Fluid Mechanics, SG2214, HT2009
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Exercise 5: Exact Solutions to the Navier-Stokes Equations I

Example 1: Plane Couette Flow

Consider the flow of a viscous Newtonian fluid between two parallel plates located at $y = 0$ and $y = h$. The upper plane is moving with velocity U. Calculate the flow field.

Assume the following:
Steady flow:

$$\frac{\partial}{\partial t} = 0$$

Parallel, fully-developed flow:

$$v = 0, \quad \frac{\partial u_i}{\partial x} = 0$$

Two-dimensional flow:

$$w = 0, \quad \frac{\partial}{\partial z} = 0$$

No pressure gradient:

$$\frac{\partial p}{\partial x_i} = 0$$

The streamwise Navier-Stokes equation is

$$\frac{\partial u}{\partial t} + (\bar{u} \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u \ ,$$

can be simplified using the above assumptions. We get

$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = A \quad \Rightarrow \quad u = Ay + B \ .$$

With the boundary conditions

$$u(0) = 0 \quad \Rightarrow \quad B = 0, \quad u(h) = U \quad \Rightarrow \quad A = U/h$$

we finally obtain

$$\boxed{u(y) = \frac{Uy}{h} \ .}$$

Example 2: Plane Poiseuille Flow (Channel Flow)

Consider the flow of a viscous Newtonian fluid between two solid boundaries at $y = \pm h$ driven by a constant pressure gradient $\nabla p = [-P, 0, 0]$. Show that

$$u = \frac{P}{2\,\mu}(h^2 - y^2), \quad v = w = 0.$$

Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\dfrac{1}{\rho} \nabla p + \nu \nabla^2 \bar{u} \\ \nabla \cdot \bar{u} = 0. \end{cases}$$

Boundary conditions:

$$\bar{u}(y = \pm h) = 0$$

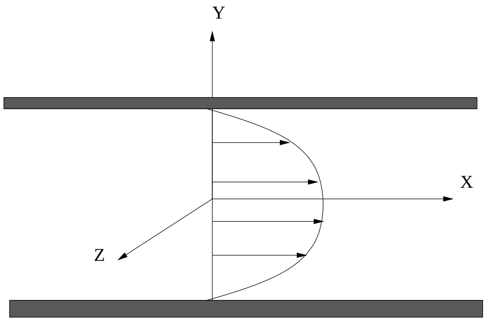

Figure 1: Coordinate system for plane Poiseuille flow.

- We are considering stationary flow and thus $\frac{\partial \bar{u}}{\partial t} = 0$.
- The constant pressure gradient implies $\bar{u} = \bar{u}(y)$. Changes of $\bar{u}$ in x, z would require a changing pressure gradient in x, z.
- The continuity equation $\nabla \cdot \bar{u}$ reduces to $\frac{\partial v}{\partial y} = 0$. The boundary condition $v(y = \pm h) = 0$ then implies $v = 0$.

Consider the spanwise (z) component of the Navier-Stokes equations:

$$\underbrace{v}_{=0} \frac{\partial w}{\partial y} = \nu \frac{\partial^2 w}{\partial y^2} \quad \Rightarrow \quad w = c_1 y + c_2$$

The boundary conditions $w(y = -h) = w(y = h) = 0$ imply $c_1 = c_2 = 0$ and thus $w = 0$. We can conclude that $\bar{u} = [u(y), 0, 0]$.

Consider now the streamwise (x) component of the Navier-Stokes equations:

$$0 = \frac{P}{\rho} + \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \frac{\partial u}{\partial y} = -\frac{P}{\nu \rho} y + d_1 \quad \{\mu = \rho \nu\} \Rightarrow u(y) = -\frac{P}{2\,\mu} y^2 + d_1 y + d_2$$

The boundary conditions at $y = \pm h$ give

$$0 = -\frac{P}{2\,\mu} h^2 + d_1 h + d_2 \quad \text{and} \quad 0 = -\frac{P}{2\,\mu} h^2 - d_1 h + d_2$$

We can directly conclude that $d_1 = 0$ and this gives $d_2 = \frac{P}{2\,\mu} h^2$. The solution is thus

$$\boxed{u = \frac{P}{2\,\mu}(h^2 - y^2), \quad v = w = 0\ .}$$

Energy Dissipation in Poiseuille Flow

a) Calculate the dissipation function for the plane Poiseuille flow computed above,

$$u = \frac{P}{2\,\mu}(h^2 - y^2), \quad v = w = 0,$$

or in terms of the bulk velocity U

$$u = \frac{3U}{2h^2}(h^2 - y^2), \quad v = w = 0.$$

The mass-flow rate through the channel is

$$Q = \int_{-h}^{h} u dy = 2Uh \ .$$

The dissipation function is defined as (dissipation to heat due to viscous stresses)

$$\Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j}.$$

For incompressible flows, it can be re-written as

$$\Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} = 2\mu e_{ij} (e_{ij} + \xi_{ij}) = 2\mu e_{ij} e_{ij},$$

where we used the fact that $e_{ij}\xi_{ij} = 0$.

