Exercise 7: Exact solution for energy equation and axisymmetric Flow

Example 1: Exact solution for energy equation

Consider plane Poiseuille flow in a straight channel with walls at \( y = \pm h \). The temperature at the lower wall is \( T(-h) = T_W + \Delta T \), whereas the upper wall is at \( T(h) = T_W \). The velocity field is

\[
    u = \frac{3U}{2} (h^2 - y^2), \quad v = w = 0,
\]

see recitation 5.

a) Derive and plot the temperature distribution.

Let us consider the energy equation for incompressible fluid.

\[
    \rho_0 c_p \frac{D}{Dt} T = \nabla \cdot (K \nabla T) + \Phi
\]

In this case, the equation of state is simply \( \rho = \rho_0 \), and \( c_p = c_v \). We also assume \( K \) to be independent of the temperature \( T \).

We can therefore assume a steady solution and a fully developed field: \( \frac{\partial}{\partial t} T = 0; \frac{\partial}{\partial x} T = 0 \). The material derivative term is then

\[
    \frac{D}{Dt} T = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = 0 + u \cdot 0 + 0 \cdot \frac{\partial T}{\partial y} = 0,
\]

while the diffusion term reduces to \( K \nabla^2 T = K \frac{\partial^2 T}{\partial y^2} \). Finally, as shown in recitation 5, the dissipation function \( \Phi \) reduces for this case to

\[
    \Phi = 2\mu e_{ij} e_{ij} = 2\mu \left[ \left( \frac{1}{2} \frac{\partial u}{\partial y} \right)^2 + \left( \frac{1}{2} \frac{\partial u}{\partial y} \right)^2 \right] = \mu \left( \frac{\partial u}{\partial y} \right)^2.
\]

Summarising the energy equations to be solved is

\[
    0 = \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{K} \left( \frac{\partial u}{\partial y} \right)^2,
\]

with boundary conditions

\[
    \begin{align*}
    T(y = -h) &= T_W + \Delta T \\
    T(y = h) &= T_W
    \end{align*}
\]
Let us introduce dimensionless variables $y^* = \frac{y}{h}$, $T^* = \frac{T-T_W}{\Delta T}$ and $u^* = \frac{u}{U}$. Re-writing equation (1), one obtains

$$0 = \frac{\partial^2 T^*}{\partial y^*^2} + \frac{\mu U^2}{K \Delta T} \left( \frac{\partial u^*}{\partial y^*} \right)^2. \quad (2)$$

The following adimensional numbers can then be introduced

$$\frac{\mu U^2}{K \Delta T} = \frac{\mu c_p U^2}{K c_p \Delta T} = \frac{\nu}{\kappa} = Pr,$$

where $Pr$ denotes the Prandtl number, the ratio between the kinematic viscosity $\nu$ and the thermal diffusivity $\kappa$. $\frac{U^2}{c_p \Delta T} = E$ is the Eckert number defined mainly for fluids. Indeed, the same expression can be rewritten for gases in terms of the Mach number $M$ and the adiabatic constant $\gamma$ as

$$\frac{U^2}{c_p \Delta T} = \frac{M^2 u_w^2}{c_p \Delta T} = \frac{M^2 \gamma R T_W}{c_p \Delta T} = (\gamma - 1) \frac{M^2 T_W}{\Delta T}.$$

Summarising the energy equations in adimensional form is

$$0 = \frac{\partial^2 T^*}{\partial y^*^2} + Pr E \left( \frac{\partial u^*}{\partial y^*} \right)^2, \quad (3)$$

with boundary conditions

$$\begin{cases}
T(y^* = -1) = 1 \\
T(y^* = 1) = 0
\end{cases}$$

Recalling the expression for the velocity field in adimensional variables $u^* = \frac{3}{2} \left(1 - y^*^2\right)$, $\frac{\partial u^*}{\partial y^*} = -3y^*$ and $\Phi = 9y^*^2$. Therefore equation (3) can be integrated to yield

$$\frac{\partial^2 T^*}{\partial y^*^2} + Pr E 9y^*^2 = 0,$$

$$\frac{\partial T^*}{\partial y^*} + Pr E 9y^*^3 = C,$$

$$T^* + Pr E 3y^*^4 = Cy^* + D.$$

$C$ and $D$ are determined imposing the boundary conditions at $y^* = \pm 1$.

