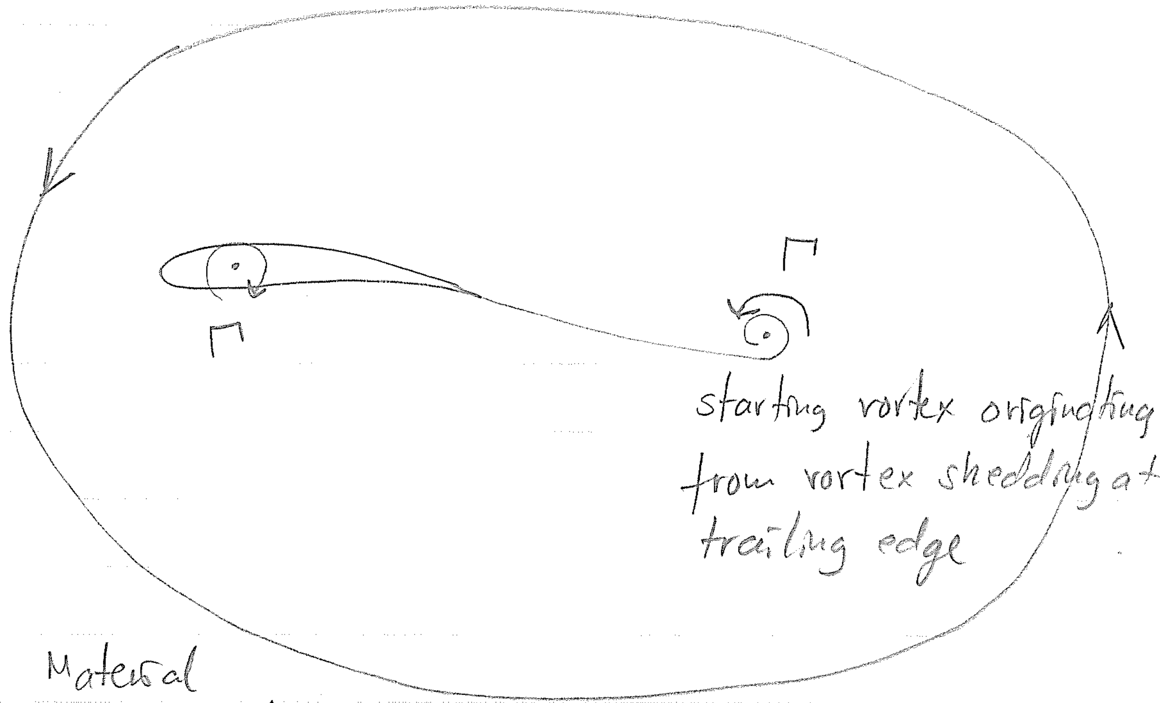
Generation of vorticity around airfoil.

