A simple model for the turbulent shear stress.

Total average stress $\quad -P\delta_{ij} + \overline{\tau}_{ij} \underbrace{- \rho\overline{u_i u_j}}_{\text{"turbulent stress"}}$

Reynolds stress

Average Newtonian stress

$$\overline{\tau}_{ij} = 2\mu E_{ij} \quad ; \quad E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Average turbulent stress. model

$$-R_{ij} = -\overline{u_i u_j} = -\frac{2}{3}\overline{k}\,\delta_{ij} + 2\nu_T E_{ij}$$

$$\frac{2}{3}\overline{k} = \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{3}$$

ν_T: turbulent kinematic viscosity

Typically $\nu_T \gg \nu = \mu/\rho$

Suppose $\nu_T = V_T\, l$

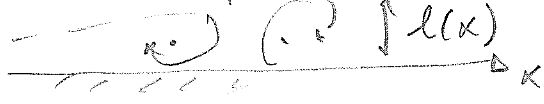
V_T: typical velocity fluctuation $V_T \sim \sqrt{\overline{k}}$

l: Prandtl's mixing length on which fluid particle retains it's momentum

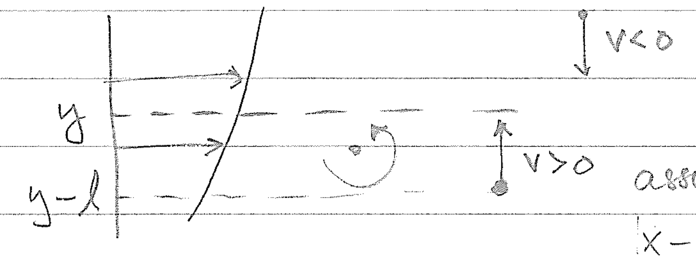
Compare kinetic gas theory $\nu \approx \overline{c}\,\lambda$

$\overline{c}$ is molecular speed, λ is mean free path

$l(\underline{x}, t)$ often taken from geometrical considerations.



Typical velocity fluctuation u_T



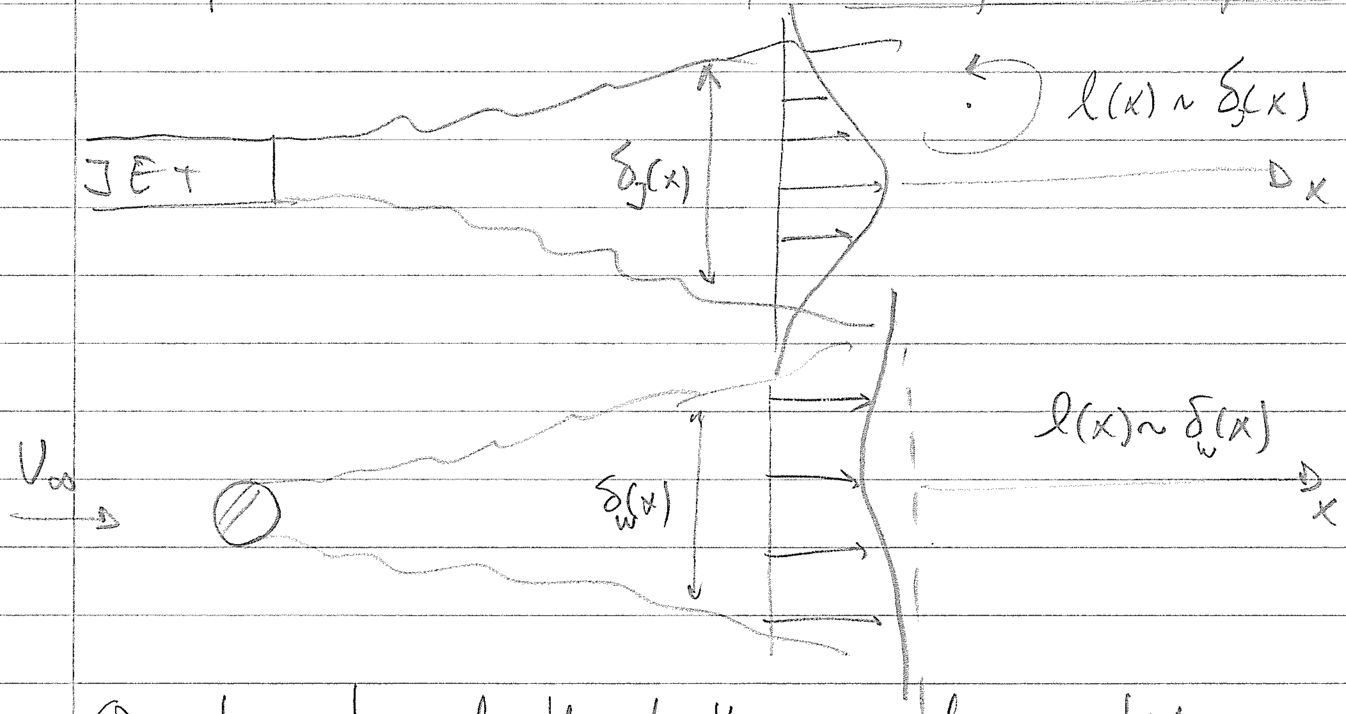
assume particle keeps its x-velocity on average

$$u_T \approx U(y-l) - U(y) \simeq -\frac{dU}{dy} l$$

$$u_T \approx U(y+l) - U(y) \simeq \frac{dU}{dy} l$$

$$-\overline{uv} \sim v_T l \frac{dU}{dy}$$

Simple model suitable for wall free shear flows.