The deformation tensor for the Poiseuille flow becomes $e_{ij} = 1/2 \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and therefore

$$\Phi = 2\mu \left[\left(\frac{1}{2} \frac{\partial u}{\partial y} \right)^2 + \left(\frac{1}{2} \frac{\partial u}{\partial y} \right)^2 \right] = \mu \left(\frac{\partial u}{\partial y} \right)^2 .$$

b) Calculate the total dissipation for unit area

$$\phi = \int_{-h}^{h} \Phi \, dy = \int_{-h}^{h} \mu \left(-\frac{3U}{h^2} y \right)^2 dy = \frac{6\mu U^2}{h}.$$

c) Write the mechanical energy equation for this flow. Integrate over the channel width and relate the total dissipation ϕ to the pressure gradient and the mass flux.

The mechanical energy equation is obtained by multiplying the Navier-Stokes equations by u_i (the energy is $\rho(1/2)u_i u_i$). One gets

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u_i u_i \right) = \rho F_i u_i - u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j}.$$

Considering the Poiseuille flow and re-writing the last term as

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \Phi,$$

the energy equation reduces to

$$0 = uP + \frac{\partial}{\partial y} (u\tau_{xy}) - \Phi.$$

Integrating across the channel each term in the expression above, one obtains for the first term

$$\int_{-h}^{h} uP dy = P \int_{-h}^{h} u dy = QP,$$

where Q is the flow rate. This term represents the work rate by pressure forces.

The second term

$$\int_{-h}^{h} \frac{\partial}{\partial y} (u\tau_{xy}) \, dy = [(u\tau_{xy})]_{-h}^{h} = 0$$

due to the no-slip boundary conditions.

The third term is the total dissipation $\phi = \int_{-h}^{h} \Phi \, dy$ defined above. Summarising

$$0 = QP - \int_{-h}^{h} \Phi \, dy.$$

One can check the results, using the expression for ϕ obtained in b). Just recall that

$$Q = \int_{-h}^{h} u dy = 2Uh,$$

and the pressure gradient can be expressed in terms of U as $P = \frac{3\mu U}{h^2}$. Therefore $QP = 6\mu U^2/h = \phi$.

Example 3: Poiseuille Flow (Pipe Flow)

Consider the viscous flow of a fluid through a pipe with a circular cross-section given by $r = a$ under the constant pressure gradient $P = -\dfrac{\partial p}{\partial z}$. Show that

$$u_z = \frac{P}{4\mu}(a^2 - r^2) \quad u_r = u_\theta = 0.$$

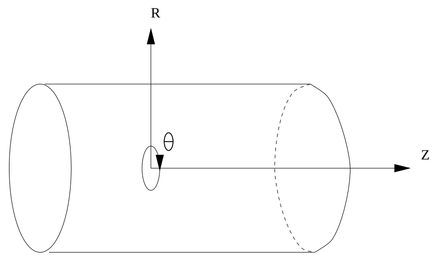

Figure 2: Coordinate system for Poiseuille flow.

Use the Navier-Stokes equations in cylindrical coordinates (see lecture notes)

$$\frac{\partial u_r}{\partial t} + (\bar{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_\theta}{\partial t} + (\bar{u} \cdot \nabla) u_\theta + \frac{u_r \, u_\theta}{r} = -\frac{1}{\rho \, r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\frac{\partial u_z}{\partial t} + (\bar{u} \cdot \nabla) u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 u_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r \, u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \ .$$

We know that $\dfrac{\partial p}{\partial \theta} = 0$ and $\dfrac{\partial p}{\partial r} = 0$ and can directly see that $u_r = u_\theta = 0$ satisfys the two first equations. From the continuity equation we get

$$\frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad u_z = u_z(r, \theta) \quad \text{only.}$$

Considering a steady flow we get from the axial component of the Navier-Stokes equations

$$(\bar{u} \cdot \nabla) u_z = u_z \frac{\partial u_z}{\partial z} \quad \Rightarrow \quad u_z \frac{\partial u_z}{\partial z} = \frac{1}{\rho} P + \nu \nabla^2 u_z \ .$$

But we know that $\frac{\partial u_z}{\partial z} = 0$ from the continuity equation. We get

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = -\frac{P}{\mu} \quad \Rightarrow \quad \frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = -\frac{P}{\mu}r$$