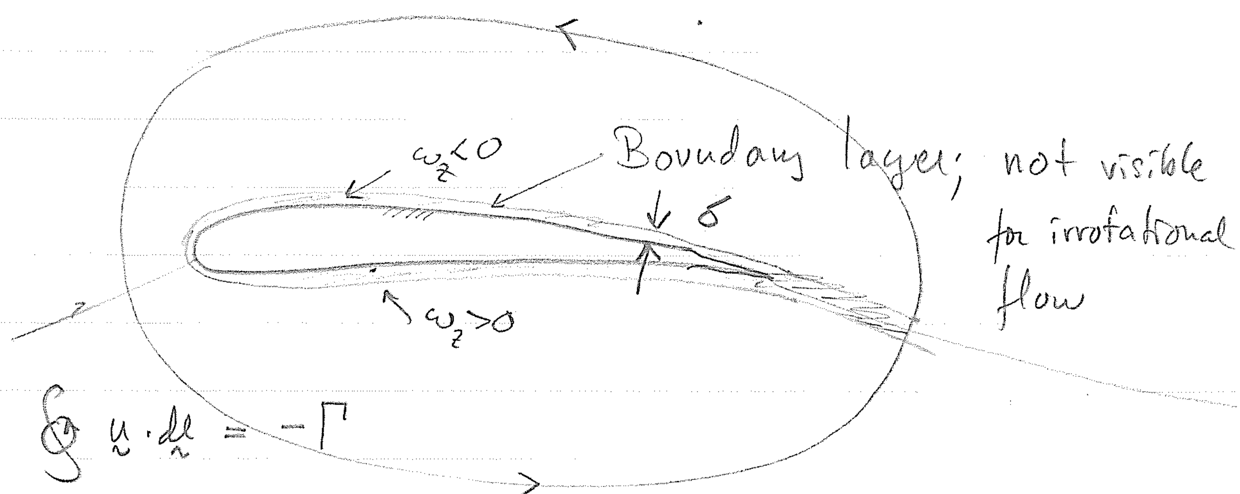


Material curve $C(t)$

Circulation $\oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l} = 0$ according to Kelvin

There can be no net circulation generated for curve $C(t)$ along which $\underset{\sim}{u}(\underset{\sim}{x},t)$ is irrotational.

Circulation around airfoil

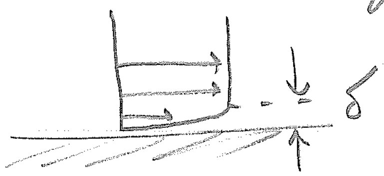


$$\oint \underset{\sim}{u} \cdot d\underset{\sim}{l} = -\Gamma$$

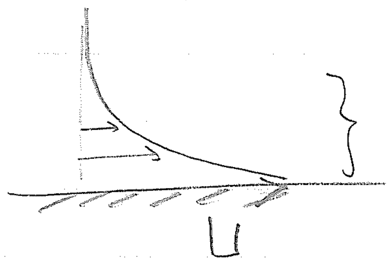
$$= \iint_S \omega_z \, dS$$

Boundary layers

In an essentially inviscid flow, $Re \gg 1$, estimate thickness of viscous region, δ, along solid wall.



Stoke's problem with diffusion of momentum



$$\Big\} \; \delta \sim \sqrt{\nu t}$$

Time for passage of the wing $t_L \sim L/U$

$$\Rightarrow \delta \sim \sqrt{\frac{\nu L}{U}} = L\sqrt{\frac{\nu}{UL}} = \frac{L}{\sqrt{Re}} \quad ; \quad Re = \frac{UL}{\nu}$$

Note: $\dfrac{\delta}{L} \sim \dfrac{1}{\sqrt{Re}} \to 0$ as $Re \to \infty$

$\Rightarrow$ large velocity gradients near walls (vorticity)

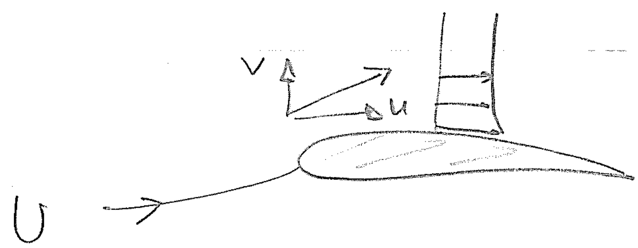
Derive approximate equation from N.-S. valid in b.l.

