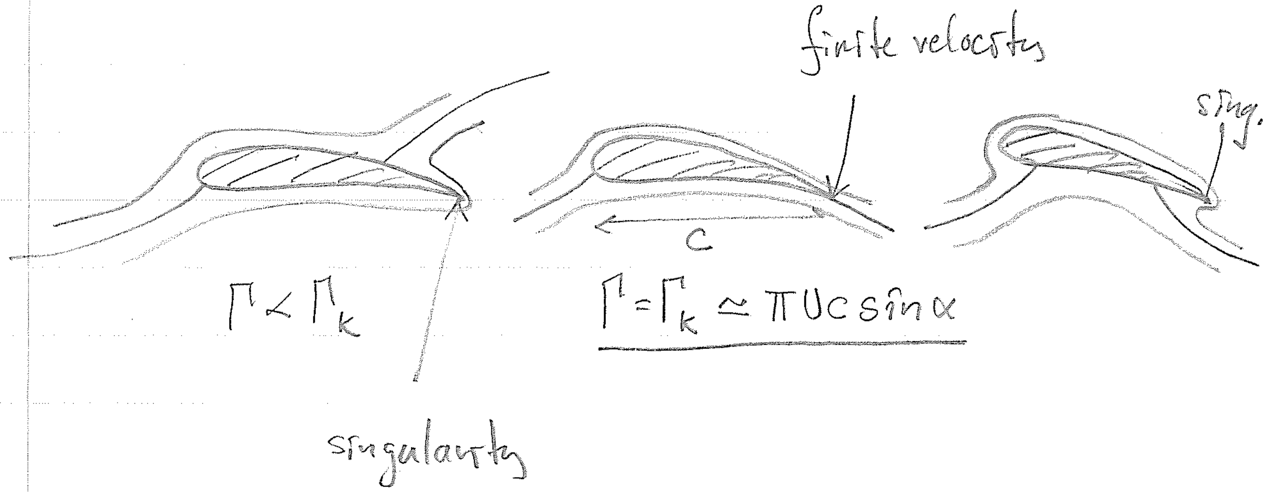
proportional to circulation

Kutta - Joukowski lift theorem

Result for circular cylinder holds for any cylinder shape

$$\text{drag } D' = 0, \quad \text{lift } L' = \rho \Gamma U_\infty$$

Γ is determined by the shape of the cylinder



Kutta condition on circulation gives finite velocity at rear edge.

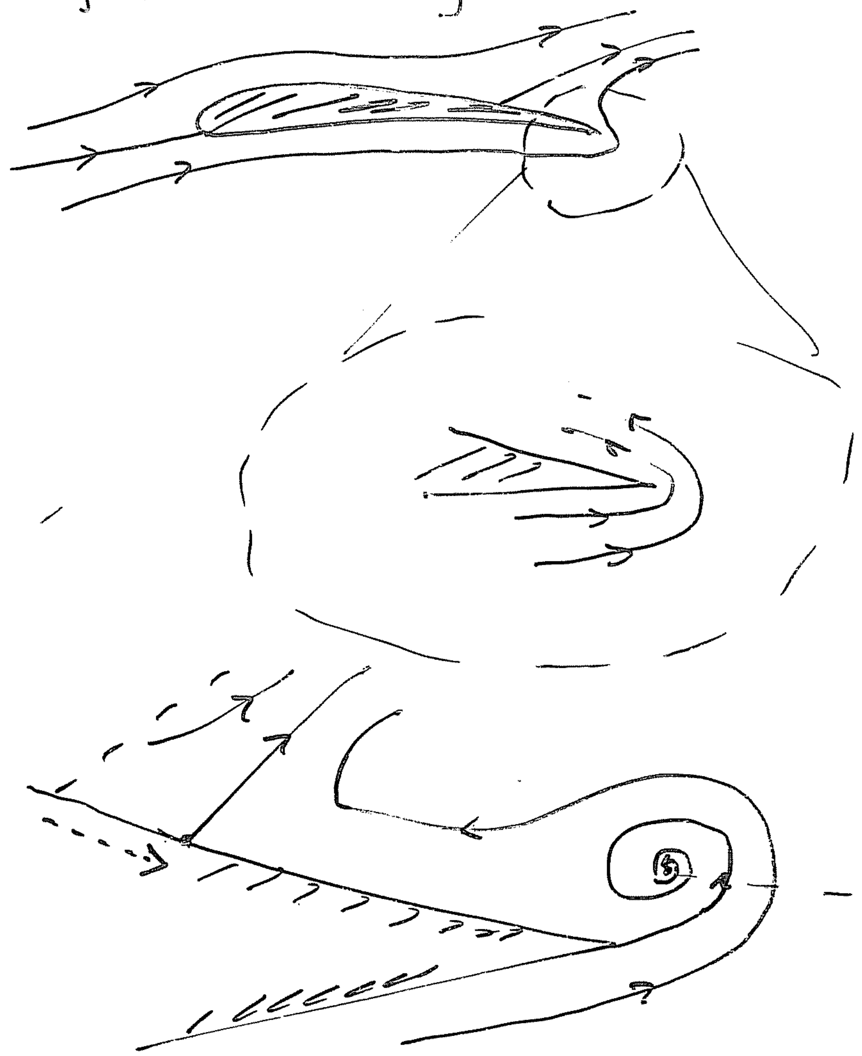
In reality, the circulation is generated by vortex shedding as a result of viscous effects.