Integrate once in r gives

$$r\frac{\partial u_z}{\partial r} = -\frac{P}{2\,\mu}r^2 + c_1 \quad \Rightarrow \quad \frac{\partial u_z}{\partial r} = -\frac{P}{2\,\mu}r + \frac{c_1}{r}\ ,$$

and integrating again we get

$$u_z = -\frac{P}{4\,\mu}r^2 + c_1\,\ln(r) + c_2 \quad \text{using the boundary conditions } u_z(r=0) < \infty \Rightarrow c_1 = 0\ .$$

We also have $u_z = 0$ at $r = a$ and this gives $c_2 = \frac{P\,a^2}{4\,\mu}$ and we finally get

$$\boxed{u_z = \frac{P}{4\,\mu}(a^2 - r^2)\ .}$$

Example 4: Asymptotic Suction Boundary Layer

Calculate the asymptotic suction boundary layer, where the boundary layer over a flat plate is kept parallel by a steady suction V_0 through the plate.

Assumptions:
Two-dimensional flow:

$$\frac{\partial}{\partial z} = 0, \quad w = 0$$

Parallel, fully-developed flow:

$$\frac{\partial}{\partial x} = 0$$

Steady flow:

$$\frac{\partial}{\partial t} = 0$$

Momentum equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Normal momentum equation gives

$$\frac{\partial p}{\partial y} = 0$$

Boundary conditions:

$$y = 0: \quad u = 0, \quad v = -V_0$$
$$y \to \infty: \quad u \to U_\infty$$

Continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = -V_0$$

Streamwise momentum equation at $y \to \infty$

$$-V_0\frac{\partial U_\infty}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 U_\infty}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0$$

Resulting streamwise momentum equation

$$-V_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{V_0}{\nu} \frac{\partial u}{\partial y}$$

Characteristic equation

$$\lambda^2 = -\frac{V_0}{\nu} \lambda \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -\frac{V_0}{\nu}$$

$$u(y) = A + B e^{-V_0 y/\nu}$$

With the boundary conditions at $y = 0$ and $y = \infty$ we get

$$\boxed{u(y) = U_\infty \left(1 - e^{-V_0 y/\nu}\right)}.$$

Example 5: Flow on an Inclined Plate

Two incompressible viscous fluids flow one on top of the other down an inclined plate at an angle α (see figure 3). They both have the same density ρ, but different viscosities μ_1 and μ_2. The lower fluid has depth h_1 and the upper h_2. Assuming that viscous forces from the surrounding air is negligible and that the pressure on the free surface is constant, show that

$$u_1(y) = \left[(h_1 + h_2)y - \frac{1}{2}y^2\right] \frac{g \sin(\alpha)}{\nu_1}.$$

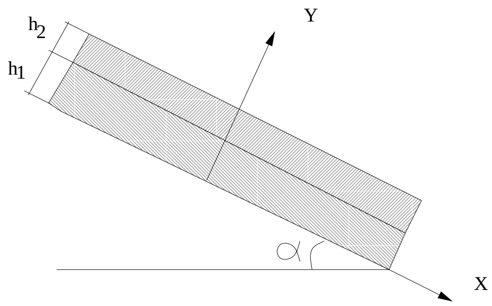

Figure 3: Coordinate system for flow down an inclined plate.

Make the ansatz $\bar{u}_1 = [u_1(y), 0, 0]$ and $\bar{u}_2 = [u_2(y), 0, 0]$. The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{gives} \quad \frac{\partial v}{\partial y} = 0 \Rightarrow v = c \quad \text{and the boundary condition at } y = 0 \text{ give } v = 0.$$

- Layer 1:

$$\text{N–S} \cdot \bar{e}_y : 0 = -\frac{1}{\rho} \frac{\partial p_1}{\partial y} - g \cos(\alpha) \quad \Rightarrow \quad p_1 = -\rho g \cos(\alpha) y + f_1(x)$$

$$\text{N–S} \cdot \bar{e}_x : 0 = -\frac{1}{\rho} f_1'(x) + \nu_1 \frac{d^2 u_1}{dy^2} + g \sin(\alpha) \quad \Rightarrow \quad f_1'(x) = c_1$$

- Layer 2:

$$\text{N–S} \cdot \bar{e}_y : 0 = -\frac{1}{\rho} \frac{\partial p_2}{\partial y} - g \cos(\alpha) \quad \Rightarrow \quad p_2 = -\rho g \cos(\alpha) y + f_2(x)$$

$$\text{N–S} \cdot \bar{e}_x : 0 = -\frac{1}{\rho} f_2'(x) + \nu_2 \frac{d^2 u_2}{dy^2} + g \sin(\alpha) \quad \Rightarrow \quad f_2'(x) = c_2$$