δ follows from this analysis

Division of flow
i) outer irrotational part

ii) boundary layer part close to walls, where the no slip condition is fulfilled

i) Irrotational, inviscid flow



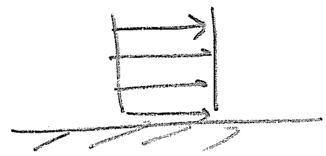
typical scales $u \sim v \sim U$

lengthscale $\sim L$

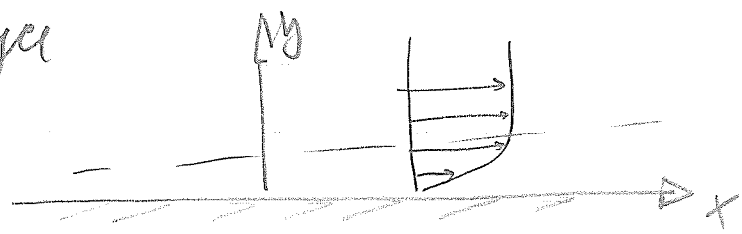
$$\begin{cases} \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\nabla p + \cancel{\dfrac{1}{Re}\nabla^2 \underset{\sim}{u}} \qquad Re \to \infty \\ \nabla \cdot \underset{\sim}{u} = 0 \end{cases}$$

Solved by e.g. potential flow theory.

This solution does not satisfy the no slip condition:



ii) Boundary layer



typical scales

$$u \sim U$$
$$v \sim V \ll U$$
$$x \sim L$$
$$y \sim \delta \ll L$$
$$p \sim \varrho U^2$$

Determine V from

$$\underbrace{\frac{\partial u}{\partial x}}_{\sim \frac{U}{L}} + \underbrace{\frac{\partial v}{\partial y}}_{\frac{V}{\delta}} = 0 \quad \Rightarrow V \sim \frac{\delta}{L} U \ll U$$

y-momentum ↑:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial^2 v}{\partial y^2}$$

$$\sim \quad U\frac{\delta U}{L^2} \qquad \left(\frac{\delta U}{U}\right)^2 \frac{1}{\delta} \qquad \frac{U^2}{\delta} \qquad \frac{\nu \delta U}{L^3} \qquad \frac{\nu \delta U}{L \delta^2}$$

$$\left(\frac{\delta}{L}\right)^2 \qquad \left(\frac{\delta}{L}\right)^2 \qquad 1 \qquad \frac{1}{Re}\left(\frac{\delta}{L}\right)^2 \qquad \frac{1}{Re}$$

when $Re \to 0$ only pressure term ~ 1

$$\Rightarrow \quad \frac{\partial p}{\partial y} = 0 \qquad p = p_e(x)$$

↑ edge of b.l. (external flow)

Pressure constant in b.l. given by inviscid outer flow.

x - momentum $\rightarrow x$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\sim \quad \frac{U^2}{L} \qquad \frac{\delta}{L}\frac{U^2}{\delta} \qquad \frac{U^2}{L} \qquad \nu\frac{U}{L^2} \qquad \nu\frac{U}{\delta^2}$$

$$\sim \quad 1 \qquad 1 \qquad 1 \qquad 1/Re \qquad 1/Re\,(L/\delta)^2$$

Balance as $Re \rightarrow \infty \Rightarrow \frac{1}{Re}\left(\frac{L}{\delta}\right)^2 \sim 1$

$\Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$ consistent with previous estimate

Boundary layer equations:

$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{dp_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

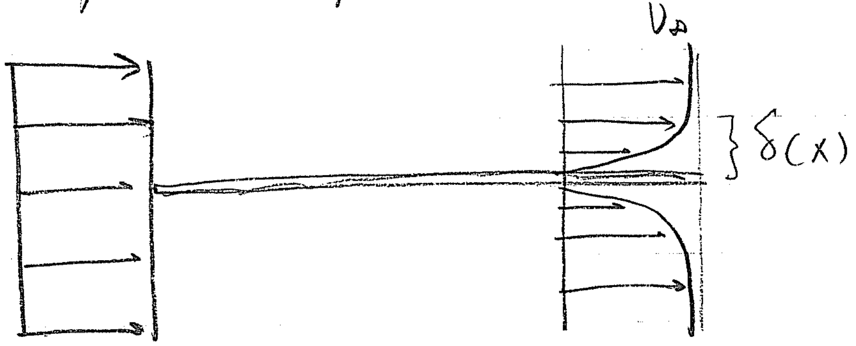
Parabolic in x

start at x_0 : $u(x=x_0, y) = u_{in}(y)$

boundary cond.: $\begin{cases} u(x, y=0) = v(x, y=0) = 0 \\ u(x, y\rightarrow\infty) = U_e(x) \end{cases}$

