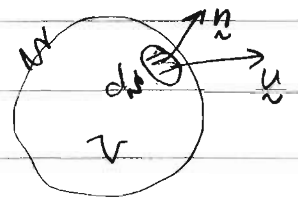
# Complete set of conservation equations.

Conservation of




1) mass : Continuity eq.
2) momentum ; Newton's law of motion
3) energy ; 1:st law of thermodynamics

1) $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0 \quad \text{Cont. eq.}$$

2) $$\rho \frac{D u_i}{D t} = \rho f_i + \frac{\partial T_{ik}}{\partial x_k} \quad \text{Cauchy's eq.}$$

In 2) let $T_{ik} = -p\delta_{ik} + 2\mu \bar{e}_{ik} + \mu_B \bar{\bar{e}}_{ik}$

$$\frac{\partial T_{ik}}{\partial x_k} = -\frac{\partial p}{\partial x_k}\delta_{ik} + \frac{\partial}{\partial x_k}\left(2\mu(T)\bar{e}_{ik}\right) + \frac{\partial}{\partial x_k}\left(\mu_B(T)\frac{\delta_{ik}}{3} e\right.$$

$$= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left(2\mu(T)\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\delta_{ik}\frac{\partial u_j}{\partial x_j}\right]\right)$$

$$+ \frac{\partial}{\partial x_i}\left(\mu_B(T)\frac{1}{3}\frac{\partial u_j}{\partial x_j}\right)$$

In 2) ⇒ Navier - Stoke's equations

– For incompressible fluid $\frac{\partial u_j}{\partial x_j} = 0$

– Assume $\mu$ = constant

$$\frac{\partial \tau_{ik}}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) =$$

$$= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \mu \frac{\partial}{\partial x_i}\left(\underbrace{\frac{\partial u_k}{\partial x_k}}_{=0}\right)$$

<u>Navier-Stoke's eq.</u> for incompr. fluid

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \rho f_i + \mu \nabla^2 u_i$$

$\Leftrightarrow$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \underbrace{-\nabla p}_{\text{net pressure force}} + \underbrace{\rho \underline{f}}_{\text{body force}} + \underbrace{\mu \nabla^2 \underline{u}}_{\text{net viscous force}}$$

per unit volume

## Conservation of energy

1:s law of thermodynamics

- Rate of change of total energy (thermal + mechanical) of material fluid volume equals rate of energy recieved by transport of heat and execution of work.

$$\frac{D}{Dt}\int_{V(t)} \varrho\left(e+\frac{1}{2}u_k u_k\right) dV = \{RTT\} =$$

$$= \frac{d}{dt}\int_{V} \varrho(\underset{\substack{\uparrow\\ \text{thermal}}}{e}+\underset{\substack{\uparrow\\ \text{mechanical}}}{\tfrac{1}{2}u_k u_k})\, dV + \underbrace{\oint\!\!\!\oint_S \varrho\left(e+\tfrac{1}{2}u_k u_k\right)u_j n_j\, dS}_{\substack{\text{net outflow of}\\ \text{energy with}\\ \text{velocity field}}}$$

energy per unit mass

$$= \underbrace{\int_V u_k \varrho f_k\, dV}_{\substack{\text{work rate by}\\ \text{body force}}} + \underbrace{\oint\!\!\!\oint_S u_k T_{kj} n_j\, dS}_{\substack{\text{work rate by}\\ \text{surface force}}} - \underbrace{\oint\!\!\!\oint_S q_j n_j\, dS}_{\substack{\text{net energy out-}\\ \text{flux by heat}\\ \text{conduction}}}$$

$$\int_V \varrho\frac{D}{Dt}\left(e+\frac{1}{2}u_k u_k\right) dV = \int_V\left(u_k\varrho f_k + \frac{\partial}{\partial x_j}\left(u_k T_{kj} - q_j\right)\right) dV$$

Differential form of total energy eq.

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2} u_k u_k\right) = u_k \rho f_k + \frac{\partial}{\partial x_j}\left(u_k T_{kj}\right) - \frac{\partial}{\partial x_j} q_j$$

Fouriers law for <u>heat conduction</u> (diffusion of heat)

$$q_j = -k \frac{\partial T}{\partial x_j} \quad ; \quad \underset{\sim}{q} = -k \nabla T$$

heat flux density vector $\underset{\sim}{q}$ [J/m$^2$ s]

thermal conductivity $k(T)$ $\quad k_{H_2O} \sim 0{,}6$ J/msK

$k_{air} \sim 0{,}025$ J/msK

thermal diffusivity $\kappa = k/\rho c_p$ [m$^2$/s]

$$-\frac{\partial q_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k(T)\frac{\partial T}{\partial x_j}\right) = \left\{\begin{matrix} k = \\ \text{const.}\end{matrix}\right\} = k \frac{\partial^2 T}{\partial x_j \partial x_j}$$