One was found that the spreading rates

$$\underbrace{\frac{D}{Dt}\delta_J(x)}_{\sim U_c(x)\frac{d\delta_J}{dx}} \sim U_c(x) \quad ; \quad \underbrace{\frac{D}{Dt}\delta_w(x)}_{\sim U_\infty \frac{d\delta_w}{dx}} \sim \Delta U_c(x)$$

$$\Rightarrow \quad \frac{d\delta_j}{dx} = \text{const.} \qquad \frac{d\delta_w}{dx} = \frac{\Delta U_c(x)}{U_\infty}\,\text{const.}$$

or $\delta_j = \text{const.}\; x$

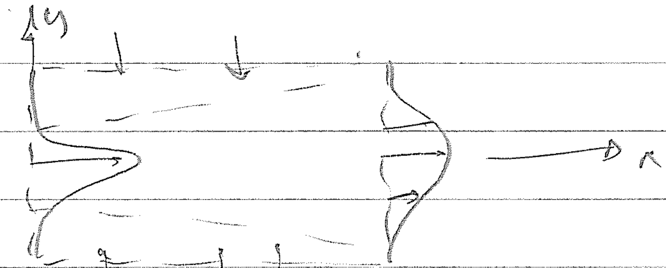
Self-preservation far downstream

JET $U(x,y) = U_c(x)\, F_j(y/\delta_j(x))$

WAKE $\Delta U(x,y) = \Delta U_c(x)\, F_w(y/\delta_w(x))$

Combined with momentum theorem

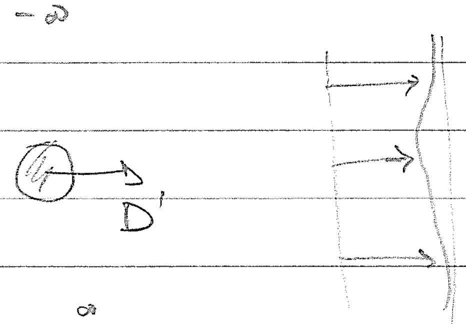
JET



Momentum flux $M = \int_{-\infty}^{\infty} \rho U^2 dy = \text{const.}$

(although $\int_{-\infty}^{\infty} \rho U dy \neq \text{const.}$)

WAKE



$U = U_\infty - \underbrace{\Delta U(x,y)}_{\ll U_\infty}$

$$\text{Drag } D' = \rho U_\infty^2 \int_{-\infty}^{\infty} \frac{U}{U_\infty}\left(1-\frac{U}{U_\infty}\right) dy = \rho U_\infty^2 \int_{-\infty}^{\infty} \left(1-\frac{\Delta U}{U_\infty}\right)\frac{\Delta U}{U_\infty}\, dy$$

$$\simeq \rho U_\infty \int_{-\infty}^{\infty} \Delta U\, dy$$

So, JET

$$M = \int_{-\infty}^{\infty} \rho U_c^2 F_J(\eta)\, d\eta\, \delta_J(x) = \rho U_c^2(x)\, \delta(x) \underbrace{\int_{-\infty}^{\infty} F_J(\eta)\, d\eta}_{\text{const.}}$$

$$\Rightarrow \rho U_c^2(x) \underbrace{\delta(x)}_{\sim x} = M / \text{const.}$$

$$\Rightarrow U_c^2 = \frac{M}{\rho x} / \text{const.} \qquad U_c(x) = \sqrt{\frac{M}{\rho x}}\ \text{const.}$$

& WAKE

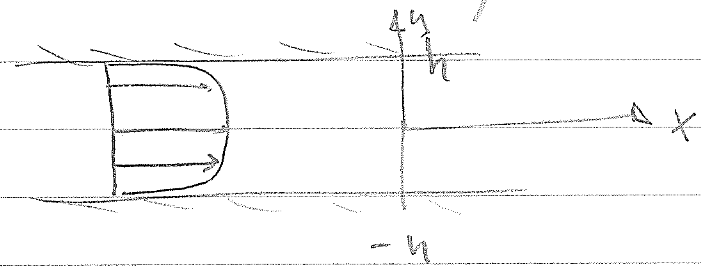
$$D' = \rho U_\infty \int_{-\infty}^{\infty} \Delta U_c(x) F_w(\eta)\, d\eta\, \delta_w(x) =$$