$U_e(x)$ is outer inviscid flow evaluated at the <u>wall.</u>
(since $\delta/L \rightarrow 0$ as $Re \rightarrow \infty$)
Pressure, $p_e(x)$, given by Bernoulli's eq. for outer flow.
$p_e(x) + \frac{1}{2}\varrho U_e^2(x) = \text{const.}$

Blausius flow on flat plate



$U_e(x) = U_\infty$

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \underbrace{U_e\dfrac{dU_e}{dx}}_{=0} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

$x = 0: \quad u = U_\infty$

$y = 0: \quad u = v = 0 \; ; \quad y \to \infty: \quad u \to U_\infty$

Find similarity solution

$$u = U_\infty f'(\eta) \qquad \eta = y/\delta(x)$$

$$= \frac{\partial \psi}{\partial y} \qquad \text{Streamfunction } \psi = \int^y U_\infty f'(\eta)\, dy =$$

$$= \int^\eta U_\infty f'(\eta)\, d\eta \cdot \delta(x) = U_\infty \delta(x) f(\eta)$$

$$v = -\left(\frac{\partial \psi}{\partial x}\right)_y = -\left(\frac{\partial \psi}{\partial x}\right)_\eta - \left(\frac{\partial \psi}{\partial \eta}\right)_x \left(\frac{\partial \eta}{\partial x}\right)_y =$$

$$= -U_\infty \left[\delta'(x) f(\eta) + \cancel{\delta(x)} f' \left(-\eta \frac{\delta'}{\cancel{\delta}}\right) \right] =$$

$$= U_\infty \delta'(x) \left[-f + \eta f' \right]$$

u & v satisfies $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

$$\frac{\partial u}{\partial x} = U_\infty f'' \frac{\partial \eta}{\partial x} = -U_\infty f'' \eta \frac{\delta'}{\delta}$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \frac{\partial \eta}{\partial y} = U_\infty f'' \frac{1}{\delta} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = U_\infty f''' \frac{1}{\delta^2}$$

$\Rightarrow$

$$\cancel{-U_\infty^2 f' f'' \eta \frac{\delta'}{\delta}} + U_\infty^2 \frac{\delta'}{\delta}[-f + \cancel{\eta f'}] f'' = \nu U_\infty f''' \frac{1}{\delta^2}$$

$$\Rightarrow \quad f''' + \underbrace{\frac{U_\infty}{\nu} \delta\delta'}_{=\text{const. for similarity solution}} f f'' = 0$$

$$\frac{d}{dx}\frac{\delta^2}{2} = \frac{\nu}{U_\infty} \cdot \text{const.} = \frac{\nu}{U_\infty}\frac{1}{2} \Rightarrow \delta^2 = \frac{\nu x}{U_\infty} + C$$