$$= \underbrace{k \nabla^2 T}_{\text{diffusion of heat}}$$

Net work rate by surface forces

$$\frac{\partial}{\partial x_j}\left(u_k T_{kj}\right) = \underset{\substack{\uparrow \\ \text{translational} \\ \text{work rate} \\ \uparrow}}{u_k \frac{\partial T_{kj}}{\partial x_j}} + \underset{\substack{\uparrow \\ \text{deformation} \\ \text{work rate} \\ \uparrow}}{T_{kj}\frac{\partial u_k}{\partial x_j}}$$

we will show : changes mech. energy | changes thermal energy

Mechanical energy eq. $u_k\left(\rho \frac{D}{Dt} u_k = \rho f_k + \frac{\partial T_{kj}}{\partial x_j}\right)$

$$\Rightarrow \quad \rho \frac{D}{Dt}\left(\frac{u_k u_k}{2}\right) = u_k \rho f_k + \underline{\underline{u_k \frac{\partial T_{kj}}{\partial x_j}}}$$

Subtract from total energy

$$\Rightarrow \quad \rho \frac{D}{Dt} e = \underline{\underline{T_{kj}\frac{\partial u_k}{\partial x_j}}} - \frac{\partial q_j}{\partial x_j}$$

Insert $T_{kj} = -p\delta_{kj} + \tau_{kj}$

$$T_{kj}\frac{\partial u_k}{\partial x_j} = \underbrace{-p\frac{\partial u_k}{\partial x_k}}_{\text{work rate}} + \underbrace{\tau_{kj} e_{kj}}_{\text{work rate}} + \underbrace{\underbrace{\tau_{kj}}_{s}\,\underbrace{\xi_{kj}}_{as}}_{=0}$$

by pressure from isotropic exp. (reversible)

by visc. str. from deformation (irreversible)

Dissipation function $\Phi = \tau_{kj} e_{kj} = \ldots =$

$$= 2\mu \bar{e}_{kj} \bar{e}_{kj} + \mu_B \left(\frac{\partial u_k}{\partial x_k}\right)^2 > 0$$

Energy equations in differential form:

$$\varrho \frac{De}{Dt} + p \frac{\partial u_k}{\partial x_k} = -\frac{\partial q_j}{\partial x_j} + \Phi$$

$$\varrho \frac{D}{Dt}\left(\frac{u_k u_k}{2}\right) + u_k \frac{\partial p}{\partial x_k} = u_k \varrho f_k + \frac{\partial}{\partial x_j}\left(u_k \tau_{kj}\right) - \Phi$$

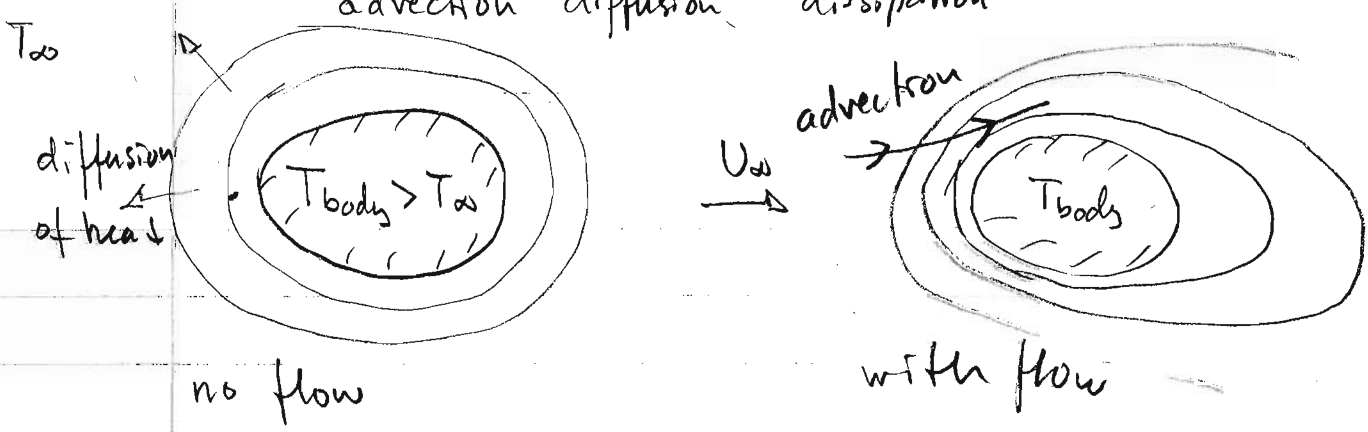
Perfect gas $e = c_v T$, $p = \varrho R T$

Incompressible fluid $\varrho = \varrho_0 = \text{const.}$ $e = cT$

$$\varrho_0 c_p \frac{DT}{Dt} = k \nabla^2 T + \Phi$$

or

$$\frac{\partial T}{\partial t} + \underbrace{\underset{\sim}{u} \cdot \nabla T}_{\text{advection}} = \underset{\text{diffusion}}{\kappa \nabla^2 T} + \underset{\text{dissipation}}{\Phi / \varrho_0 c_p}$$