$$D = \frac{1}{2} \left(1 + \frac{3}{2} Pr E \right); \quad C = \frac{1}{2}$$

Finally the temperature field can be written as

$$T^* = \frac{1}{2} + \frac{3}{4} Pr E - \frac{1}{2} y^* - \frac{3}{4} Pr E \left( y^*^4 \right)$$

$$T^* = \frac{1}{2} (1 - y^*) + \frac{3}{4} Pr E \left(1 - y^*^4 \right),$$

and in dimensional form

$$T = 1 + \frac{1}{2} \frac{\Delta T}{T_W} \left(1 - \frac{y}{h} \right) + \frac{3}{4} Pr \frac{U^2}{c_p T_W} \left(1 - \left(\frac{y}{h} \right)^4 \right).$$

The solution is composed of two parts, the first, linear in $y$, is the temperature distribution one would obtain in the presence of a temperature difference between the two walls. The second, fourth order contribution, is the heating due to dissipation in the fluid.
b) Write the heat flux at the wall. Determine the value of \( PrE \) for which the heat flux at the lower wall is zero.

\[
\frac{\partial T^*}{\partial y^*} \bigg|_{y^*=-1} = \left[ \frac{1}{2} + \frac{3}{4} PrE (-4y^*^3) \right] \bigg|_{y^*=-1} = \frac{1}{2} (-1 + 6 PrE).
\]

The wall heat flux is

\[
q_y = -K \frac{\partial T}{\partial y} \bigg|_{y=-h} = -\frac{K}{h} \Delta T \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=-1} = -\frac{K}{h} \Delta T \frac{1}{2} (-1 + 6 PrE).
\]

Normalising

\[
\frac{q_y}{\rho c_p U \Delta T} = -\frac{K \Delta T}{h \rho c_p U \Delta T} (-1/2 + 3 PrE) = -\frac{\kappa}{U h} (-1/2 + 3 PrE) = -\frac{1}{Re Pr} (-1/2 + 3 PrE).
\]

Thus \( q_y = 0 \) if \( PrE = 1/6 \).

**Example 2: Axisymmetric Flow**

Consider incompressible and rotationally symmetric flow with no mass source at the symmetry axis. The velocity component in the direction of the axis of symmetry and the vorticity is given.

\[ u_z = \gamma z \quad \text{and} \quad \omega = \omega(r) \hat{e}_z. \]

1) Compute \( u_r(r, z) \) and \( u_\theta(r, z) \) for the two cases when

\[ a) \quad \omega(r) = 0 \quad \text{and} \quad b) \quad \omega(r) = \omega_0 e^{-r^2/\alpha^2}. \]

The vorticity is given,

\[ \omega = \nabla \times \vec{u} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix} \]

Look at the different components:

\[ \hat{e}_r : \quad \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = \omega \cdot \hat{e}_r = 0 \quad \Rightarrow \quad u_\theta = u_\theta(r) \]
\[ \bar{e}_\theta : \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = \bar{\omega} \cdot \bar{e}_\theta = 0 \quad \Rightarrow \quad u_r = u_r(r) \]

\[ \bar{e}_z : \frac{1}{r} \left[ \frac{\partial (ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right] = \bar{\omega} \cdot \bar{e}_z = \omega(r) \]

The rotational symmetry implies \( \frac{\partial}{\partial \theta} = 0 \). From the \( \bar{e}_z \) component we get,

\[ \frac{\partial}{\partial r} (ru_\theta) = r \omega(r) \quad (4) \]

We also have the incompressibility condition:

\[ \nabla \cdot \vec{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \]

From this we get,

\[ \frac{\partial}{\partial r} (ru_r) = -r \gamma \quad (5) \]

Equation (2) is valid both for case a) and b). Case a) \( \omega(r) = 0 \)

\[ (1) \Rightarrow \frac{\partial}{\partial r} (ru_\theta) = 0 \quad \Rightarrow \quad u_\theta = \frac{A}{r} \]

\[ (2) \Rightarrow \frac{\partial}{\partial r} (ru_r) = -r \gamma \quad \Rightarrow \quad \text{integrate the rhs} \quad ru_r = -\frac{r^2 \gamma}{2} + B \]

\[ \Rightarrow \quad u_r = -\frac{\gamma r}{2} + \frac{B}{r} \]

There should be no mass source at \( r = 0 \). The mass flow is,

\[ Q = \int_0^{2\pi} u_r r \, d\theta = 2\pi B - 2\pi \frac{\gamma r^2}{2}. \]

When \( r \to 0 \) then \( Q \to 2\pi B \) which gives \( B = 0 \).

\[ u_r = -\frac{\gamma r}{2} \quad \text{for both a) and b)!} \]
The circulation $\Gamma$ is,

$$\Gamma = \int_0^{2\pi} u_\theta r \, d\theta = 2\pi A \Rightarrow A = \frac{\Gamma}{2\pi}$$