$$\delta(0) = 0 \Rightarrow \boxed{\delta = \sqrt{\nu x / U_\infty}}$$

$$\boxed{f''' + \frac{1}{2} f f'' = 0} \quad \text{Blausius equation}$$

$$f(0) = f'(0) = 0 \quad ; \quad f'(\eta \to \infty) = 1$$

$$\delta'(x) = \frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}}$$

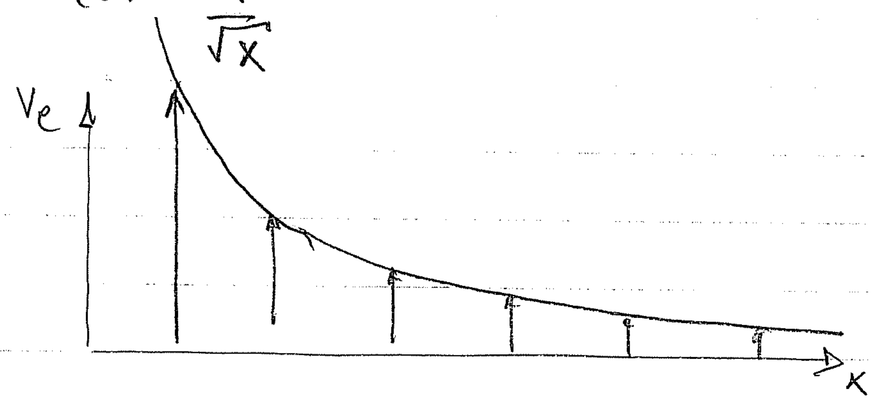
$$v = U_\infty \sqrt{\frac{\nu}{U_\infty x}} \frac{1}{2}[\eta f' - f] \qquad v \ll U_\infty \text{ as } \frac{U_\infty x}{\nu} \gg 1$$

numerical solution {see OH}

boundary layer thickness $\delta_{99}(x) \sim \sqrt{x}$

$$f'(\eta_{99}) = 0.99 \Rightarrow \eta_{99} = 4.9 = \frac{\delta_{99}(x)}{\sqrt{\nu x / U_\infty}}$$

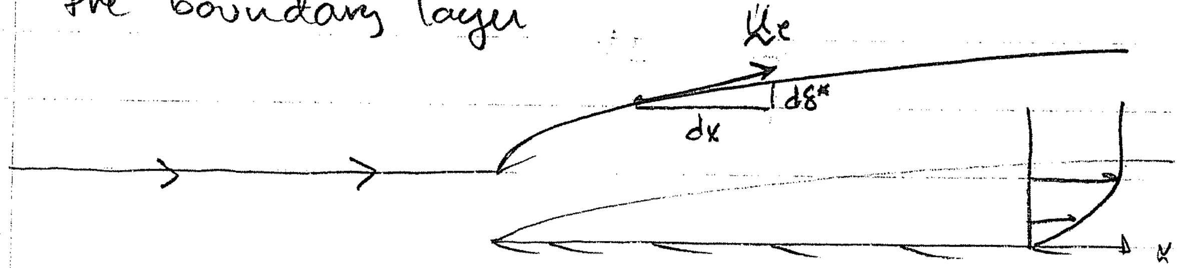
note: $V_e(x) \sim \frac{1}{\sqrt{x}}$



Boundary layer approx. invalid as $x \to 0$

we require $Re_x = \frac{U_\infty x}{\nu} \gg 1$; $x \gg \nu / U_\infty$

Displacement of inviscid outer flow due to the boundary layer



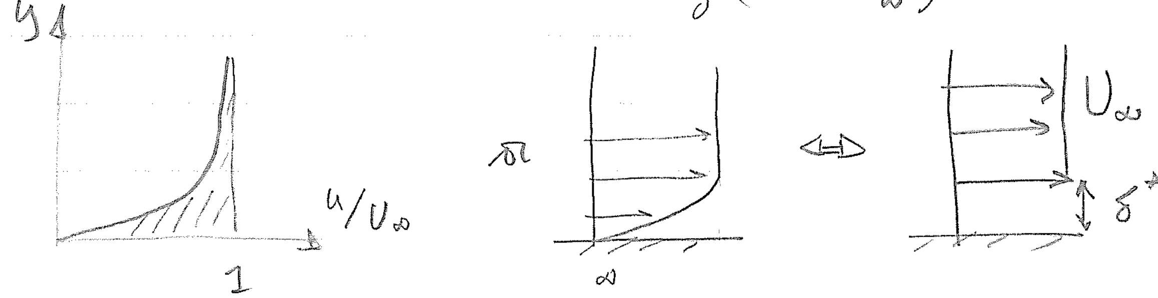
streamline inclination $$\frac{d\delta^*}{dx} = \frac{V_e(x)}{U_\infty} = 0.8604 \sqrt{\frac{\nu}{x U_\infty}}$$

total displacement $$\delta^* = \int_0^x \frac{V_e(x)}{U_\infty} dx = 0.8604 \cdot 2 \cdot \sqrt{\frac{\nu x}{U_\infty}}$$