Enthalpy $h = e + p/\varrho$ inserted into thermal en. eq.

$$\Rightarrow \quad \varrho \frac{D}{Dt} h - \frac{D}{Dt} p = -\frac{\partial q_j}{\partial x_j} + \Phi$$

Perfect gas $h = c_p T$ , $p = \varrho R T$

X

Details of dissipation function derivation:

$$\Phi = \tau_{ij} e_{ij} = \left(2\mu \bar{e}_{ij} + \mu_B \delta_{ij} \bar{\bar{e}}_{kk}\right)\left(\bar{e}_{ij} + \frac{1}{3}\delta_{ij} \bar{\bar{e}}_{kk}\right)$$

$$= 2\mu \bar{e}_{ij}\bar{e}_{ij} + 2\mu \frac{1}{3} \underbrace{\bar{e}_{ii}}_{=0} \bar{\bar{e}}_{kk} + \mu_B \bar{\bar{e}}_{kk} \underbrace{\bar{e}_{ii}}_{=0} + \mu_B \bar{\bar{e}}_{kk}^{\,2} \underbrace{\frac{\delta_{ii}}{3}}_{=3}$$

$$= 2\mu \underbrace{\bar{e}_{ij}\bar{e}_{ij}}_{\geq 0} + \mu_B \underbrace{\bar{\bar{e}}_{kk}^{\,2}}_{\geq 0} \quad ; \quad \bar{\bar{e}}_{kk} = \frac{\partial u_k}{\partial x_k}$$

X

Details of enthalpy derivation

$$\frac{D}{Dt} h = \frac{D}{Dt} e + \frac{1}{\varrho}\frac{Dp}{Dt} + p \underbrace{\frac{D}{Dt}\frac{1}{\varrho}}_{-\frac{1}{\varrho^2}\frac{D\varrho}{Dt} = \frac{1}{\varrho}\frac{\partial u_k}{\partial x_k}}$$

$$\frac{D h}{Dt} = \underline{\underline{\frac{De}{Dt}}} + \frac{1}{\varrho}\frac{Dp}{Dt} + \underline{\underline{\frac{p}{\varrho}\frac{\partial u_k}{\partial x_k}}}$$

# Boussinesque approximation

$$\rho \frac{D\underset{\sim}{u}}{Dt} = -\nabla p + \rho \underset{\sim}{g} + \mu \nabla^2 \underset{\sim}{u}$$

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = k \nabla^2 T + \Phi$$

Assume $\rho \approx \rho_0 = \text{constant}$, but

$$\rho \underset{\sim}{g} = (\rho_0 - \rho_0 \alpha (T - T_0)) \underset{\sim}{g}$$

coefficient of thermal expansion

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Mainly hydrostatic balance $0 = -\nabla p_0 + \rho_0 \underset{\sim}{g}$

Small departure $p = p_0(\underset{\sim}{x}) + p'$, $p' \ll p_0$, $\underset{\sim}{u}$

$$\rho_0 \frac{D}{Dt}\underset{\sim}{u} = -\nabla p' - \rho_0 \alpha (T - T_0) \underset{\sim}{g} + \mu \nabla^2 \underset{\sim}{u}$$

$$\rho_0 C_p \frac{D}{Dt} T - \underset{\sim}{u} \cdot \nabla p_0 = k \nabla^2 T + \Phi$$

$$\rho_0 C_p \left(\frac{\partial T}{\partial t} + \underset{\sim}{u} \cdot \nabla T\right) + \underbrace{\underset{\sim}{u} \cdot \rho_0 \underset{\sim}{g}}_{\sim U \rho_0 g} = k \nabla^2 T + \Phi$$

$$\sim \rho_0 C_p \frac{U \Delta T}{L}$$

$$\frac{U \varrho_0 g}{\varrho_0 C_p U \Delta T / L} \sim \frac{gL}{C_p \Delta T} \sim \frac{10\ \mathrm{m/s^2}\ 1\,\mathrm{m}}{10^3\ \mathrm{J/kgK}\ 1\,\mathrm{K}} = 10^{-2} \ll 1$$

$$\Phi \sim \mu \left(\frac{U}{L}\right)^2 \sim \frac{\mu U}{L^2} \cdot U \sim \varrho_0 \alpha \Delta T g U$$

$$\frac{\Phi}{\varrho_0 C_p \underline{u} \cdot \nabla T} \sim \frac{\cancel{\varrho_0} \alpha \Delta T g \cancel{U}}{\cancel{\varrho_0} C_p \cancel{U} \Delta T / L} \sim \underbrace{\alpha \Delta T}_{\ll 1} \underbrace{\frac{gL}{C_p \Delta T}}_{\ll 1} \ll 1$$

$$\varrho_0 C_p \frac{DT}{Dt} = k \nabla^2 T \quad \text{for Boussinesq approx.}$$