{Show: Generation of circulation}

GENERATION OF CIRCULATION

Batchelor 1967
An introduction to Fluid Dynamics

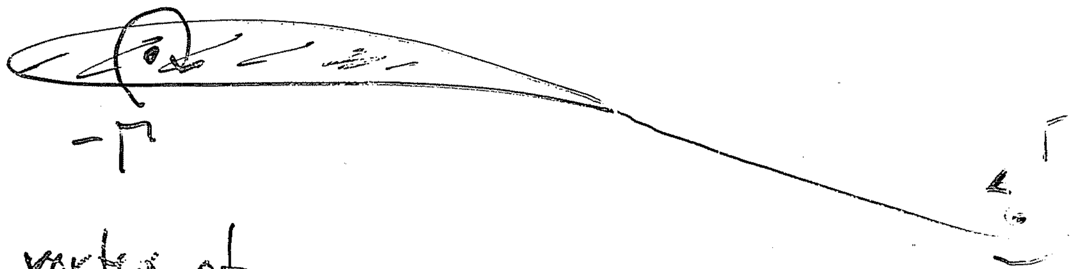
Recently started wing



vorticity sheet from boundary layer

Γ

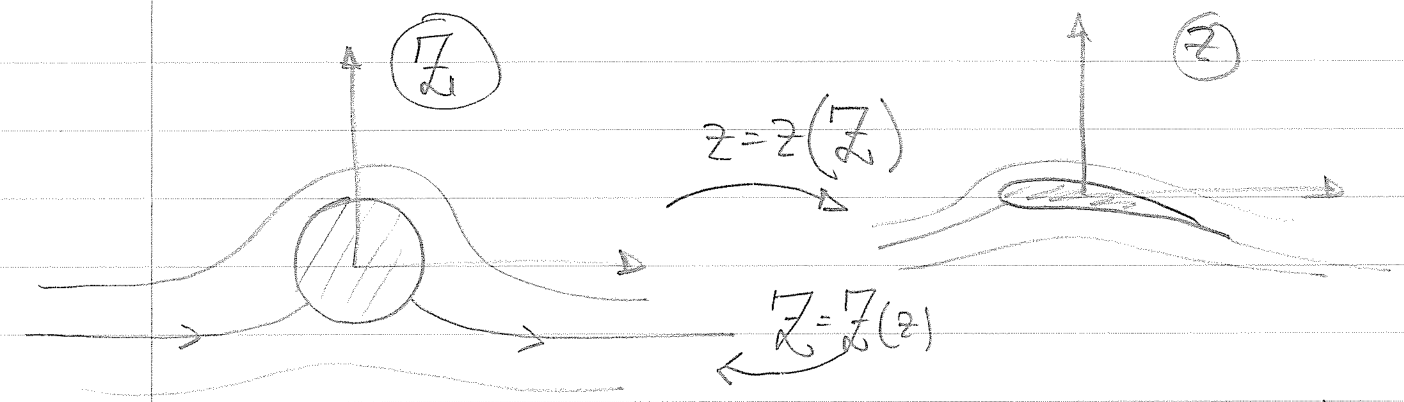
generates spiral vortex sheet of total circulation Γ.



"mirror" vortex of circulation $-\Gamma$

Starting vortex

Conformal mapping : Transforming one flow into another



Body of which you know $F(Z)$

Velocity potential for mapped body $F(Z(z))$

Velocity around mapped body

$$u - iv = W(z) = \frac{dF}{dz} = \frac{dF}{dZ}\frac{dZ}{dz} = \frac{\frac{dF}{dZ}}{\frac{dz}{dZ}}$$

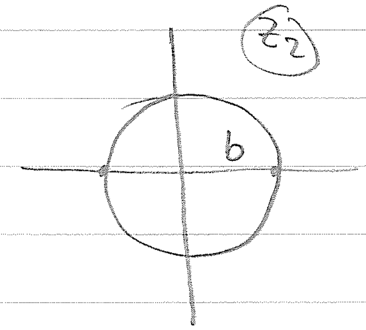
If $z(Z)$ and $F(Z)$ are analytic

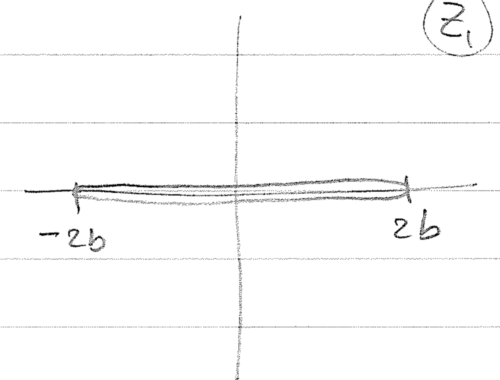
then $F(z)$ is also analytic.