Displacement thickness δ^*

Blausius $\dfrac{\delta^*}{x} = \dfrac{1.7208}{\sqrt{Re_x}}$

General definition $\delta^* \triangleq \int_0^\infty \left(1 - \dfrac{u}{U_\infty}\right) dy$



$$\int_0^\infty (U_\infty - u)\, dy = U_\infty \delta^*$$

check $\dfrac{d\delta^*}{dx} = \dfrac{d}{dx}\int_0^\infty \left(1 - \dfrac{u}{U_\infty}\right) dy = \int_0^\infty -\dfrac{1}{U_\infty}\dfrac{\partial u}{\partial x}\, dy = \int_0^\infty \dfrac{1}{U_\infty}\dfrac{\partial v}{\partial y}\, dy$

$$= \frac{V_e(x)}{U_\infty}$$

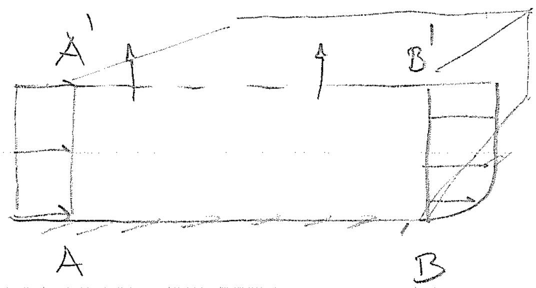
Skin friction $\tau_w = \tau_{xy}(y=0) = \mu \dfrac{\partial u}{\partial y}(y=0) =$

$$= \mu U_\infty \frac{\partial}{\partial y}\left(f'(\eta)\right)_{\eta=0} = \frac{\mu U_\infty}{\delta(x)} f''(0) = \frac{\mu U_\infty}{\sqrt{\nu x / U_\infty}} f''(0)$$

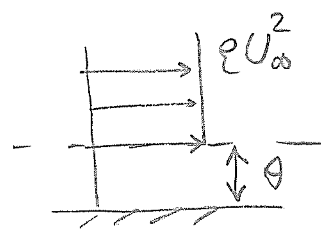
$$= \varrho U_\infty^2 \sqrt{\frac{\nu}{U_\infty x}}\, f''(0) = \{\text{Blausius}\} = \frac{0.332}{\sqrt{Re_x}} \varrho U_\infty^2 \sim \frac{1}{\sqrt{x}}$$

Drag $D' = \int_0^x \tau_w dx = 0.664 \sqrt{\dfrac{\nu x}{U_\infty}}\, \varrho U_\infty^2$

Momentum loss thickness $\vartheta(x) = 0.664\sqrt{\frac{\nu x}{U_\infty}}$



loss of momentum $\Longleftrightarrow$



General definition $\theta(x) \triangleq \int_0^\infty \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy$

x-momentum flux density in x-direction $= \rho u u$

— " — in y-direction $= \rho u v$

$$\underbrace{\frac{d}{dt}\int_V \rho \underset{\sim}{u}\, dV}_{=0} + \oiint_S \rho \underset{\sim}{u}(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS = \int_V \rho \underset{\sim}{f}\, dV + \oiint_S \underset{\sim}{P}\, dS$$

$\to x$: $\oiint_S \rho u(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS = -D_x'\cdot\text{width}$

$$\int_B^{B'} \rho u u\,dy + \int_{A'}^{B'} \rho U_\infty V_e(x)\,dx + \int_A^{A'}\rho U_\infty(-U_\infty)\,dy = -D_x'$$

$$\underbrace{\frac{d}{dt}\int_V \rho\, dV}_{=0} + \oiint_S \rho(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS = 0$$

$$\int_B^{B'}\rho u\,dy + \int_{A'}^{B'}\rho V_e(x)\,dx + \int_A^{A'}\rho(-U_\infty)\,dy = 0$$