Case b) $\omega(r) = \omega_0 e^{-r^2/a^2}$,

$$\frac{\partial}{\partial r} (r u_\theta) = r \omega_0 e^{-r^2/a^2}$$

Integrate the right hand side:

$$r u_\theta = -\frac{\omega_0 a^2}{2} e^{-r^2/a^2} + C \Rightarrow u_\theta = -\frac{\omega_0 a^2}{2r} e^{-r^2/a^2} + \frac{C}{r}$$

Look at the circulation,

$$\Gamma(r) = \int_0^{2\pi} u_\theta r \, d\theta = 2\pi(-\frac{\omega_0 a^2}{2} e^{-r^2/a^2} + C)$$

$$\Rightarrow \text{for } r = 0 \text{ we get } \Gamma_0 = 2\pi(-\frac{\omega_0 a^2}{2} + C) \Rightarrow C = \frac{\Gamma_0}{2\pi} + \frac{\omega_0 a^2}{2}$$

This gives

$$u_\theta = \frac{\Gamma_0}{2\pi r} + \frac{\omega_0 a^2}{2r} \left[ 1 - e^{-r^2/a^2} \right]$$

2) Consider circles $C(t)$ following the flow with a radius $R(t)$ and position $Z(t)$. Compute $R(t)$ and $Z(t)$. Can any of the flow cases be inviscid?

$$R(0) = R_0 \quad Z(0) = Z_0$$

$$\frac{dR}{dt} = u_r(R) = -\frac{1}{2} \gamma R \Rightarrow R(t) = R_0 e^{-\frac{1}{2} \gamma t}$$

$$\frac{dZ}{dt} = u_z(Z) = \gamma Z \Rightarrow Z(t) = Z_0 e^{\gamma t}$$

The circulation is then,

$$\Gamma(t) = \int_0^{2\pi} u_\theta(R) R \, d\theta = 2\pi Ru_\theta(R) = \begin{cases} \Gamma \quad a) \\ \Gamma_0 + \pi \omega_0 a^2 \left[ 1 - e^{-R^2/a^2} \right] \quad b) \end{cases}$$

The circulation is constant for a) but not for b). This means that the flow in a) can be inviscid.
Example 3

Show that the inviscid vorticity equation

$$\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \nabla)\bar{u}$$

reduces to the equation

$$\frac{D}{Dt} \left( \frac{\omega}{r} \right) = 0$$

in the case of axisymmetric flow

$$\vec{u} = u_r(r,z,t)\hat{e}_r + u_z(r,z,t)\hat{e}_z.$$ 

The vorticity in an axisymmetric flow

$$\bar{\omega} = \nabla \times \bar{u} = \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{e}_\theta = \omega \hat{e}_\theta$$

Study the right hand side of the inviscid vorticity equation

$$(\bar{\omega} \cdot \nabla)\bar{u} = \omega \frac{\partial}{\partial \theta} \left( u_r(r,z,t)\hat{e}_r + u_z(r,z,t)\hat{e}_z \right) =$$

$$\frac{\omega}{r} \frac{\partial u_r}{\partial \theta} \hat{e}_r + \frac{\omega}{r} u_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\omega}{r} \frac{\partial u_z}{\partial \theta} \hat{e}_z + \frac{\omega}{r} u_z \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\omega}{r} u_r \hat{e}_\theta$$

The left hand side of the inviscid vorticity equation gives

$$\frac{D\bar{\omega}}{Dt} = \frac{\partial \bar{\omega}}{\partial t} + (\bar{u} \cdot \nabla)\bar{\omega} = \left\{ \bar{\omega} = \omega \hat{e}_\theta \right\} = \left( \frac{\partial \omega}{\partial t} + \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \omega \right) \hat{e}_\theta$$

This gives that the inviscid vorticity equation now is

$$\frac{\partial \omega}{\partial t} + \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \omega = \frac{\omega}{r} u_r$$

Multiply by $\frac{1}{r}$

$$\frac{\partial}{\partial t} \left( \frac{\omega}{r} \right) + \frac{1}{r} \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \omega - \frac{\omega}{r^2} u_r = 0$$

Notice that

$$-\frac{\omega}{r^2} u_r = \omega u_r \frac{1}{r} \quad \text{and that} \quad \omega u_z \frac{1}{r} = 0$$

This means we can write

$$\frac{\partial}{\partial t} \left( \frac{\omega}{r} \right) + \frac{1}{r} \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \omega + \omega \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \frac{1}{r} = 0$$

And thus we have

$$\frac{\partial}{\partial t} \left( \frac{\omega}{r} \right) + \frac{1}{r} \left( u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \frac{\omega}{r} = \frac{D}{Dt} \left( \frac{\omega}{r} \right) = 0$$