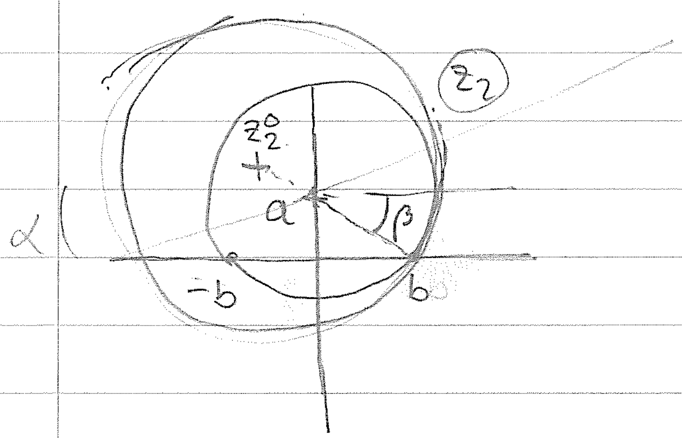
Study the transformation

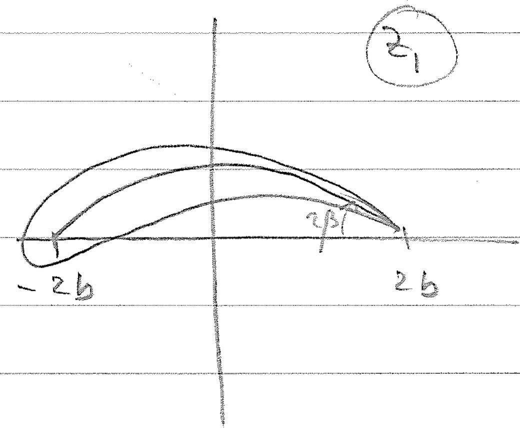
$$z_1 = z_2 + \frac{b^2}{z_2} = b e^{i\theta_2} + \frac{b^2}{b e^{i\theta_2}} = b(e^{i\theta_2} + e^{-i\theta_2}) = 2b\cos\theta_2$$

circle in z_2-plane : $z_2 = b e^{i\theta_2}$

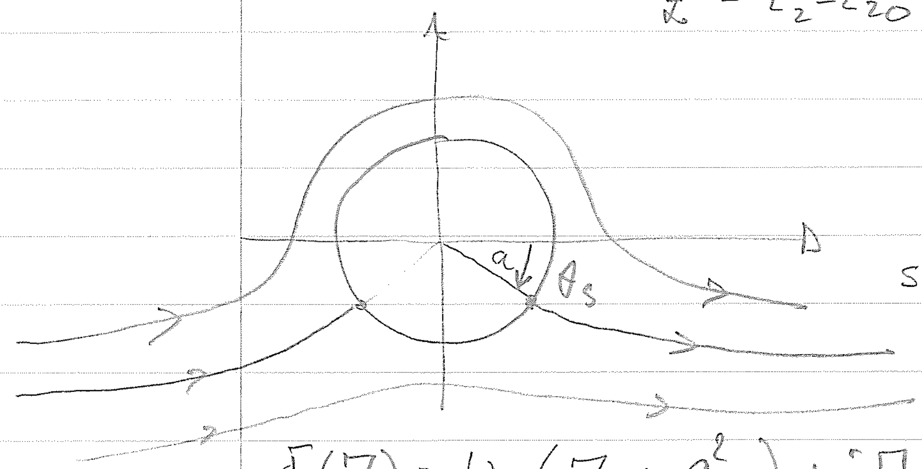








$$Z = z_2 - z_{20}$$



$$\sin\theta_s = \frac{\Gamma}{4\pi a U_\infty}$$

$$F(Z) = U_\infty\left(Z + \frac{a^2}{Z}\right) + \frac{i\Gamma}{2\pi}\ln\frac{Z}{a}$$