$$= \rho U_\infty \underbrace{\Delta U_c(x)}_{\sim U_\infty \frac{d\delta_w}{dx}} \delta_w(x) \underbrace{\int_{-\infty}^{\infty} F_w(\eta)\, d\eta}_{\text{const.}}$$

$$\delta_w \frac{d\delta_w}{dx} = \frac{D'}{\rho U_\infty^2} / \text{const.}$$

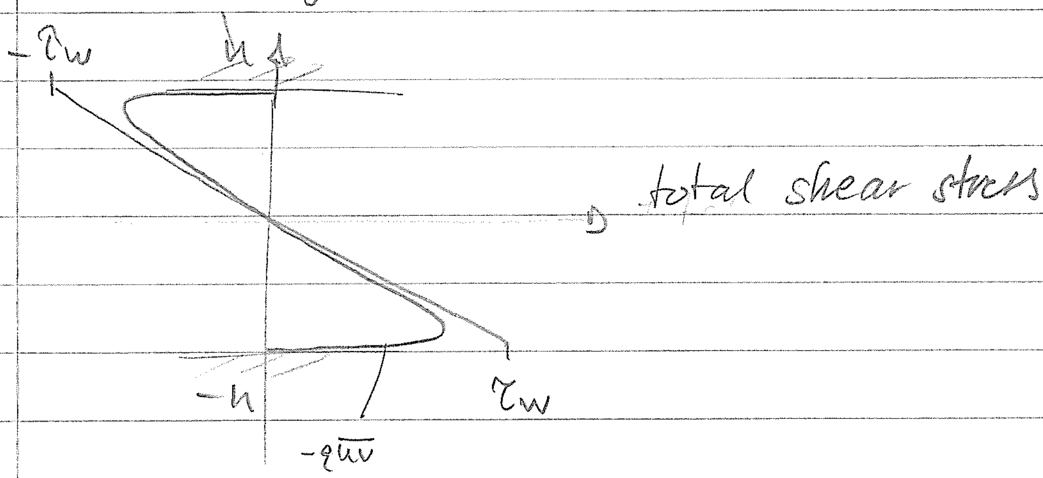
$$\Rightarrow \begin{cases} \delta_w(x) = \sqrt{\dfrac{D' x}{\rho U_\infty^2}}\ \text{const.} \\[2ex] \Delta U_c(x) = U_\infty \sqrt{\dfrac{D'}{\rho U_\infty^2 x}}\ \text{const.} \end{cases}$$

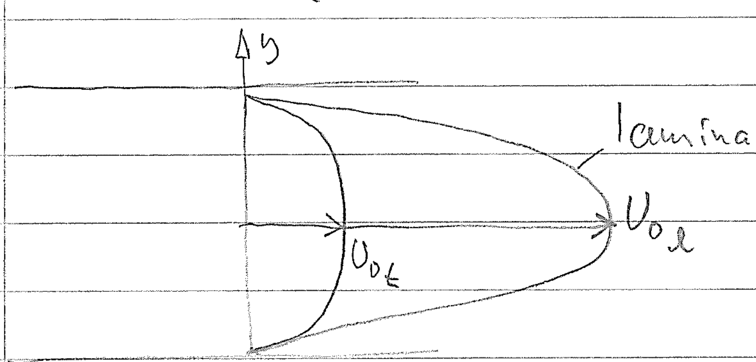
Turbulent channel flow



Fully developed $\Rightarrow$ total shear stress =

$$= \mu \frac{dU}{dy} - \rho \overline{uv} = -\frac{y}{h}\tau_w$$



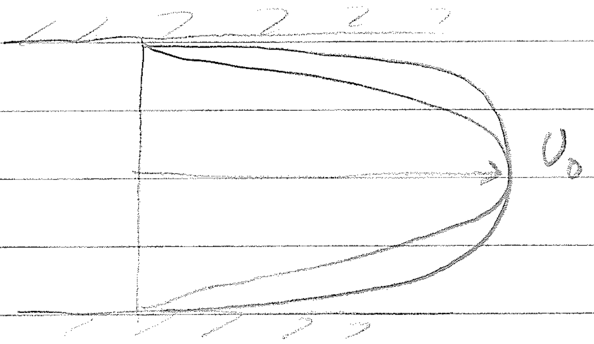


$\overline{uv} = 0$ at wall $\therefore$ $\tau_w = \mu \frac{dU}{dy}(y=-h)$

turbulent flow with same τ_w as laminar case gives $U_{0t} < U_{0l}$.

turbulent flow with the same $Re = \frac{U_0 h}{\nu}$

$\Rightarrow$ different τ_w



$\tau_{w\,turb} \gg \tau_{w\,lam.}$

Structure of wall bounded shear flows

Introduce friction velocity $u_\tau = \sqrt{\tau_w/\rho}$

length $l^+ = \nu/u_\tau$

Define $y^+ = \frac{y+h}{l^+}$, $U^+ = \frac{U}{u_\tau}$

if $Re_\tau = \frac{u_\tau h}{\nu} \gg 1$ then