The pressure at the free surface $y = h_1 + h_2$ is p_0:

$$p_0 = -\rho g \cos(\alpha)(h_1 + h_2) + f_2(x) \quad \Rightarrow \quad f_2 = p_0 + \rho g(h_1 + h_2)\cos(\alpha) \quad \Rightarrow \quad f_2^{'} = 0$$

The pressure is continuous at $y = h_1$:

$$p_0 + \rho g h_2 \cos(\alpha) = -\rho g h_1 \cos(\alpha) + f_1(x) \quad \Rightarrow \quad f_1 = p_0 + \rho g(h_1 + h_2)\cos(\alpha) \quad \Rightarrow \quad f_1^{'} = 0$$

This gives the pressure:

$$p_1(y) = p_2(y) = p(y) = -\rho\, g\, \cos(\alpha) y + p_0 + \rho\, g\, \cos(\alpha)(h_1 + h_2)$$

We now have two momentum equations in x:

$$0 = \nu_1 \frac{d^2 u_1}{dy^2} + g\sin(\alpha) \quad (1)$$

$$0 = \nu_2 \frac{d^2 u_2}{dy^2} + g\sin(\alpha) \quad (2)$$

And four boundary conditions:

BC1: No slip on the plate: $u_1(0) = 0$

BC2: No viscous forces on the free surface: $\mu_2 \left.\frac{du_2}{dy}\right|_{y=h_1+h_2} = 0$

BC3: Force balance at the fluid interface: $\mu_1 \left.\frac{du_1}{dy}\right|_{y=h_1} = \mu_2 \left.\frac{du_2}{dy}\right|_{y=h_1}$

BC4: Continous velocity at the interface: $u_1|_{y=h_1} = u_2|_{y=h_1}$

$$(1) \Rightarrow \quad \frac{du_1}{dy} = -\frac{g}{\nu_1} y \sin(\alpha) + c_{11} \quad \Rightarrow \quad u_1 = -\frac{g}{2\,\nu_1} y^2 \sin(\alpha) + c_{11} y + c_{12}$$

$$(2) \Rightarrow \quad \frac{du_2}{dy} = -\frac{g}{\nu_2} y \sin(\alpha) + c_{21} \quad \Rightarrow \quad u_2 = -\frac{g}{2\,\nu_2} y^2 \sin(\alpha) + c_{21} y + c_{22}$$

$$BC1 \Rightarrow \quad c_{12} = 0$$

$$BC2 \Rightarrow \quad \mu_2(-\frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha) + c_{21}) = 0 \quad \Rightarrow \quad c_{21} = \frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha)$$

$$BC3 \Rightarrow \quad \mu_1(-\frac{g}{\nu_1} y \sin(\alpha) + c_{11}) = \mu_2(-\frac{g}{\nu_2} y \sin(\alpha) + c_{21}) \quad \{\mu = \nu\rho\} \quad \Rightarrow c_{11} = \frac{\mu_2}{\mu_1} c_{21} = \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha)$$

$$BC4 \Rightarrow \quad -\frac{g}{2\,\nu_1} h_1^2 \sin(\alpha) + \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha) h_1 = -\frac{g}{2\,\nu_2} h_1^2 \sin(\alpha) + \frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha) h_1 + c_{22}$$

$$\Rightarrow c_{22} = g\sin(\alpha)\left(\frac{h_1^2}{2} - (h_1 + h_2)h_1\right)\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)$$

This gives us the velocities:

$$u_1(y) = -\frac{g}{2\,\nu_1} y^2 \sin(\alpha) + \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha) y$$

$$\boxed{u_1(y) = \frac{g\sin(\alpha)}{\nu_1}\left[(h_1 + h_2)y - \frac{1}{2}y^2\right]}$$

$$u_2(y) = -\frac{g\sin(\alpha)}{2\,\nu_2} y^2 + \frac{g\sin(\alpha)}{\nu_2}(h_1 + h_2)y + g\sin(\alpha)\left(\frac{h_1^2}{2} - (h_1 + h_2)h_1\right)\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)$$

$$\boxed{u_2(y) = \frac{g\sin(\alpha)}{\nu_2}\left[(h_1 + h_2)y - \frac{1}{2}y^2\right] + g\sin(\alpha)\left[\frac{h_1^2}{2} - (h_1 + h_2)h_1\right]\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)}$$

The velocity in layer 1 does depend on h_2 but not on the viscosity in layer 2. This is because the depth is important for the tangential stress boundary condition at the interface, unlike the viscosity. There is no acceleration of the upper layer and thus the tangential stress must be equal to the gravitational force on the upper layer which depends on h_2 but not on ν_2.