Kutta condition : rear stagnation point must coincide with the trailing edge
i.e. $\theta_s = \beta$

$$\Rightarrow \sin\beta = \frac{\Gamma_k}{4\pi a U_\infty}$$

Lift force $L' = \rho\Gamma_k U_\infty = \rho U_\infty^2 4\pi a \sin\beta$

$$u - iv = \frac{dF}{dz_1} = \frac{dF}{dZ}\bigg/\frac{dz_1}{dZ} =$$

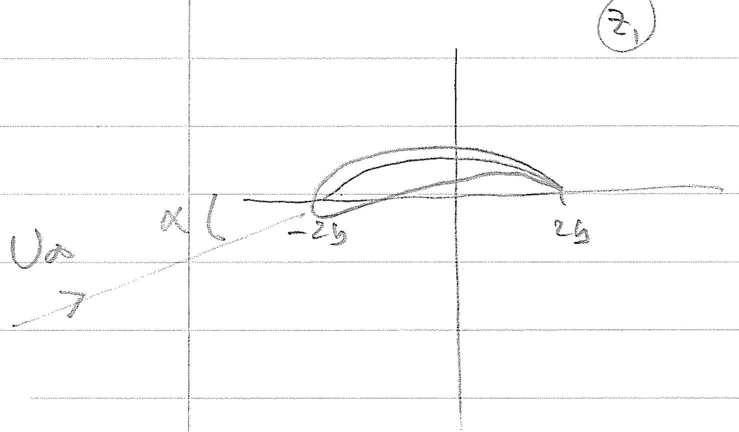
$$= \frac{dF}{dZ}\bigg/\frac{dz_1}{dz_2}\underbrace{\frac{dz_2}{dZ}}_{=1} = \frac{U_\infty\left(1 - \frac{a^2}{Z^2}\right) + \frac{i\Gamma_k}{2\pi}\frac{1}{Z}}{1 - b^2/z_2^2}$$

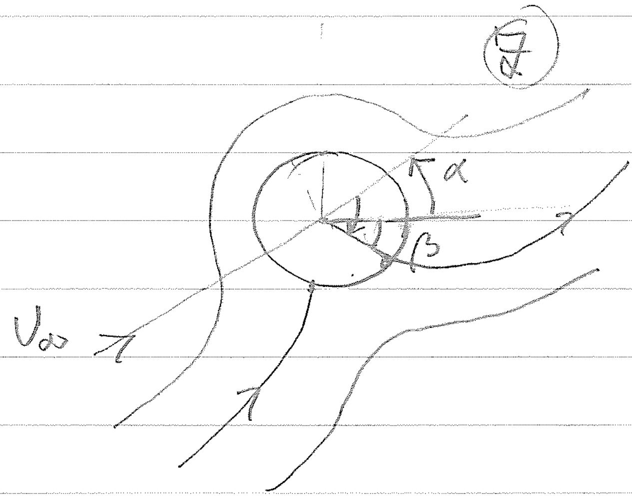
At trailing edge $z_2 = b \Rightarrow \frac{dF}{dz_1} \to \infty$

unless $\frac{dF}{dZ} = 0$ at $Z = a e^{-i\beta}$,

OK, because of Kutta condition.

Angle of attack α:





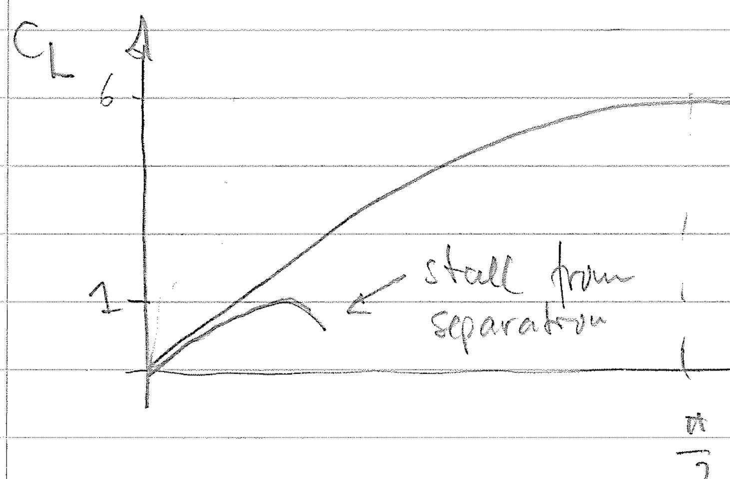
$$\Gamma_k = 4\pi a U_\infty \sin(\alpha+\beta)$$

$$L' = \rho U_\infty^2 \, 4\pi a \sin(\alpha+\beta)$$

Lift coefficient $\quad C_L = \dfrac{L'}{\frac{1}{2}\rho U_\infty^2 c} \qquad$ cord $c \approx 4b$

$$= 2\pi \frac{a}{b} \sin(\alpha+\beta)$$

$$\left(a = (1+\varepsilon)\frac{b}{\cos\beta} \Rightarrow C_L = 2\pi \frac{(1+\varepsilon)}{\cos\beta} \sin(\alpha+\beta) \right)$$



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